# RECOVERABILITY OF SUBGRAPHS

J.T. RUSSELL

ABSTRACT. We present a number of flexible definitions for subgraphs of a graph-with-boundary and prove several straightforward recovery results. Our main goal is to clearly present some vocabulary and fundamental results for dealing with subgraphs.

Let $G = (V, V_B, E)$ denote a graph with boundary throughout this paper.

**Definition 1.** Let $G^+ = G \wedge \{v_1, v_2, ...v_k\}$ denote the graph where the vertices $v_1$, $v_2$, ..., $v_k$ are promoted to the boundary, as in [1]. We say $G^+$ is a *promotion* of $G$.

**Definition 2.** A *subgraph* of $G$ is a triple $(V', V'_B E')$ such that $V'$ is a subset of $V$, $V'_B$ is any subset of $V'$ that contains $V'_B \cap V'$, and $E'$ is a relation on $V'$ such that if $p \sim_{E'} q$ then $p \sim_E q$. That is, a subgraph is a set of vertices and some collection of the edges from $G$ that join vertices in $V'$, with additional boundary vertices chosen arbitrarily.

We say a subgraph $G_1$ *contains* a subgraph $G_2$ if $G_2$ is a subgraph of $G_1$; i.e. $V_1 \supset V_2$, $V_{1B} \supset V_{2B}$, and $E_1 \supset E_2$.

**Definition 3.** A *simple subgraph* of $G$ is a subgraph $(V', V'_B E')$ such that $V'_B = V' \cap V_B$.

**Definition 4.** A *full subgraph* of $G$ is a subgraph $(V', V'_B E')$ such that $E'$ is simply the restriction of $E$ to $V'$. That is, it contains all edges from $G$ that link vertices in $V'$.

**Definition 5.** A subgraph $G'$ of $G$ is *interior-connected* if every pair of vertices in $G'$ is connected through the interior of $G'$. I.e., for every $p$, $q$ in $S$ there exists a path joining $p$ and $q$ that goes through no boundary vertices except possibly $p$ or $q$ themselves.

**Definition 6.** An *interior-connected-component* (ICC) $G'$ of $G$ is a maximal interior-connected simple subgraph. That is, $G'$ is an interior-connected simple subgraph of $G$ such that if $H$ is an interior-connected simple subgraph that contains $G'$ then $H = G'$.

Each interior-connected component is a full subgraph. Every interior node $i$ of $G$ is contained in a unique interior-connected component, which we will denote $ICC_G(i)$. The same is not true for every boundary node: a boundary node may belong to several ICC's. The intersection of two ICC's contains only boundary nodes and boundary-boundary edges.

---

*Date*: July 23, 2003.

It has come to my attention that some of the definitions and results in this paper can be found incidentally in various forms in a number of papers, including [3], [4], and [5]. These results were produced independently in this paper, but more importantly they are assembled together in what I hope is a useful and easily accessible form.

**Definition 7.** A *Simon-subgraph* of $G$ is a full subgraph $G' = (V', V'_B, E')$ such that $V'_B$ is a union of $V' \cap V_B$ with the set of all vertices in $V'$ that are adjacent to some vertex outside of $V'$.

Note that a Simon-subgraph is fully determined by its set of vertices.

**Proposition 8.** *A Simon-subgraph $G' = (V', V'_B, E')$ of $G$ is a simple subgraph of $G \wedge V'_B$.*

**Definition 9.** Given a subgraph $G' = (V', V'_B, E')$ in $G$, its *complement* is the simple subgraph $H = (W, W_B, F)$ such that $F$ includes every edge not in $E'$, and $W$ includes all vertices not in $V'$ and all vertices that are endpoints of edges in $F$.

Note that a subgraph and its complement are *not disjoint*. Rather, their intersection consists of all vertices that are adjacent to points inside and outside of $V'$. We will refer to these points as the *topological boundary* of the subgraph, as distinct from the *electrical boundary*, which is $V'_B$. The *Simon boundary* of a subgraph is the union of its topological boundary with the boundary inherited from $G$; i.e., the Simon boundary is the set of all vertices in the topological boundary or in $V_B \cap V'$. Thus a Simon-subgraph is precisely a full subgraph whose electrical boundary coincides with its Simon boundary.

The intersection of a subgraph with its complement contains no edges.

**Definition 10.** An *isolated subgraph* $G'$ of $G$ is a simple subgraph such that no interior node of $G'$ is connected through the interior to any interior node of its complement. Equivalently, an isolated subgraph is one whose electrical boundary contains its topological boundary.

**Proposition 11.** *Any ICC of $G$ or any union of ICC's of $G$ is an isolated subgraph of $G$.*

**Proposition 12.** *Every Simon-subgraph $(V', V'_B, E')$ of $G$ is an isolated subgraph of $G \wedge V'_B$.*

**Proposition 13.** *Let $G'$ be an isolated subgraph of $G$ such that every boundary node is connected-through-the-interior to an interior node. Then $G'$ is a union of ICC's of $G$. In particular,*

$$G' = \bigcup_{i \in \text{int}\, G'} ICC_G(i) \tag{1}$$

*Proof.* Every interior vertex in $G'$ is contained in its own ICC, and since each boundary vertex is connected-through-the-interior to an interior node, it is contained in the ICC of that node. So $G'$ is contained in the union. On the other hand, if $v$ is a vertex not in $G'$, then any path from an interior node of $G'$ to $v$ must go through a boundary node. So $v$ is not contained in any of the ICC's. Thus the union is contained in $G'$; hence equality. □

**Lemma 14.** *Suppose that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ consists only of boundary nodes (and no edges). Then if $G$ is recoverable, $G_1$ and $G_2$ are recoverable.*

*Proof.* Let $\Lambda_1$ and $\Lambda_2$ be the respective response matrices for $G_1$ and $G_2$. Then we can compute $\Lambda$, the response matrix for $G$, in terms of $\Lambda_1$ and $\Lambda_2$. If $A$ designates the vertices contained only in $G_1$, $B$ the vertices in the intersection, and $C$ the vertices only in $G_2$, then

$$\Lambda = \begin{bmatrix} (\Lambda_1)_{AA} & (\Lambda_1)_{AB} & 0 \\ (\Lambda_1)_{BA} & (\Lambda_1)_{BB} + (\Lambda_2)_{BB} & (\Lambda_2)_{BC} \\ 0 & (\Lambda_2)_{CB} & (\Lambda_2)_{CC} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \tag{2}$$

Suppose we are given $\Lambda_1$. Then we can assign arbitrary positive conductivities to the edges in $G_2$ and compute the corresponding response matrix $\Lambda_2$. Next, add together $\Lambda_1$ and $\Lambda_2$ to obtain $\Lambda$ for the combined graph. Since $G$ is recoverable, we can recover all of the conductivities in $G$ from $\Lambda$—in particular, we recover all of the conductivities in $G_1$. □

By applying Lemma 14 repeatedly, we also obtain

**Corollary 15.** *If $G = G_1 \cup G_2 \cup ... \cup G_n$ and for each $i$ and $j$ the intersection $G_i \cap G_j$ contains only boundary nodes, then each $G_i$ is recoverable.*

Since (by definition) the intersection of an isolated subgraph with its complement contains only boundary nodes, we can apply Lemma 14 to obtain

**Corollary 16.** *Any isolated subgraph of a recoverable graph is recoverable.*

Thus by Proposition 11 we obtain

**Corollary 17.** *Any ICC (or any union of ICC's) of a recoverable graph is recoverable.*

And finally,

**Corollary 18.** *Any Simon-subgraph of a recoverable graph is recoverable.*

*Proof.* Let $G'$ be a Simon-subgraph of $G$, and $G$ be recoverable. By Proposition 12, $G'$ is an isolated subgraph of a promotion of $G$. By [1], any promotion of a recoverable graph is recoverable. Thus $G'$ is an isolated subgraph of a recoverable graph, and by Corollary 16, $G'$ is recoverable. □

## References

[1] Johnston, Andrew. "Modification of the Boundary Set for Solving the Inverse Problem"; 2003.
[2] Simon Pai's talk, 2003.
[3] Card, Ryan. "Using Network Amalagamation and Separation to Solve the Inverse Problem"; 2000.
[4] Ingerman, David. "Theory of Equivalent Networks and Some of Its Applications"; 1992.
[5] Carlson, Thomas. "A Natural Representation for a Class of Circular Planar Graphs"; 1999.

Stanford University / University of Washington REU
*E-mail address*: jefe@stanford.edu