CONNECTIONS AND DETERMINANTS: A GEOMETRIC FORMULATION

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Abstract. This paper contains a geometric argument for some of the connection-determinant relations discussed in section 3.7 of the book *Inverse Problems for Electrical Networks*by Edward B. Curtis and James A. Morrow.

In James A. Morrow and Edward B. Curtis' book [1], they prove the following useful property of response matrices Λ of an electrical network: If P and Q are two disjoint sets of boundary vertices with $|P| = |Q| = n$, such that $det(\Lambda(P;Q)) \neq 0$, then there exists an n-connection from P to Q. The proof they give is via a rather abstract determinantal formula. It is my hope that a more geometric argument for the same property might be an aid to intuition. In this paper, a more direct geometric proof of the theorem is obtained.

To do this we will make use of the concept of cutsets of a graph:

Definition 1. (Cutset, Minimal Cutset): *Given a graph $G = \{V, E\}$ (where V is the set of vertices and E is the set of edges) and two disjoint sets of vertices $P, Q \subset V$ a* cutset *of the pair P, Q is a set $R \subset V$ such that all paths from a vertex $p \in P$ to a vertex $q \in Q$ intersect R. A* minimal cutset *of the pair P, Q is a cutset of P, Q of minimal order among all such cutsets.*

This allows us to quote and use a theorem about cutsets, Menger's n-Arc Theorem:

Theorem 1. (Menger, 1927 [2]): *Let G be a graph with A and B two disjoint n-tuples of graph vertices. Then either G contains n pairwise disjoint AB-paths, or the minimal cutset of A, B in G has order $d < n$.*

Finally, before we begin the proof we introduce the following notation:

Notation. *Where A is a matrix, $\vec{u}$ is a vector, and P and Q are two sets of vertices, let $A(P;Q)$ be the submatrix of A with row indices in P and column indices in Q. Let $\vec{u}(P)$ be the sub-vector of $\vec{u}$ with indices in P, and let $\vec{u}[P]$ be the sub-vector of $\vec{u}$ excluding entries with indices in P.*

We are now prepared to give a geometric proof for the connection-determinant theorem.

Theorem 2. *If P and Q are two disjoint sets of boundary vertices with $|P| = |Q| = n$, such that $det(\Lambda(P;Q)) \neq 0$, then there exists an n-connection from P to Q.*

Proof. Let $\Gamma = (G, \gamma)$ be an electrical network on a graph with boundary $G = \{V, \partial V, E\}$, and suppose that P and Q are two disjoint sets of boundary vertices with $|P| = |Q| = n$, such that $det(\Lambda(P;Q)) \neq 0$.

Let $\hat{G}$ be the graph formed from G by removing all boundary vertices other than the vertices in P or Q, and all edges incident to those vertices. Let $R = \{r_1, ..., r_d\}$ be a minimal cutset of the pair P, Q, in $\hat{G}$. Therefore, in G, any path from a vertex $p_i \in P$ to a vertex $q_j \in Q$ must pass through some vertex $r_k \in R$, or a boundary vertex $s \in \partial V - (P \cup Q)$. Let $\vec{v} \in \mathbb{R}^n$ be arbitrary, and let $\vec{u} \in \mathbb{R}^n$ be the vector $\vec{u} = \Lambda(P;Q)^{-1}\vec{v}$. Let $\vec{u}' \in \mathbb{R}^{|\partial V|}$ be the vector with $\vec{u}'(Q) = \vec{u}$ and $\vec{u}'[Q] = \vec{0}$. Consider the vector $\vec{u}'$ to be a vector of boundary voltages on ∂V; this gives rise to a solution of the Dirichlet problem - let $\vec{w} \in \mathbb{R}^d$ be the vector of voltages on the vertices $r_1...r_d$ in the Dirichlet solution. Note that the current into the network at the set of nodes P is given by $(\Lambda\vec{u}')(P) = \Lambda(P,Q)\vec{u} = \vec{v}$.

Now consider the network $\Gamma' = (G', \gamma)$ obtained from $\Gamma = (G, \gamma)$ by declaring all the vertices in R to be boundary vertices, and let Λ' be the response matrix of this new network. The graph G' has the same set of vertices V, but a new set of boundary vertices $\partial V' = \partial V \cup R$. Apply a boundary voltage $\vec{w}'$ to Γ' equal to $\vec{u}'$ on ∂V and $\vec{w}$ on R. (It is possible that $\partial V \cap R$ will be non-empty, but $\vec{u}'(i) = \vec{w}(i)$ on any shared vertices i because $\vec{w}$ is part of the solution to the Dirichlet problem with boundary voltage $\vec{u}'$, so this boundary voltage is well-defined.) This boundary voltage gives rise to the same voltages on each vertex in V as the boundary voltage given by $\vec{u}'$ on the old graph Γ, so the current response is the same - in particular, the currents into the graph at the vertices in P are given by $\vec{v}$. The current is given by $\Lambda'\vec{w}'$, so $(\Lambda'\vec{w}')(P) = \vec{v}$. Since $Q + R - P$ is the set of vertices where $\vec{w}'$ is nonzero:

$$(\Lambda'\vec{w}')(P) = \Lambda'(P, Q + R - P)\vec{w}'(Q + R - P) = \vec{v}$$

and since $\vec{v}$ was chosen arbitrarily, we have:

$$Rank(\Lambda'(P, Q + R - P)) = n$$

There is no connection through G' from any vertex $q \in Q - R$ to any vertex $p \in P - R$, so $\Lambda'(P - R; Q - R) = 0$ (where 0 here is the $|P - R| \times |Q - R|$ 0-matrix). Because $Rank(\Lambda'(P; Q + R - P)) = n$, the rows of $\Lambda'(P; Q + R - P)$ are linearly independent. Ignoring the rows of $\Lambda'(P; Q + R - P)$ that correspond to vertices in $P \cap R$, we have:

$$Rank(\Lambda'(P - R; Q + R - P)) = |P - R|$$

Note that since $\Lambda'(P - R; Q - R) = 0$:

$$Rank(\Lambda'(P - R; Q + R - P)) = Rank(\Lambda'(P - R; R - P)) = |P - R|$$

Finally, have:

$$|P| - |P \cap R| = |P - R| = Rank(\Lambda'(P - R; R - P)) \leq |R - P| = |R| - |P \cap R|$$

Since $n = |P| \leq |R| = d$, the degree of the minimal cutset of the pair P, Q is at least n; by Menger's n-Arc theorem there is an n-connection between P and Q. □

References

[1] Curtis, Edward B. & Morrow, James A., *Inverse Problems for Electrical Networks*, Series on Applied Mathematics - Vol. 13, World Scientifc, New Jersey, 2000.

[2] Menger, Karl, *Kurventheorie*, Teubner, Berlin, Germany, 1932.

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