2^n TO 1 GRAPHS

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Abstract. We begin by examining the triangle in triangle network as a basis for understanding the properties of the n-gon in n-gon networks. These networks can also be viewed as cycles of n $\bigstar_4$'s connected at two boundary nodes. These graphs have the property that the inverse problem does not have a unique solution, but rather two solutions. This paper analyzes the properties of the response matrix and the recurrence relation used in solving the $\bigstar - K$ transformed network and generalizes the properties of 2^m to 1 networks composed entirely of $\bigstar_n$'s.

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1. Introduction

A graph with boundary and multiplicity (or a *multi-graph*) is a triple $G = (V, V_B, M)$ where V is a set of nodes, V_B is any subset of V, designated as the boundary nodes, and M is the *edge-multiplicity* function, defined on pairs of vertices such that $M(i, j)$ is the number of edges joining nodes i and j. M is always a non-negative integer, and $M(i, j) = M(j, i)$ [6].

We say i is adjacent to j, or $i \sim j$, if $M(i, j)$ is positive, and we say they are connected by a single edge (or a *singleton*) when $M(i, j) = 1$. If $M(i, j) = m$ is greater than one, we say that (i, j) are joined by a *cable of multiplicity* m. If $M(i, j)$ is zero or one for all pairs (i, j), we say that G is *simple* [6].

A *network* is a pair $\Gamma = (G, \gamma)$ where G is a graph and γ is a positive conductivity function defined on all cables in G. If G is a simple graph, then γ is defined on G's edges. If G is not simple, then γ assigns a conductivity to each *cable* of G, not to individual edges [6]. The *Kirchhoff matrix* for Γ, denoted K, is defined such that

$$K_{ij} = \begin{cases} \gamma_{ij} & i \sim j \\ -\sum_{k \neq i} K_{ik} & i = j \\ 0 & i \nsim j \text{ and } i \neq j \end{cases} \tag{1}$$

K is symmetric and negative semi-definite [3]. Suppose that we write

$$K = \begin{array}{cc} & \begin{array}{cc} \partial & \text{int} \end{array} \\ \begin{array}{c} \partial \\ \text{int} \end{array} & \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \end{array} \tag{2}$$

Then the *response matrix* of Γ is defined in terms of the Kirchhoff matrix as the Schur complement

$$\Lambda = K/K(I;I) = A - BC^{-1}B^\top \tag{3}$$

Given a graph G and its response matrix Λ, the inverse problem is the recovery of the original Kirchhoff matrix associated with G from Λ. A graph is called *recoverable* if it has a unique Kirchhoff matrix that maps to a given response. A graph is *n to 1 recoverable* whenever there are n distinct Kirchhoff matrices on the same graph which produce the same response matrix.

Definition 1.1. An *n-star*, denoted $\bigstar_n$, is a graph with boundary that has n boundary nodes and one interior node, where each boundary node is connected by a single edge to the interior node and there are no other edges.

Definition 1.2. The *complete graph* on n vertices, denoted K_n, is a graph with n boundary nodes, no interior nodes, and a single edge connecting every pair of nodes [6].

The response matrix of a complete graph is $\Lambda = (\lambda_{ij})$, where λ_{ij} is equal to the conductivity on the edge joining i and j. Conversely, any response matrix can be thought of as corresponding to a complete graph with conductivies equal to the entries. Thus every network Γ is response-equivalent to the complete graph whose conductivities are equal to the entries in Γ's response matrix. In particular, every n-star is response-equivalent to a complete graph, and there is a simple relationship between the conductivities of the two networks.

Consider networks on $\bigstar_n$ and K_n. Denote the n conductivities on $\bigstar_n$ by γ_1, γ_2, ..., γ_n, and the $n(n-1)/2$ conductivities on K_n by λ_{ij}. Then we can transform the star to the K that corresponds to the star's response matrix, which we calculate by interiorizing the central node. This yields the formula

$$\lambda_{ij} = \frac{\gamma_i \gamma_j}{\sum_{k=1}^{n} \gamma_k} \tag{4}$$

We refer to this as the $\bigstar - K$ transformation.

Not every complete graph (i.e., not every response matrix) corresponds to a $\bigstar$. Therefore, the characterization of the response matrices of a $\bigstar_n$ network is needed.

Theorem 1.3. *A network on a complete graph is response-equivalent to a (unique) $\bigstar$ if and only if its conductivities satisfy the following property:*

$$\lambda_{ij}\lambda_{kl} = \lambda_{ik}\lambda_{jl} \textit{ for all pairwise distinct } i, j, k, l. \tag{5}$$

Proof. For proof see [6]. $\square$

This condition is the same as the determinantal condition corresponding to the fact that a star has no two-connections. It can also be interpreted geometrically. Consider any quadrilateral formed by edges in a complete graph. The determinantal condition says that the conductivities on each pair of opposite edges have the same product. Hence we call equation (5) *the quadrilateral condition* [6].

The motivation for the examination of 2 to 1 graphs came from papers by Tracy Lovejoy and Jeffrey Russell, which apply the $\bigstar - K$ tool in order to solve the inverse problem. It is important to realize that all graphs mentioned in this paper are really multi-graphs. The K is essentially the response matrix found by taking the Schur complement of the original Kirchhoff matrix with respect to the interior nodes. Denote the n conductivities on $\bigstar_n$ by γ_1, γ_2, ..., γ_n, and the $n(n-1)/2$ conductivities on K_n by $\lambda_{ij} \geq 0$. The sign convention used in this paper is opposite of that used by Curtis and Morrow. We have adopted the convention of considering all off-diagonal conductivities in our Kirchhoff matrix to be positive. Thus terms on the diagonal are negative, chosen so that row sums are zero. The formula for recovering the conductivities in $\bigstar_n$ is:

$$\gamma_i = -\lambda_{ii} + \frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}} = \sum_{i \neq j} \lambda_{ij} + \frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}} \tag{6}$$

where i, j, k are pairwise distinct[4]. In order to recover a particular edge of a star embedded in a network, all conductivities from the response matrix required for calculation must be known. When cables of multiplicity greater than 1 are created in $\bigstar - K$ transformations of adjacent stars in the original network, it is difficult to recover the individual conductivities of these edges since only their sum appears as a single entry in the response matrix. These cables allow for non-unique solutions of the inverse problem, hence n to 1 networks.

2. The n-gon in n-gon Networks

Figure 1 is a simple triangle in triangle network. It can be perceived as three $\bigstar_4$'s centered at interior nodes 7, 8, and 9, connected in a cycle. After applying $\bigstar - K$ transformation on the network, the resulting network becomes three K_4's connected in a cycle, a completely connected graph of six boundary nodes without interior nodes, where edges (1,4), (2,5), and (3,6) are cables of multiplicity 2.

Definition 2.1. An n-gon in n-gon (or polygon in polygon) graph is a sequence $C_1C_2C_3...C_nC_1$ of n $\bigstar_4$'s where two boundary nodes of C_i are identified with two boundary nodes of C_{i+1} such that the resulting graph is a cycle resembling one polygon embedded in another.

The convention for numbering nodes of an n-gon in n-gon graph will be to start from the inner polygon and number the boundary nodes clockwise around the polygon. Then number the boundary nodes of the outer polygon starting from the node aligned with node 1. The interior nodes are then numbered clockwise beginning with the interior node adjacent to node 1 and node n. (Refer to figures 1, 2 and 3.)

To recover the single edge α in the $\bigstar - K$ transformed network in figure 1, we can use the quadrilateral conditions on the K_4 transformed from the $\bigstar_4$ centered at node 7 in figure 1 and continue in a cycle around the triangle counterclockwise

[4], resulting in the equation:

$$\alpha = \lambda_{14} - \cfrac{\lambda_{12}\lambda_{45}}{\lambda_{25} - \cfrac{\lambda_{23}\lambda_{56}}{\lambda_{36} - \cfrac{\lambda_{13}\lambda_{46}}{\alpha}}}. \tag{7}$$

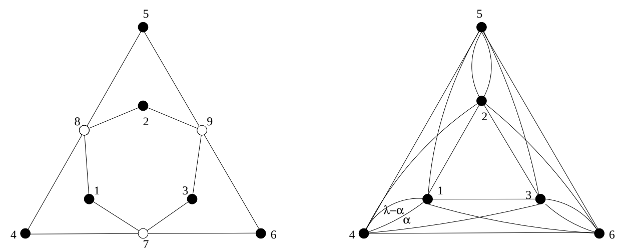

FIGURE 1. The triangle in triangle network before and after $\bigstar - K$ transformation.

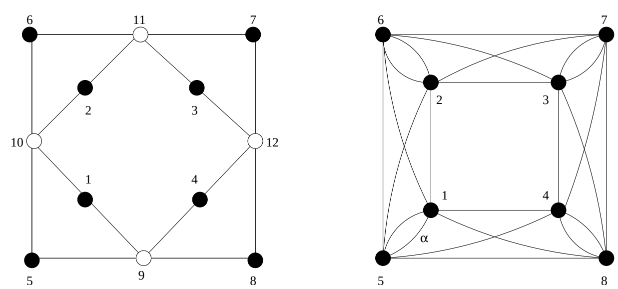

FIGURE 2. The square in square network before and after $\bigstar - K$ transformation.

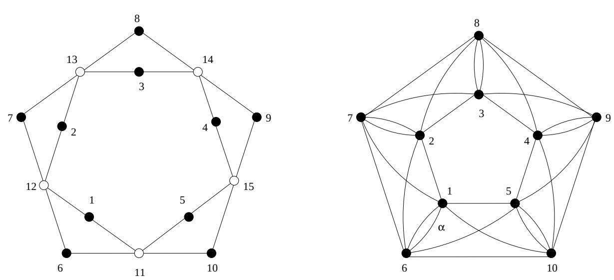

FIGURE 3. The pentagon in pentagon network before and after $\bigstar - K$ transformation.

Simplifying the equation results in a quadratic in terms of entries in the response matrix. In particular, all polygon in polygon networks will be studied in a similar manner by examining a continued fraction that can be expressed as a quotient of terms, which satisfy a recursive determinantal relation. The continued fraction is:

$$a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \cfrac{b_5}{a_5 + \dots \cfrac{b_k}{a_k}}}}} = \frac{p_k}{q_k} \tag{8}$$

where p_k and q_k are defined recursively as:

$$\begin{aligned} p_k &= a_k p_{k-1} + b_k p_{k-2} \text{ and,} \\ q_k &= a_k q_{k-1} + b_k q_{k-2} \end{aligned} \tag{9}$$

and the initial values are defined to be $p_0 = 1$, $p_1 = a_1$, $q_1 = 1$, and $q_2 = a_2$ [2]. By inspection of equations (7) and (9), a generalization of the continued fraction in terms of α and entries of the response matrix Λ for arbitrary n-gon in n-gon networks, where $k = n+1$, follows by noticing that $a_k = \alpha$ and $\frac{p_k}{q_k} = \alpha$. The parameter α satisfies:

$$\alpha = \frac{\alpha p_{k-1} + b_k p_{k-2}}{\alpha q_{k-1} + b_k q_{k-2}}. \tag{10}$$

We can rewrite this equation in quadratic form:

$$\alpha^2 q_{k-1} + \alpha(b_k q_{k-2} - p_{k-1}) - b_k p_{k-2} = 0. \tag{11}$$

These p_k's and q_k's are determinants of a tri-diagonal matrix [2] which is a sub-matrix of the response matrix Λ. For any matrix M, $M(i,j;n,m)$ is the sub-matrix of M formed by the i^{th} and j^{th} rows and the n^{th} and m^{th} columns. We will define P_j to be the tri-diagonal matrix of which p_j is the determinant, and Q_j analogously. Thus $q_n = det\Lambda(2, n+3, 4...; n+2, 3, n+4...)$ [4]. For any n-gon in n-gon network, the continued fraction will terminate at the index $k = n+1$, and p_k is the determinant of the following $n+1$ by $n+1$ matrix where α is the bottom rightmost entry:

$$P_k = \begin{bmatrix} \lambda_{1,n+1} & \lambda_{1,2} & 0 & \dots & 0 & 0 & 0 \\ \lambda_{n+1,n+2} & \lambda_{2,n+2} & \lambda_{2,3} & 0 & \dots & 0 & 0 \\ 0 & \lambda_{n+2,n+3} & \lambda_{3,n+3} & \dots & 0 & 0 & 0 \\ 0 & \vdots & \ddots & \dots & 0 & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & & \dots & \lambda_{n-1,2n-1} & \lambda_{n-1,n} & 0 \\ 0 & 0 & \dots & 0 & \lambda_{2n-1,2n} & \lambda_{n,2n} & \lambda_{1,n} \\ 0 & 0 & \dots & 0 & 0 & \lambda_{n+1,2n} & \alpha \end{bmatrix}. \tag{12}$$

Note that $a_j = \lambda_{j,n+j}$ is the j^{th} diagonal entry, and $b_j = -\lambda_{j,j+1}\lambda_{n+j,n+1+j}$, a product of off-diagonal terms. For any matrix M, let $M[i;j]$ denote the sub-matrix of M formed by removing the i^{th} row and j^{th} column. The recursively defined p's and q's are all sub-determinants of this matrix: $p_{k-1} = |P_k[n+1; n+1]| = |P_{k-1}|$, $p_{k-2} = |P_k[n, n+1; n, n+1]|$. Similarly, $q_{k-1} = |P_k[1, n+1; 1, n+1]| = |P_{k-1}[1,1]|$, and $q_{k-2} = |P_{k-2}[1,1]|$. In order for the solutions to equation (11) to be positive real conductivities, it is important to determine the signs of the determinants p_{k-1},

p_{k-2}, q_{k-1}, and q_{k-2}. As we have adopted the convention of considering all off-diagonal conductivities in our Kirchhoff matrix to be positive, all of the entries in matrix P_k are positive.

3. Tri-Diagonal Matrices

The following results all rely on some important properties unique to the determinants of tri-diagonal matrices. We have already mentioned that these determinants obey a recursive relation. The result of interest is that redistribution across the main diagonal doesn't change determinants. Recall that b_i stands for the negative of the product of the two off-diagonal terms across the diagonal term a_i.

Claim 3.1. Given a tri-diagonal matrix M, which is $n \times n$ with diagonal entries $a_1, a_2, ...a_n$, if $\widetilde{M}$ is a new matrix such that M and $\widetilde{M}$ have the same diagonal entries, then $detM = det\widetilde{M}$ if $b_n = \tilde{b}_n$.

Proof. The proof goes by induction on n. Assume $p_i = \tilde{p}_i$ and $q_i = \tilde{q}_i$ for all $i < n$. The base case is trivial since $p_1 = \tilde{p}_1 = a_1$ and $q_2 = \tilde{q}_2 = a_2$ by assumption. Now $|M| = p_n = a_n p_{n-1} + b_n p_{n-2}$ and $|\widetilde{M}| = \tilde{p}_n = a_n \tilde{p}_{n-1} + \tilde{b}_n \tilde{p}_{n-2}$, which in turn equals $a_n p_{n-1} + \tilde{b}_n p_{n-2}$ by the induction hypothesis. Thus $p_n = \tilde{p}_n$ if $b_n = \tilde{b}_n$. □

Intuitively, this explains that in the determinant of a tri-diagonal matrix, any off-diagonal term only occurs in product with its off-diagonal pair.

Corollary 3.2. *Given matrices M and $\widetilde{M}$ as in Claim 3.1, M and $\widetilde{M}$ have the same eigenvalues.*

Proof. Let M and $\widetilde{M}$ be the matrices with diagonal entries $a_1, a_2, ..., a_n$ such that the products of the off-diagonal terms are all the same as in Claim 3.1. Construct new matrices M_λ and $\widetilde{M_\lambda}$ such that the diagonal entries are now $a_1 - \lambda, a_2 - \lambda, ..., a_n - \lambda$. By Claim 3.1, $detM_\lambda = det\widetilde{M_\lambda}$. Written as polynomials in λ, these determinants are the characteristic polynomials of M and $\widetilde{M}$, and are in fact the same polynomial. Thus M and $\widetilde{M}$ have the same eigenvalues. □

Corollary 3.3. *Any tri-diagonal matrix M with all real entries has all real eigenvalues.*

Proof. Redistribution across the diagonal of the matrix M may be done to produce a matrix $\widetilde{M}$ such that $\widetilde{M}$ is symmetric. Since all of the eigenvalues of $\widetilde{M}$ are real, by Corollary 3.2, M has all real eigenvalues. □

4. Properties of n-gon in n-gon Networks

This section assumes that we have a response matrix which comes from a n-gon in n-gon network with positive real conductivities in order to prove necessary sign conditions.

Theorem 4.1. *If $\Lambda = (\lambda_{ij})$ is the response matrix of an n-gon in n-gon network with conventionally numbered nodes and positive real conductivities, and the matrix P_k with $k = n + 1$ is formed from the entries in Λ, then for all $j \leq k$, the determinants p_j and q_j are strictly positive.*

Before proving this theorem, it is important to go back and understand the original Kirchhoff matrix from which our response matrix is derived in order to better understand the entries in P_k. The polygon in polygon is a flower, which means that there are no boundary to boundary edges, nor boundary spikes. Also, there are no connections between interior nodes. Hence, in the Kirchhoff matrix, thought of in block form written in the usual way, both A and C are diagonal. Thus all entries are completely specified by the entries in B, since the entries in A and C are found by requiring row sums to be zero. Due to the nature of the n-gon in n-gon network, B in the Kirchhoff matrix looks like the matrix below.

$$\begin{bmatrix}
\gamma_{1,2n+1} & \gamma_{1,2n+2} & 0 & 0 & \dots & \dots & 0 \\
0 & \gamma_{2,2n+2} & \gamma_{2,2n+3} & 0 & \dots & \dots & 0 \\
0 & 0 & \gamma_{3,2n+3} & \gamma_{3,2n+4} & 0 & \dots & 0 \\
\vdots & \vdots & & \ddots & \ddots & & \vdots \\
0 & \dots & \dots & \dots & 0 & \gamma_{n-1,3n-1} & \gamma_{n-1,3n} \\
\gamma_{n,2n+1} & 0 & 0 & \dots & 0 & 0 & \gamma_{n,3n} \\
\gamma_{n+1,2n+1} & \gamma_{n+1,2n+2} & 0 & 0 & \dots & \dots & 0 \\
0 & \gamma_{n+2,2n+2} & \gamma_{n+2,2n+3} & 0 & 0 & \dots & 0 \\
0 & 0 & \gamma_{n+3,2n+3} & \gamma_{n+3,2n+4} & 0 & \dots & 0 \\
\vdots & \vdots & & \ddots & \ddots & & \vdots \\
0 & \dots & 0 & 0 & 0 & \gamma_{2n-1,3n-1} & \gamma_{2n-1,3n} \\
\gamma_{2n,2n+1} & 0 & 0 & \dots & 0 & 0 & \gamma_{2n,3n}
\end{bmatrix}$$

The entries in C are the negative of the column sums of B which for convenience will be denoted

$$C = \begin{bmatrix}
-\sigma_1 & 0 & \dots & 0 \\
0 & -\sigma_2 & 0 & \dots \\
\vdots & 0 & \ddots & 0 \\
0 & \dots & 0 & -\sigma_n
\end{bmatrix}.$$

After taking the Schur complement given by the formula $\Lambda = A - B^T C^{-1} B$, the formulas for the entries in P_k can easily be written down. The diagonal terms for $i \leq n$ are given by:

$$\lambda_{i,n+i} = \frac{\gamma_{i,2n+i}\gamma_{n+i,2n+i}}{\sigma_i} + \frac{\gamma_{i,2n+i\mathrm{mod}(n)+1}\gamma_{n+i,2n+i\mathrm{mod}(n)+1}}{\sigma_{(i+1)\mathrm{mod}(n)}} = \beta_{i,i} + \beta_{i,i+1}. \tag{13}$$

Thus we define

$$\beta_{i,i} = \frac{\gamma_{i,2n+i}\gamma_{n+i,2n+i}}{\sigma_i},$$

and

$$\beta_{i,i+1} = \frac{\gamma_{i,2n+i\mathrm{mod}(n)+1}\gamma_{n+i,2n+i\mathrm{mod}(n)+1}}{\sigma_{(i+1)\mathrm{mod}(n)}}$$

where these β's represent single edges in the cables of multiplicity 2. The off diagonal terms defined for $i < n$ are given by the formulas:

$$\begin{aligned}
\lambda_{i,i+1} &= \tfrac{\gamma_{i,2n+i+1}\gamma_{i+1,2n+i+1}}{\sigma_{i+1}} \\
\lambda_{n+i,n+i+1} &= \tfrac{\gamma_{n+i,2n+i+1}\gamma_{n+i+1,2n+i+1}}{\sigma_{i+1}}
\end{aligned} \tag{14}$$

Substituting equation (13) written in terms of β's for space economy, into equation (12), P_{k-1} can be rewritten. At this point, we look at equations (13) and (14), in order to recognize that :

$$\lambda_{i,i+1}\lambda_{n+i,n+i+1} = \beta_{i,i+1}\beta_{i+1,i+1} = b_{i+1} \tag{15}$$

Equation (15) is the quadrilateral condition involving two single edges in opposite cables of multiplicity 2 and two connecting edges. By Claim 3.1 redistribution across the main diagonal preserves the determinants p_j and q_j, for all j, since for all i the value of b_i is left unchanged. Substituting the result of equation (15) into the matrix P_{k-1} such that all entries are written in terms of the variable β yields the matrix $\widetilde{P}_{k-1}$=

$$\begin{bmatrix} \beta_{1,1}+\beta_{1,2} & \beta_{1,2} & 0 & \dots & 0 \\ \beta_{2,2} & \beta_{2,2}+\beta_{2,3} & \beta_{2,3} & 0 & \dots \\ 0 & \beta_{3,3} & \beta_{3,3}+\beta_{3,4} & \dots & 0 \\ 0 & \vdots & \ddots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \beta_{k-2,k-2}+\beta_{k-2,k-1} & \beta_{k-2,k-1} \\ 0 & \dots & 0 & \beta_{k-1,k-1} & \beta_{k-1,k-1}+\beta_{k-1,k} \end{bmatrix}.$$

With $\widetilde{P}_{k-1}$ in this form, it is obvious that the diagonal terms are equal to the sum of the off diagonal terms, except for the first and last rows whose diagonal terms are strictly greater than the row sums. Clearly every principal 2×2 determinant is greater than zero. The proof that p_i and $q_i > 0$, for all i is motivated by the Gerschgorin Circle Theorem.

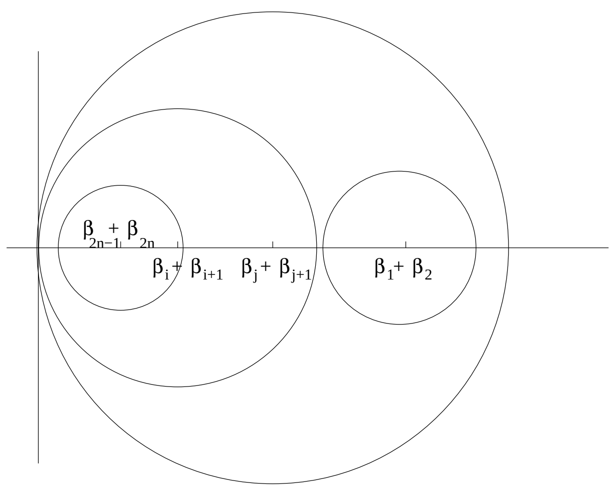

FIGURE 4. Illustration of Gerschgorin circles for P_{k-1}.

Theorem 4.2 (Gerschgorin Circle Theorem)**.** [1] *Let A be a square complex matrix. Around every element a_{ii} on the diagonal of the matrix, we draw a circle with radius the sum of the norms of the other elements on the same row $\sum_{j\neq i}|a_{ij}|$. Such circles are called Gershgorin disks. Every eigenvalue of A lies in the union of these Gershgorin disks.*

Proof. For proof see [1]. □

Proof. (of Theorem 4.1) P_k is a square real matrix. The eigenvalues are all real by corollary 3.3 and the diagonal entries are all greater than or equal to the row sums. By Gerschgorin Circle Theorem, all eigenvalues of P_{j-1} for $j < k$ are greater than or equal to zero. This implies that $p_{j-1} \geq 0$ for all such j. The proof that in fact all eigenvalues are strictly greater than zero goes by contradiction. Assume there exists j such that $p_j = 0$, meaning at least one of the eigenvalues of P_j is equal to zero. The recursion relation says that $p_{j+1} = a_{j+1}p_j + b_{j+1}p_{j-1} = b_{j+1}p_{j-1}$, however, since $b_{j+1} < 0$ and both p_{j-1} and $p_{j+1} \geq 0$, if $p_{j-1} \neq 0$, then p_{j+1} will be negative, which can not happen. Thus both p_{j-1} and $p_{j+1} = 0$. Continuing this argument replacing within the recursion relation $j-1$ and $j+1$ for j forces all $p_i = 0$ for all $i < k$. This is a contradiction since

$$p_2 = \begin{vmatrix} \beta_{1,1} + \beta_{1,2} & \beta_{1,2} \\ \beta_{2,2} & \beta_{2,2} + \beta_{2,3} \end{vmatrix} = \beta_{1,2}\beta_{2,2} + \beta_{1,1}\beta_{2,3} + \beta_{1,2}\beta_{2,3} > 0.$$

Therefore no eigenvalue of any $P_j = 0$, thus all $p_j > 0$. The proof that all $q_j > 0$ follows the same argument. □

Recall equation (11): $\alpha^2 q_{k-1} + \alpha(b_k q_{k-2} - p_{k-1}) - b_k p_{k-2} = 0$. Let $a = q_{k-1}$, $b = b_k q_{k-2} - p_{k-1}$, and $c = -b_k p_{k-2}$ in order to talk about the quadratic formula in terms of the usual variables:

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{16}$$

Theorem 4.1 shows that $a > 0$. Since for all j, $-b_j > 0$, $c > 0$ also. Similarly, $b < 0$. Therefore, $|\sqrt{b^2 - 4ac}| < |b|$. Thus both solutions of the quadratic formula are real and positive when the discriminant is greater than or equal to zero. Since we made the assumption that our response matrix comes from a Kirchhoff matrix with positive real conductivities as entries and the edge α came from the $\bigstar - K$ transformation on the $\bigstar_4$ centered at interior node $2n+1$, one of the roots α of (11) is $\frac{\gamma_{1,2n+1}\gamma_{n+1,2n+1}}{\sigma_1}$. It will be proven in section 5 that the discriminant must always be greater than or equal to zero. In conclusion, whenever a response matrix comes from an n-gon in n-gon network, there will always be two positive (not necessarily different) real solutions for α. Since α is used in solving for the conductivities of the original network, there are two Kirchhoff matrices corresponding to two n-gon in n-gon networks for each valid response matrix.

Remark 4.3. The proof of Theorem 4.1 utilized specific quadrilateral conditions in our cabled graph. However, any quadrilateral condition would suffice. This choice was arbitrary. The proof only relied on Claim 3.1 which allows redistribution across the main diagonal of the tri-diagonal matrix. This allows us to generalize our results to conclude that anytime one uses the quadrilateral conditions in a cycle of K_4's to get a continued fraction as detailed above, the resulting equation will always be a quadratic equation that yields two real positive solutions .

5. Conditions for a Unique Solution

The goal of this section is to understand the conditions required on the conductivities in the original Kirchhoff matrix of an n-gon in n-gon network for there to be one unique solution. Our initial approach to the problem assumes that, given an n-gon in n-gon network with positive real conductivities, the quadratic equation (11) has a unique solution (single root of multiplicity 2). Thus, knowing that

$\alpha = \frac{\gamma_{1,2n+1}\gamma_{n+1,2n+1}}{\sigma_1}$ and assuming α is the only solution yields $\alpha^2 = \frac{-b_{n+1}p_{n-1}}{q_n}$. Squaring the first equation and setting both forms of α^2 equal results in the equation:

$$\frac{p_{n-1}}{q_n} = \frac{\gamma_{1,2n+1}\gamma_{n+1,2n+1}}{\gamma_{n,2n+1}\gamma_{2n,2n+1}} \tag{17}$$

Study of equation (17) gives us the intuition to rewrite the recursive formulas for the determinants of P_{k-1}. This allows us to prove directly the conditions necessary for unique recoverability of the n-gon in n-gon networks. The results that follow are based on special properties of the tri-diagonal matrix $\widetilde{P}_{k-1}$, the modified matrix associated with P_{k-1} in Section 4.

Claim 5.1. The recursive determinantal relations, for $det(\widetilde{P}_{k-1}) = p_{k-1}$ may be rewritten as:

$$p_k = \beta_{1,1}q_k + \prod_{i=1}^{k}\beta_{i,i+1} \tag{18}$$

$$q_k = \beta_{k,k+1}q_{k-1} + \prod_{i=2}^{k}\beta_{i,i} \tag{19}$$

Proof. The proof goes by induction. Assume for all $1 \le k < k+1$, $p_k = \beta_{1,1}q_k + \prod_{i=1}^{k}\beta_{i,i+1}$. The base case is simple, because $p_1 = \beta_{1,1}q_1 + \beta_{1,2} = \beta_{1,1} + \beta_{1,2}$ is the first entry in the matrix $\widetilde{P}_k$. Since $p_{k+1} = a_{k+1}p_k + b_{k+1}p_{k-1}$ by definition,

$$p_{k+1} = (\beta_{k+1,k+1} + \beta_{k+1,k+2})p_k - \beta_{k,k+1}\beta_{k+1,k+1}p_{k-1}.$$

Then by the inductive hypothesis,

$$p_{k+1} = (\beta_{k+1,k+1} + \beta_{k+1,k+2})(\beta_{1,1}q_k + \prod_{i=1}^{k}\beta_{i,i+1})$$

$$-\beta_{k,k+1}\beta_{k+1,k+1}(\beta_{1,1}q_{k-1} + \prod_{i=1}^{k-1}\beta_{i,i+1}),$$

which in turn equals

$$\beta_{1,1}[(\beta_{k+1,k+1}\beta_{k+1,k+2})q_k - \beta_{k,k+1}\beta_{k+1,k+1}q_{k-1}] + \prod_{i=1}^{k+1}\beta_{i,i+1}.$$

Since $a_{k+1} = \beta_{k+1,k+1} + \beta_{k+1,k+2}$ and $b_{k+1} = -\beta_{k,k+1}\beta_{k+1,k+1}$,

$$p_{k+1} = \beta_{1,1}q_{k+1} + \prod_{i=1}^{k+1}\beta_{i,i+1}$$

and this completes the proof of the first relation. Assume for all $2 \le k < k+1$, $q_k = \beta_{k,k+1}q_{k-1} + \prod_{i=2}^{k}\beta_{i,i}$. The base case is simple, because $q_2 = \beta_{2,3}q_1 + \beta_{2,2} = \beta_{2,3} + \beta_{2,2}$. Since $q_{k+1} = a_{k+1}q_k + b_{k+1}q_{k-1}$ by definition,

$$q_{k+1} = (\beta_{k+1,k+1} + \beta_{k+1,k+2})q_k - \beta_{k,k+1}\beta_{k+1,k+1}q_{k-1}.$$

$$= \beta_{k+1,k+2}q_k + \beta_{k+1,k+1}q_k - \beta_{k,k+1}\beta_{k+1,k+1}q_{k-1}.$$

By the inductive hypothesis,

$$(\beta_{k+1,k+1})q_k = (\beta_{k+1,k+1}\beta_{k,k+1})q_{k-1} + \beta_{k+1,k+1}\prod_{i=2}^{k}\beta_{i,i}.$$

Substitution yields

$$q_{k+1} = (\beta_{k+1,k+2})q_k + \prod_{i=2}^{k+1}\beta_{i,i},$$

which concludes the proof. □

Theorem 5.2. *The discriminant* $\sqrt{b^2-4ac}$ *for a* n*-gon in* n*-gon network is always real.*

Proof. Recall that

$$\begin{aligned} a &= q_{k-1} \\ b &= b_k q_{k-1} - p_{k-1} \\ c &= -b_k p_{k-2} \qquad\text{and} \\ b_k &= -\beta_{1,1}\beta_{k-1,k}. \end{aligned}$$

Applying equations (1) and (2) in order to write everything in terms of q_{k-2} we find that

$$\begin{aligned} a &= \beta_{k-1,k}q_{k-2} + \textstyle\prod_{i=2}^{k-1}\beta_{i,i} \\ b &= -2(\beta_{1,1}\beta_{k-1,k})q_{k-2} - \textstyle\prod_{i=1}^{k-1}\beta_{i,i} - \prod_{i=1}^{k-1}\beta_{i,i+1} \\ c &= (\beta_{1,1})^2\beta_{k-1,k}q_{k-2} + (\beta_{1,1})^2\textstyle\prod_{i=1}^{k-1}\beta_{i,i+1}. \end{aligned}$$

Thus

$$\begin{aligned} b^2-4ac &= 4(\beta_{1,1}\beta_{k-1,k})^2(q_{k-2})^2 + 4\beta_{1,1}\beta_{k-1,k}(\textstyle\prod_{i=1}^{k-1}\beta_{i,i} \\ &\quad + \textstyle\prod_{i=1}^{k-1}\beta_{i,i+1})q_{k-2} + (\prod_{i=1}^{k-1}\beta_{i,i} + \prod_{i=1}^{k-1}\beta_{i,i+1})^2 \\ &\quad -4(\beta_{1,1}\beta_{k-1,k})^2(q_{k-2})^2 - 4\beta_{1,1}\beta_{k-1,k}\textstyle\prod_{i=1}^{k-1}\beta_{i,i+1} \\ &\quad -4\beta_{1,1}\beta_{k-1,k}\textstyle\prod_{i=1}^{k-1}\beta_{i,i} - 4\prod_{i=1}^{k-1}\beta_{i,i}\prod_{i=1}^{k-1}\beta_{i,i+1} \\ &= (\textstyle\prod_{i=1}^{k-1}\beta_{i,i} - \prod_{i=1}^{k-1}\beta_{i,i+1})^2 \geq 0. \end{aligned}$$

Therefore the square root is always real. □

Corollary 5.3. *A* n*-gon in* n*-gon network has a unique solution if and only if* $\prod_{i=1}^{k-1}\beta_{i,i} = \prod_{i=1}^{k-1}\beta_{i,i+1}$ *if and only if* $\prod_{i=1}^{n}\pi_i = \prod_{i=1}^{n}\omega_i$ *where* $\beta_{i,i} = \frac{\pi_i\pi_{n+i}}{\sigma_i}$ *and* $\beta_{i,i+1} = \frac{\omega_i\omega_{n+i}}{\sigma_{(i \bmod n)+1}}$*. (Refer to Figure 5.)*

Consequently, if we denote the conductivities of an n-gon in n-gon network as in figure 5, then the condition for the network to have a unique solution is that the product of π's equals to the product of ω's. This surprisingly simple and elegant result was not expected in earlier studies.

Conjecture 5.4. We believe that the type of symmetry involved in this geometric mean of products of conductivities is related to the method of monodromy used in recovering the 2 solutions to this type of looped network. There is room for further exploration on this interpretation.

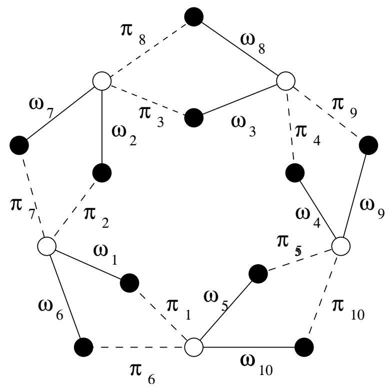

FIGURE 5. Illustration of the geometry of the $\beta_{i,j}$'s.

6. Characterization of the Response Matrices

Aided by the culmination of the extensive work that has been done on the $\bigstar - K$ transformation by Jeff Russell and Tracy Lovejoy, we were able to find and prove valid the characterization of the response matrices of n-gon in n-gon networks. We assume that we are given a Kirchhoff matrix and list the algebraic properties necessary for such a matrix to be a valid response matrix of an n-gon in n-gon network.

Theorem 6.1. *Any matrix Λ, which has the properties of a Kirchhoff matrix, is the response matrix for an n-gon in n-gon network if and only if:*

I. *For $n \geq 3$, $|\Lambda| = 2n$.*

II. *For $i < j$,*

$$\lambda_{ij} \neq 0 \iff j = i+1, n+i-1, n+i, n+i+1 \text{ or } i = 1 \text{ and } j = 2n.$$

III. *If matrix P_k is constructed as in equation (12), then*

$$\begin{aligned} &q_{k-1} > 0 && \text{and,} \\ &-\lambda_{1,n}\lambda_{n+1,2n}q_{k-2} - p_{k-1} < 0 && \text{and,} \\ &\lambda_{1,n}\lambda_{n+1,2n}p_{k-2} > 0 && \text{and,} \\ &(-\lambda_{1,n}\lambda_{n+1,2n}q_{k-2} - p_{k-1})^2 - 4\lambda_{1,n}\lambda_{n+1,2n}p_{k-2}q_{k-1} \geq 0 && \end{aligned}$$

IV. *For $i \leq n$,*

$$\lambda_{i,(i \text{ mod } n)+1}\lambda_{n+i,n+(i \text{ mod } n)+1} = \lambda_{(i \text{ mod } n)+1,n+i}\lambda_{i,n+(i \text{ mod } n)+1}$$

Proof. Certainly any matrix which does not satisfy properties I and II under any possible rearrangement of rows cannot be a response matrix for an n-gon in n-gon network. When property III is satisfied, there will be two real and positive solutions to equation (11). This is equivalent to verifying that all the quadrilateral conditions involving the cables of multiplicity 2 and the parallel edges connecting them are satisfied because it is how we found the two solutions in the first place - by using quadrilateral conditions. However, this only enforces one of the three quadrilateral conditions which are present in a K_4 transformed from a $\bigstar_4$. Thus property IV completes the set of quadrilateral conditions. And by Jeff Russell's theorem that K_4 satisfies all the quadrilateral relations iff it came from a $\bigstar_4$, we've proven that each K_4 in the network came from a $\bigstar_4$. Therefore, the original network must have been an n-gon in n-gon network because it is a cycle of $\bigstar_4$'s. Hence this list of properties completely characterizes the response matrices of n-gon in n-gon networks. □

7. The Counting Principle

The following definitions are for graphs that consist entirely of either $\bigstar_n$'s identified at certain boundary nodes or are $Y-\Delta$ equivalent to $\bigstar$'s identified at certain boundary nodes with some boundary to boundary edges which can only come from $Y-\Delta$ transformations.

Let's denote a graph G and its associated $\bigstar - K$ transformed graph $G^{\bigstar}$.

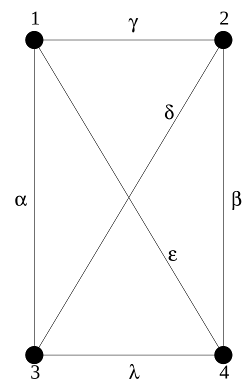

Figure 6. K_4-subset

Definition 7.1. Given a graph G, a K_4-subset of $G^{\bigstar}$ is a K_4 that is a subset of a K_n which is $\bigstar - K$ transformed from some $\bigstar_n$ in G for $n \geq 4$.

Definition 7.2. A K_4-cycle, being a subset of $G^{\bigstar}$, is a sequence $C_1C_2C_3...C_nC_1$ [5] of n K_4-subsets such that nodes 1 and 2 of C_i are identified with nodes 3 and 4 of C_{i+1}, respectively, and it satisfies the following properties:

1. The conductivity of each cable of multiplicity 2, which necessarily exist in the K_4-cycle due to the identification of nodes, can be determined either from an entry in the response matrix or through quadrilateral conditions or $Y-\Delta$ relations in $G^{\bigstar}$.
2. No edge in a cable of multiplicity 2 can be determined by quadrilateral conditions other than those in $C_1, C_2, C_3, ...,$ and C_n.

Remark 7.3. When property (1) fails, the graph is ∞ to 1.

Remark 7.4. When the properties in the definition of a K_4-cycle are satisfied, the single edges in the cables of multiplicity 2 can only be found by using the quadratic resulting from the continued fraction discovered in the polygon in polygon networks detailed earlier.

Definition 7.5. A K_4-cycle is *quadratic* when the conductivities of either edges α and β or edges ϵ and δ in each K_4-subset can be uniquely determined from relations in $G^{\bigstar}$, allowing for an at most 2 to 1 solvable K_4-cycle.

Remark 7.6. To *solve* a quadratic K_4-cycle is to solve the quadratic equation resulted from the continued fraction found in polygon in polygon networks as explained earlier. Note that all such quadratic equations will definitely produce two positive, real, but not necessarily distinct solutions, as proven earlier.

Lemma 7.7. *Two quadratic K_4-cycles whose intersection contains one or more K_4-subsets including their corresponding cables of multiplicity 2 are* quadratically equivalent *in the sense that the solutions of the quadratic equation obtained through constructing the continued fraction around one cycle completely determines the conductivities on the other cycle.*

Proof. Let α denote one of the edges in a cable of multiplicity 2 contained in the intersection of the two quadratic K_4-cycles. Solving for α as detailed in section 1 determines the conductivities of each single edge in every cable of multiplicity 2 in both quadratic K_4-cycles. □

Lemma 7.8. *Lemma 7.7 defines an equivalence relation R on the set of all quadratic K_4-cycles in the graph $G^\bigstar$.*

Proof. Given quadratic K_4-cycles p, q and r in a graph, with a relation R defined as in lemma 7.7, certainly pRp, thus it is reflexive. Whenever pRq, qRp, thus the relation is symmetric. If pRq and qRr, then since the information in p determines all information in q, which in turn determines the information in r, pRr, hence the relation is also transitive. Therefore this is an equivalence relation. □

Theorem 7.9 (The Counting Principle)**.** *If after $\bigstar - K$ transformation on graph G, the graph $G^\bigstar$ consists entirely of quadratic K_4-cycles and edges of either uniquely determined conductivities or conductivities entirely dependent on the solutions of some K_4-cycle, and n is the number of equivalence classes of quadratic K_4-cycles, then the graph is at most 2^n to 1.*

Proof. The equivalence relation in lemma 7.8 partitions the set of all quadratic K_4-cycles. Since each equivalence class yields at most two unique sets of conductivities completely independent of any other class, and it is assumed that there are n equivalence classes, and any edge outside of the equivalence classes can be either uniquely determined or is entirely dependent on the solutions of some K_4-cycle, the entire graph is at most 2^n to 1. □

8. Applications of the Counting Principle

First of all, we will define a way of denoting K_4-cycles for convenience. Implicit in the geometry of K_4-subsets is a sequence of numbers corresponding to the vertices of the quadrilateral. The sequence has four numbers, and their order corresponds to the edges which make up the quadrilateral. Thus, if a sequence 1234 corresponds to the quadrilateral of a K_4-subset, the geometry of the sequence says that the edges connecting vertex 1 to vertex 2, 2 to 3, 3 to 4 and 4 to 1 is the complete list of edges of interest in the K_4-subset quadrilateral. A K_4-cycle in this manner can be written as a sequence of such sequences, such that within every quadrilateral sequence, the first pair and last pair of numbers correspond to cables of multiplicity 2 in the graph, and the last two numbers are repeated as the first pair of numbers in the next K_4-subset sequence.

In the figures below, the tic marks on the edges are used to denote multiple edges: one tic mark denotes a cable of multiplicity 2, two a cable of multiplicity 3, etc.

Example 8.1 (The $\bigstar_n$ Cycle)**.** Since the polygon in polygon graph can be visualized as a cycle of $\bigstar_4$'s, what happens when we generalize the idea to include $\bigstar_5$'s, $\bigstar_6$'s, or even $\bigstar_n$? The following exposition will explore this idea.

Let's denote the boundary nodes of a $\bigstar_n$ to be $1, 2, 3, ..., n$.

Definition 8.2. A graph is a $\bigstar_n$ *cycle* if it is a loop $C_1 C_2 C_3 ... C_k C_1$ of k subgraphs of $\bigstar_n$ such that nodes $1, ..., \frac{n}{2}$ of C_i are identified with nodes $\frac{n}{2}+1, ..., n$ of $C_{(i+1) modk}$ when n is even, and that nodes $1, ..., \frac{n-1}{2}$ of C_i are identified with nodes $\frac{n-1}{2}+1, ..., n-1$ of $C_{(i+1) modk}$ and all node n's are identified when n is odd, respectively.

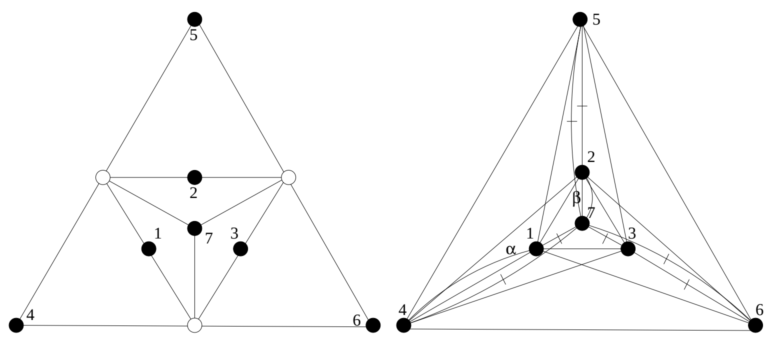

FIGURE 7. $\bigstar_5$ Cycle and its $\bigstar - K$ transformed graph.

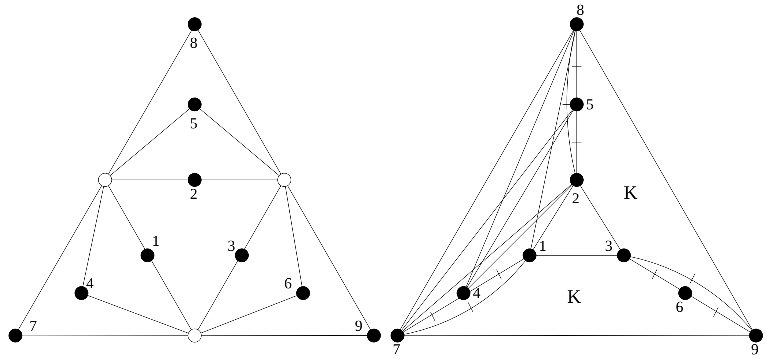

FIGURE 8. $\bigstar_6$ Cycle and its $\bigstar - K$ transformed graph.

Figure 7 is a $\bigstar_5$ cycle. Let α denote one of the edges in cable 14 of multiplicity 2 and β denote one of the edges in cable 27 of multiplicity 2. Then through certain quadrilateral conditions we can find the conductivity of β as a function of α:

$$\beta = \lambda_{27} - \frac{\lambda_{26}(\lambda_{37} - \frac{\lambda_{34}(\lambda_{17} - \frac{\alpha\beta}{\lambda_{24}})}{\lambda_{14} - \alpha})}{\lambda_{36} - \frac{\lambda_{13}\lambda_{46}}{\lambda_{14} - \alpha}}. \tag{20}$$

Simplifying equation (20) yields:

$$\beta(\lambda_{24}\lambda_{36}\lambda_{14} - \lambda_{24}\lambda_{13}\lambda_{46} - \lambda_{24}\lambda_{36}\alpha + \lambda_{36}\lambda_{34}\alpha) =$$

$$\lambda_{24}(\lambda_{27}\lambda_{36}\lambda_{14} - \lambda_{27}\lambda_{36}\alpha - \lambda_{27}\lambda_{13}\lambda_{46} - \lambda_{36}\lambda_{37}\lambda_{14} + \lambda_{36}\lambda_{37}\alpha + \lambda_{36}\lambda_{34}\lambda_{17}).$$

Expanding the cable of multiplicity 2 as $\lambda_{36} = \omega_1 + \omega_2$, and writing the cable $\lambda_{14} = \alpha + \delta$, utilizing quadrilateral conditions the coefficient of β can be simplified as:

$$\lambda_{24}\delta\omega_2 + \omega_1\lambda_{34}\alpha + \omega_2\lambda_{34}\alpha > 0.$$

This indicates that β is uniquely determined by α, whose conductivity can be determined by solving the quadratic K_4-cycle 1452-5236-3641. By symmetry, all the rest of the cables of multiplicity 2 are uniquely determined by α. Therefore, there is only one quadratic K_4-cycle in the graph and the conductivities of the rest of the edges all depend entirely on the conductivities obtained by solving the cycle. The $\bigstar_5$ cycle is at most 2 to 1.

Figure 8 is a $\bigstar_6$ cycle. The $\bigstar - K$ transformed graph contains several quadratic K_4-cycles. However, consider the quadratic K_4-cycle C=4785-8569-6974. Since all the quadratic K_4-cycles in the graph intersect at some K_4-subset with each other, by lemma 7.7, every other quadratic K_4-cycle in the graph is in the same equivalence class as C. Therefore, by theorem 7.9, the $\bigstar_6$ cycle is at most 2 to 1.

Remark 8.3. The $\bigstar_6$ cycle graph drawn here can easily be generalized to having n $\bigstar_6$'s connected in a cycle, and it would still have the same 2 to 1 property.

Proposition 8.4. *All $\bigstar_n$ cycles are at most 2 to 1. (Will be proven later)*

Example 8.5 (The Threepede Cycle)**.** This is an example of a 2 to 1 graph that, after $Y - \Delta$ transformations, becomes a graph consisting only of $\bigstar_n$'s identified at boundary nodes and additional boundary to boundary connections.

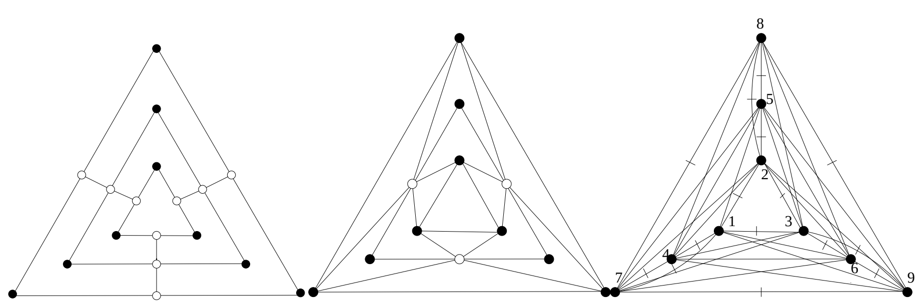

Figure 9. The Threepede Cycle, its $Y - \Delta$ equivalent, and its $\bigstar - K$ transformed graph.

Figure 9 is a threepede cycle. It can also be recognized as a $\bigstar_6$ cycle with certain boundary to boundary connections, whose conductivities can be easily recovered after we have recovered the embedded $\bigstar_6$ cycle. Therefore, following the arguments given for the $\bigstar_6$ cycle, the threepede cycle is at most 2 to 1.

Remark 8.6. This graph can easily be generalized to having n threepedes connected in a cycle, and it would still have the same 2 to 1 property.

Example 8.7 (The Race Track[4])**.** Figure 10 is the race track graph. In order to determine α, we solve the quadratic K_4-cycle 1245-4567-67910-91012. All edges in the K_4-subsets 3874 and 3892 are determined by the solutions of the quadratic K_4-cycle through quadrilateral conditions. Therefore, the race track graph is at most 2 to 1.

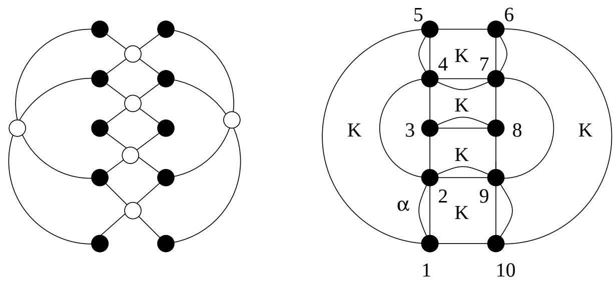

FIGURE 10. The Race Track graph and its $\bigstar - K$ transformed graph.

Remark 8.8. The graph can be easily generalized to having "longer race track" by adding more $\bigstar_4$'s to top and bottom, and the graph will still be at most 2 to 1 by the same argument.

Example 8.9 (The (n,k)-torus)**.** Figure 11, the (3,3)-torus, is an example of an at most 64 to 1 network.

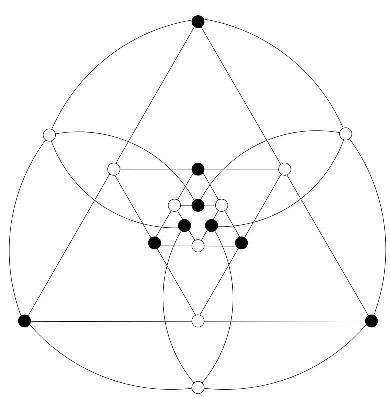

FIGURE 11. The (3,3)-torus, three triangle in triangle graphs connected in a cycle.

The triangle in triangle network can be visualized as lying on a cylinder, thus this graph can be visualized as lying on a torus.

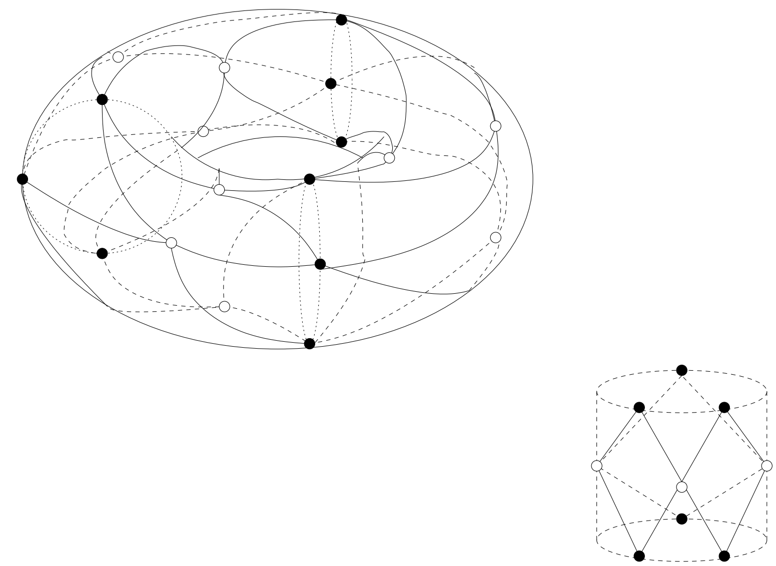

FIGURE 12. Three triangle in triangle graphs connected in a cycle as visualized on a torus.

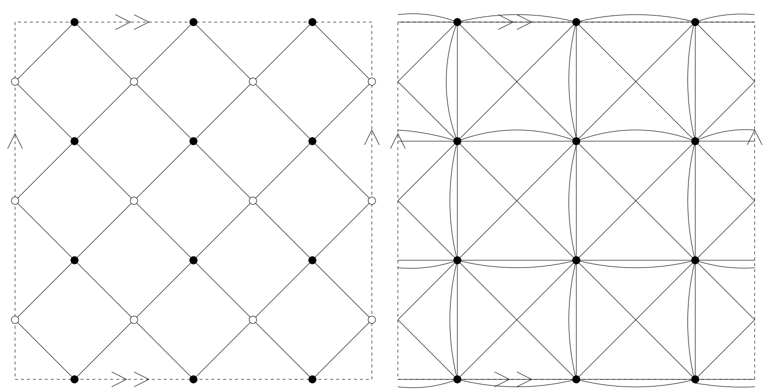

FIGURE 13. Topological equivalent of three triangle in triangle graphs connected in torus and its $\bigstar - K$ transform.

When visualized this way, the counting principle can be applied. There are 6 disjoint equivalence classes of quadratic K_4-cycles corresponding to the three cycles around each cylinder and three cycles around the torus. Thus this graph is 2^6 to 1, or 64 to 1. This can be generalized to n triangle in triangle networks connected in a cycle. There will still be three equivalence classes of quadratic K_4-loops corresponding to cycles around the torus, but now with n equivalence classes corresponding to the cycles around each of the n cylinders. Hence this is a 2^{n+3} to 1 network. This can be further generalized to k n-gon in n-gon networks connected in a cycle. There are now n equivalence classes corresponding to n quadratic K_4-cycles around the torus, and k equivalence classes corresponding to k quadratic

K_4-cycles around each cylinder. Thus this yields a network that is at most 2^{k+n} to 1.

Example 8.10 (The Spider)**.** Figure 14, the spider, is an example of a 4 to 1 network which contains cycles that are not actually K_4-cycles.

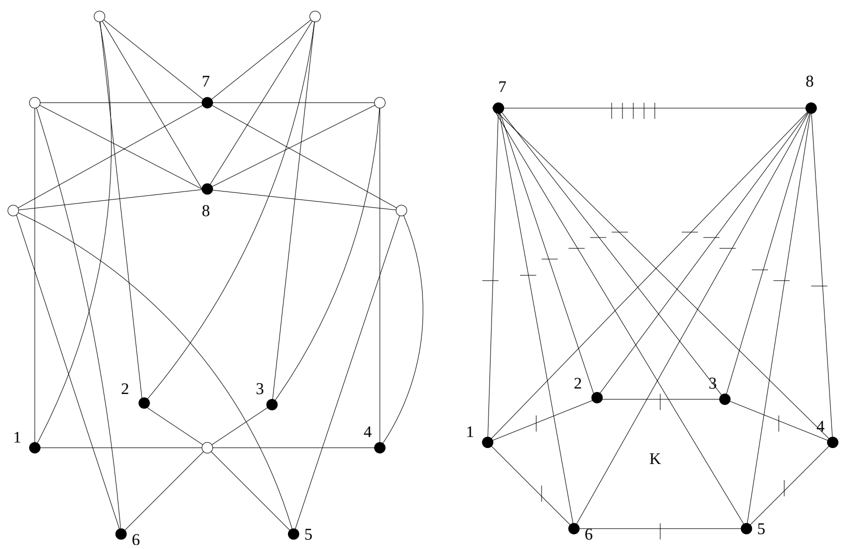

FIGURE 14. Six $\bigstar_4$'s connected to a $\bigstar_6$ before and after $\bigstar - K$ transformation.

There are several subsets of quadrilaterals which appear to be K_4-cycles, beginning and ending at the cables of multiplicity 2 that connect the nodes of the $\bigstar - K$ transformed $\bigstar_6$, which loop through the cable of multiplicity 6 connecting nodes 7 and 8. However, these are not K_4-cycles as they do not satisfy property (2) of definition 7.2. Quadrilateral conditions using the single edges on the interior of the K_6 can be used to find the single edges in each cable of multiplicity 2 around the outside. Thus each of the six edges connecting nodes 7 and 8 can be solved uniquely. Thus the only two distinct equivalence classes of quadratic K_4-cycles correspond to two cycles which utilize the long cables of multiplicity 2 connecting nodes 7 and 8 to nodes 1 through 6. The two quadratic K_4-cycles are: 8127-7238-8347-7458-8567-7618 and 7128-8237-7348-8457-7568-8617. These K_4-cycles are not equivalent because the only other quadrilaterals involving the cables of multiplicity 2 of interest are quadrilaterals such that all sides are cables of multiplicity 2, thus no information can be gained through quadrilateral conditions. Hence this network is at most 4 to 1.

9. Conclusion

Although we were able to determine when an n-gon in n-gon network has a unique solution in terms of conductivities on our original Kirchhoff matrix, we have not found a simple relationship directly from entries in the response matrix other than to demand the discriminant be zero. There is more work to be done in the characterization of unique solutions from the response matrix as well as the formulation of a geometric understanding of the conductivity pseudo-symmetry which produces unique solutions to the inverse problem.

Conjecture 9.1. In a graph composed entirely of $\bigstar_n$'s with boundary nodes identified, the only way to get a 2^n to 1 graph is through the existence of cycles.

There is also more work to be done in understanding the highly non-linear relationship between the two Kirchhoff matrices that have the same response in an n-gon in n-gon network.

Although all graphs we have considered involve multiple solutions in powers of 2, a problem of interest is to find a 3 to 1 graph, which we are certain can not possibly have a cycle of $\bigstar$'s, which would yield a quadratic.

It is important to note that none of the graphs we studied were circular planar. The theory of k-connections in [3] is based on a circular ordering and the information this provides. Another interesting problem is to try to find a connections between n to 1 graphs and permutations of k-connections in the graph, and perhaps truly understand conditions on determinants of existent k-connections in the non-planar case.

10. MATLAB Code

The following program randomly generates a Kirchhoff matrix for an n-gon in n-gon network, where n is specified by the user.

```
n = input('Please enter n for a randomly generated n-gon in n-gon network:');
K = zeros(3*n,3*n);
for i = 1:n
    K(i,2*n+i) = rand;
    K(i,2*n+mod(i,n)+1) = rand;
    K(n+i,2*n+i) = rand;
    K(n+i,2*n+mod(i,n)+1) = rand;
end
K = K + K';
for i = 1:length(K(1,:))
    K(i,i)=-sum(K(i,:));
end
n = length(K(1,:))/3;
L = getL(K, 2*n);
K
L
```

Given a Kirchhoff matrix, the following program calculates the response matrix.

```
function L = getL(K,n)
A = K(1:n,1:n);
C = K((n+1):end,(n+1):end);
B = K(1:n,(n+1):end);
L = A - B*(inv(C))*B';
```

Given a response matrix Λ, the following program checks to see if Λ is a valid response for an n-gon in n-gon network. If it is valid, the program returns the two (not necessarily distinct) matrices B_1 and B_2 that contain all information within the two Kirchhoff matrices corresponding to Λ.

```
function K = ngon(L)
isResponse = true;
if mod(length(L(1,:)),2) ~= 0
```

```
    isResponse = false;
end
n = length(L(1,:))/2;
Pk = zeros(n+1, n+1);
for i = 1:n
    Pk(i, i) = L(i, n+i);
    if i ~= n
        Pk(i, i+1) = L(i, i+1);
        Pk(i+1, i) = L(n+i, n+i+1);
    end
end
Pk(n, n+1) = L(n, 1);
Pk(n+1, n) = L(n+1, 2*n);
k = n + 1;
bk = -Pk(k-1, k)*Pk(k, k-1);
a = det(Pk(2:k-1, 2:k-1));
b = bk*det(Pk(2:k-2, 2:k-2)) - det(Pk(1:k-1, 1:k-1));
c = -bk*det(Pk(1:k-2, 1:k-2));
discriminant = sqrt(b*b - 4*a*c);
if a <= 0 | b >= 0 | c <=0 | (b*b - 4*a*c) < 0
    isResponse = false;
end
for i = 1:n
    if abs(1 - Pk(i,i+1)*Pk(i+1,i)/(L(i,n+mod(i,n)+1)*L(mod(i,n)+1,n+i)))
                                                            > 0.0000000001
        isResponse = false;
    end
end
if isResponse == true
    disp(sprintf('The response matrix given is the response matrix of an
%d-gon in %d-gon network.',n,n))
    alpha1 = (-b + discriminant)/(2*a);
    alpha2 = (-b - discriminant)/(2*a);
    B1 = ngon_getB(L,Pk,alpha1);
    B2 = ngon_getB(L,Pk,alpha2);
    B1
    B2
    K = [B1,B2];
else
    disp(sprintf('The response matrix given is NOT the response matrix of
an %d-gon in %d-gon network.',n,n))
    K = 0;
end
```

Given α as in Section 2, the following program calculates the matrix B of the Kirchhoff matrix.

```
function result = ngon_getB(L,Pk,alpha)
n = length(L(1,:))/2;
B = zeros(2*n, n);
for i = 1:n
    otheredge = L(i,n+i)-alpha;
    back = [i-1,n+i-1];
    forward = [i+1,n+i+1];
```

```
    if i == 1
        back = [n,2*n];
    elseif i == n
        forward = [1,n+1];
    end
    B(i,i) = alpha + L(i,back(1)) + L(i,back(2))
             + alpha*L(i,back(2))/L(n+i,back(2));
    B(i,mod(i,n)+1) = otheredge + L(i,forward(1)) + L(i,forward(2))
                      + otheredge*L(i,forward(2))/L(n+i,forward(2));
    B(n+i,i) = alpha + L(n+i,back(1)) + L(n+i,back(2))
               + alpha*L(n+i,back(1))/L(i,back(1));
    B(n+i,mod(i,n)+1) = otheredge + L(n+i,forward(1)) + L(n+i,forward(2))
                        + otheredge*L(n+i,forward(1))/L(i,forward(1));
    alpha = Pk(i,i+1)*Pk(i+1,i)/otheredge;
end
result = B;
```

References

[1] Chatelin, Francoise. "Eigenvalues of Matrices." Wiley, ©1993.

[2] Chrystal. *Textbook of Algebra.* –Vol. 2. Chelsea Publishing, ©1964.

[3] Curtis, Edward B., and James A. Morrow. *Inverse Problems for Electrical Networks.* Series on applied mathematics – Vol. 13. World Scientific, ©2000.

[4] Lovejoy, Tracy. "Applications of the Star-K Tool." 2003.

[5] Reichert, Nick. "The Smallest Recoverable Flower." 2004.

[6] Russell, Jeffrey. "Star and K Solve the Inverse Problem." 2003.

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