Graph Construction

Adam Chacon

August 12, 2005

Abstract

This paper explores two ways to construct a graph. One way attempts to generalize the Cut-Point Lemma in the non-circular planar case, and the other way attempts to build a graph based on the set of all connections.

Contents

1 Searching for a Cut Point Lemma

The motivation for looking for an analog to the Cut-Point Lemma on non-circular planar and nonplanar graphs comes from the simplicity of constructing medial graphs from the Cut-Point Lemma in the circular planar case. The annular symmetric examples that it has worked for so far have been just as easy as in the circular planar case. Additionally, the equation itself is similar to the Descartes-Euler Polyhedral Formula.

1.1 Background

Define circular planar, medial graph, geodesic, z-sequence, cutpoint lemma, euler's equation, Karen's theorem for circular planar z-sequences, Nick's method for drawing non-circular planar graphs on a surface of higher genus. Give examples. Give methods of attack- different embeddings, maximally connected embeddings, well-ordered z-sequence embeddings. Conclude this section.

Refer to [1] for definitions of circular planar, k-connection, well-connected, medial graph, and two-colorable.

Definition 1.1 *The $z-$ sequence of a medial graph is the ordering of the geodesics as they occur clockwise around the circle.*

This theorem will be used later on:

Theorem 1.1 *If a circular planar critical, connected graph's z-sequence is $1, ..., n, 1, ..., n$ then the graph is well-connected.*

Theorem 1.2 (Curtis and Morrow, Cut-Point Lemma) *Suppose A is a finite family of chords in the disc, and assume that A is lensless. Let X and Y be a pair of cut-points for A. Let $m(X,Y)$ be the maximum k-connection from X to Y, let $n(X,Y)$ be the largest number of black cells completely contained in X, and let $r(X,Y)$ be the number of reentrant geodesics in X. If G is circular planar, then:*

$$m(X,Y) + r(X,Y) - n(X,Y) = 0$$

Theorem 1.3 (Descrates-Euler Polyhedral Formula) *Suppose P is a polyhedron of genus zero. Let v be the number of vertices of P, E be the number of edges, and F be the number of faces. Then*

$$V - E + F = 2$$

Generally,

Theorem 1.4 *(Poincare Formula)*

$$V - E + F = \chi(g)$$

where $\chi(g) = 2 - 2g$ is the Euler Characteristic of the polyhedron, and g is the genus of the polyhedron.

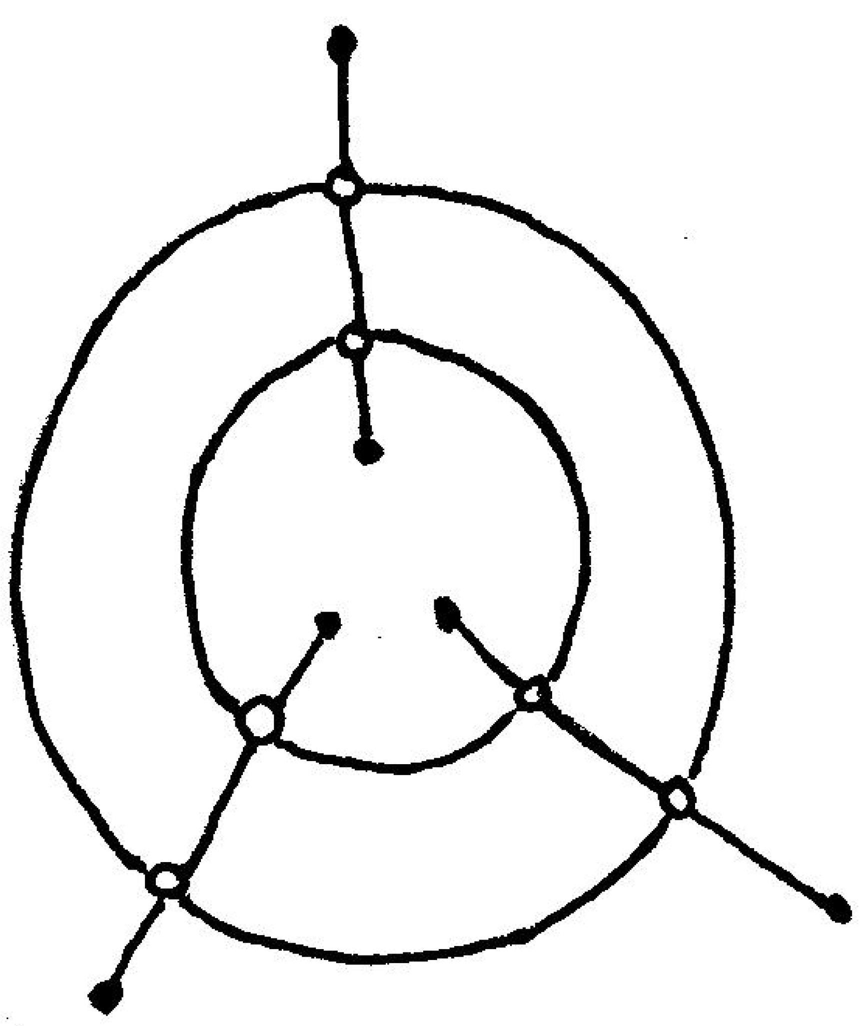
As stated earlier, there is a similarity between the Cut Point Lemma and the Polyhedral Formula:

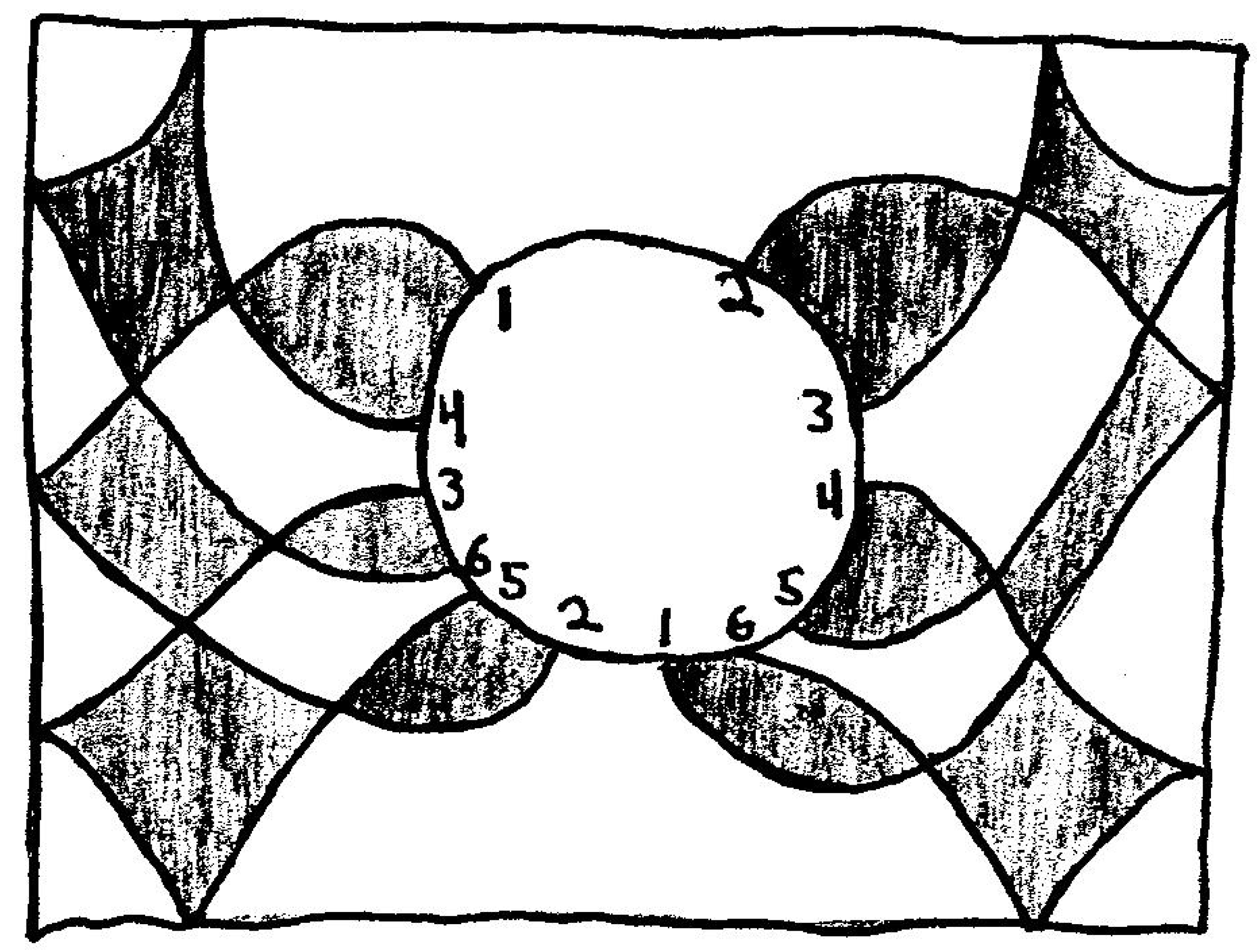
$$V - E + F = 2(\text{planar graphs})$$

$$m(X,Y) + r(X,Y) - n(X,Y) = 0(\text{circular planar graphs})$$

1.2 Examples and Methods of Approach

!





In [3]'s paper he gives several examples of medial graphs. I first looked at those, and larger classes of those graphs. For example, the annular graph with three rods intersected by two circles is typically called the "circles and rods" graph, and the square within the square is typically called the n-gon-in-n-gon graph. The 3 circles and 2 rods graph has z-sequence 123456125634. Note that recoverability does not play a role. The argument for two circles, rods, works for n circles and m rods. If we affix a cut-point X and move the other cut-point Y from cell to cell in a clockwise fashion starting from the cell adjacent to X and naively attempt to use the Cut-Point Lemma by taking a sample of the numbers m, r, and n for every new cell Y hits, we see:

1.If X is afixed on a black cell, n=0,0,1,1,2,2,...,n-2, n-2, n-1. Putting two cut points in the same cell gives trivial information, so I have disregarded this case. If X is affixed on a white cell, n= 0,1,1,2,2,...,n-2, n-2, n-1, n-1. Note that these numbers for n are the same for all medial graphs.

2.

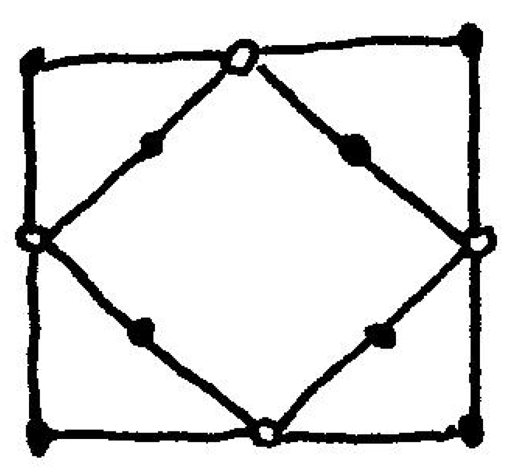
$$m = min(X, Y)$$

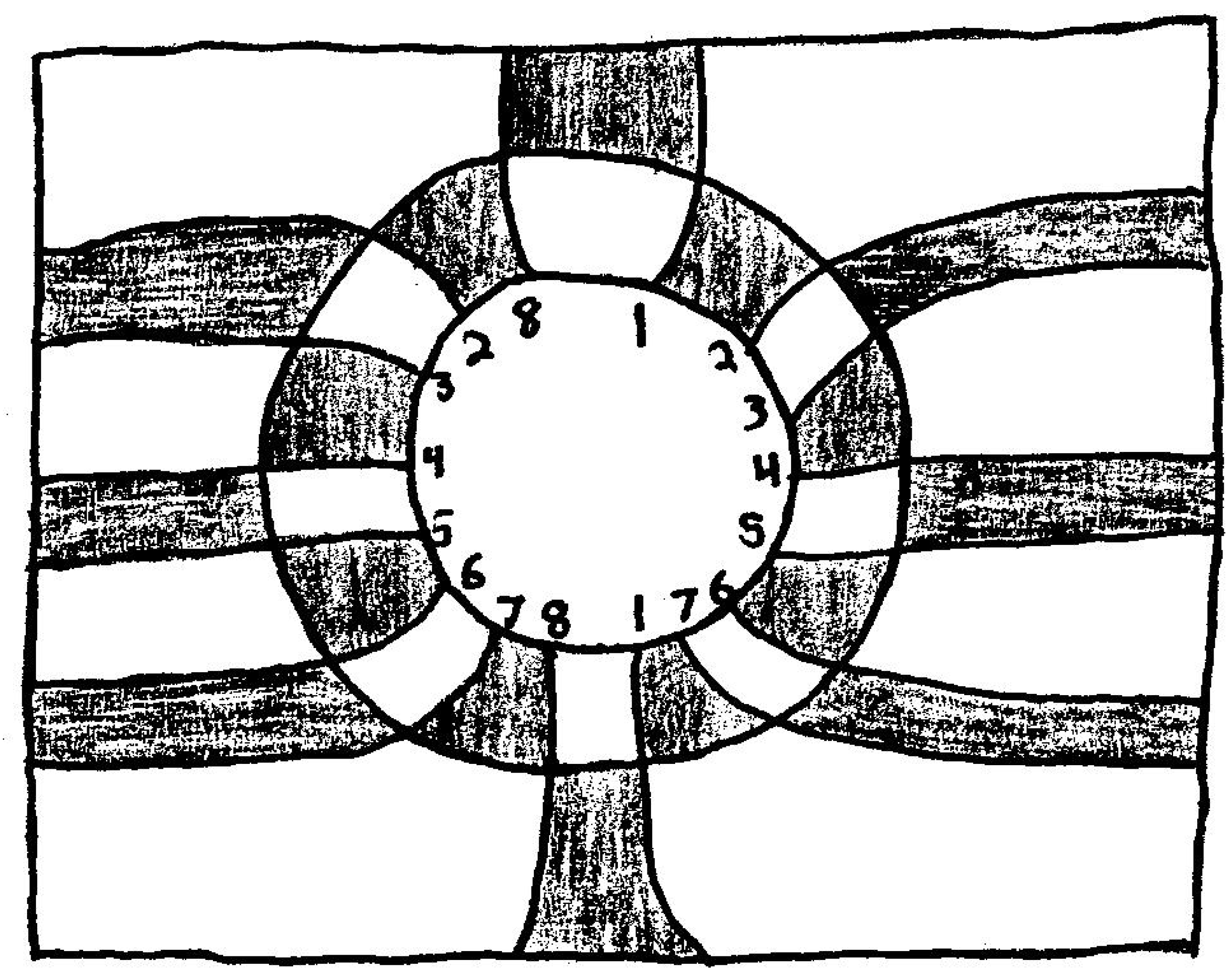
3. r is determined by the z-sequence. Wherever X was fixed corrseponds to a starting point in the z-sequence. A reentrant geodesic will simply be a number that recurs.

We see now that for

$$m - n + r = 0$$

that since n is invariant, either m or r must change. With some observation we see that if the z-sequence were 123456123456 then no matter where X is fixed the cut-point lemma holds. Before deciding how we might change the z-sequence, here is another example.





The square-in-square has z-sequence 1234567187654328. Again, the same properties hold for r and n as usual, but that isn't so for m. A more in-depth description of that property is now appropriate.

Theorem 1.5 *Assume we have the medial graph drawn and embedded by [3]'s method. The property*

$$m = min(n_{XY}, n_{YX})$$

is equivalent to well-connected.

Proof: If G is well-connected, then all the connections in the circular ordering exist. The Cut-Point Lemma is only interested in connections in a circular ordering as well. Thus with any circular pair, the greatest k-connection will be the number of boundary nodes in the smaller set of the pair. □

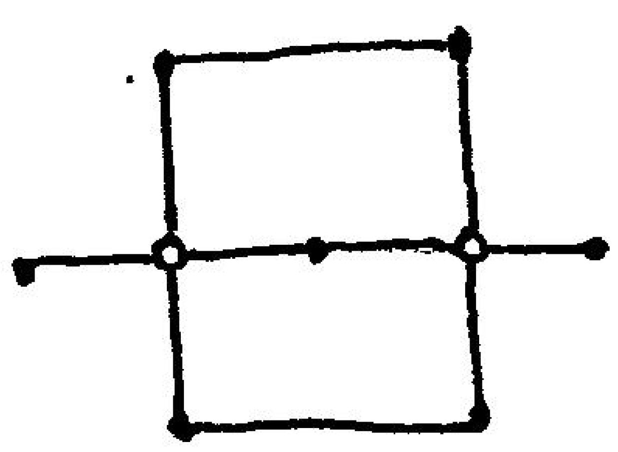
We see now that the medial graph for the square-in-square is not well-connected with the boundary ordering as is. If we have a well-connected embedding and the z-sequence is 1,...,n,1,...,n then the Cut-Point Lemma in the non-circular planar case will hold without modification. Note that z-sequence 1,...,n,1,...,n does not imply the graph is well-connected in the non-circular planar case because [2]'s proof relies on the Cut-Point lemma holding, and certainly well-connected does not imply the z-sequence is 1,...,n,1,...,n.

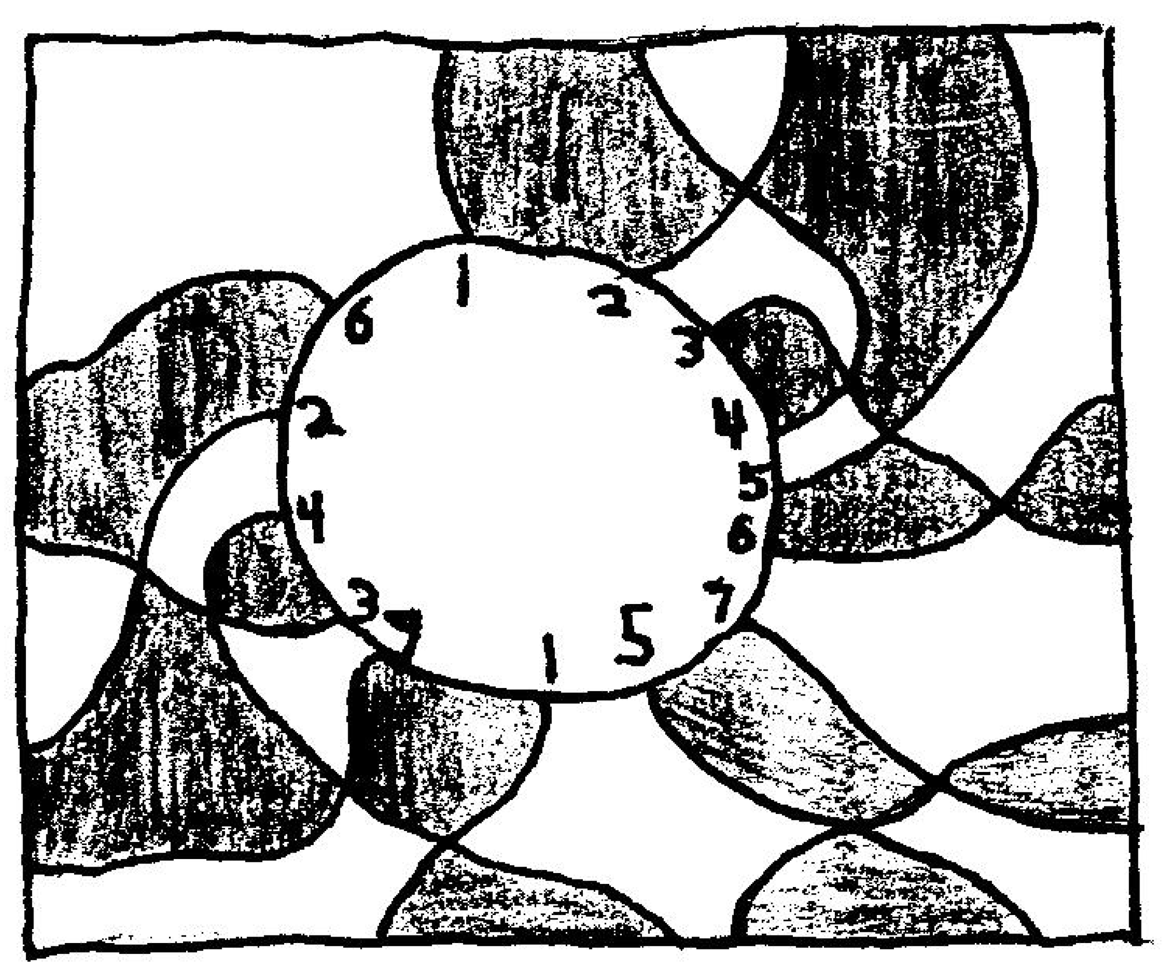
Now, for no reason other than wanting to make the z-sequence 1,...,n,1,...,n, instead of embedding the graphs on a torus, embed them on a projective plane.

The sequence is now as desired. The circles and rods is still well-connected, and the square-in-square is *now* well-connected. Thus the cut point lemma holds since both conditions were met.

1.3 Conclusion

Embedding the graphs on the projective plane brought us the right criteria for the Cut-Point Lemma only by luck. In general, noncircular planar graphs won't have the z-sequence property, and of course not all graphs are well-connected. An example:





Although we can find an ordering of the boundary nodes that is well-connected, the z-sequence has no obvious patterns, so choosing which surface to embed it in remains a mystery.

2 Determining a Graph Through Connections

The problem is now to determine a graph given all sets of 1,2,...,k-connections. The motivation for this is that it completely bypasses the medial graph and the reliance on the Cut-Point Lemma. A nice result we will lose is the $Y - \Delta$ equivalence that the medial graph has with the original graph. There is a larger set of equivalent graphs to a given set of connections. Furthermore, we are assuming we have all the connections, while the response matrix will detect only connections in a circular ordering for a circular planar graph with our current technology. Detecting connections in non-circular planar and nonplanar graphs is a current problem, as is detecting those outside the ordering in the given response matrix. With these differences in mind, I'll redefine *well – connected*:

Definition 2.1 *A graph is well – connected if all k-connections exist.*

The difference is that with the previous definition of *well – connected* it considered only connections between *circular pairs.*

2.1 Existence

Not all sets of connections correspond to graphs. An example is the set of connections that has no 1-connections but has 2-connections. Another example comes from a count of the minimum number of k-connections, if they exist.

Theorem 2.1 *If G is a connected graph with n boundary nodes, then*

$$\text{minimum number of k-connections if they exist} = \binom{n-2}{2k}\binom{2k}{k}$$

We will explore the less obvious sets of false connections. The method used for detecting connections that don't correspond to graphs is to look at connections that *don't* exist. For example, if a 4-connection $(abcd);(efgh)$ doesn't exist, we analyze the smaller broken connections, which I'll call sub-connections, that are involved in the 4-connection.

Definition 2.2 *For $1 < j < k-1$ a $sub-connection$ of a k-connection is a j-connection that involves only nodes listed in the k-connection.*

The overall attempt is to analyze each broken k-connection one-by-one by the sub-connections under it. This method, if complete, takes care of a large class of nontrivial existence problems, but not all of them. There are other (nontrivial) sets of connections that don't correspond to graphs, which will be discussed later. Back to the previous example, we are interested only in connections on opposite sides, like $(abc);(egh)$ but not $(abe);(fgh)$. Now the procedure used to determine existence was to take all the possible paths of the broken k-connection and eliminate those that conflict with the sub-connections. For a broken 3-connection $(abc);(def)$ we have the rows

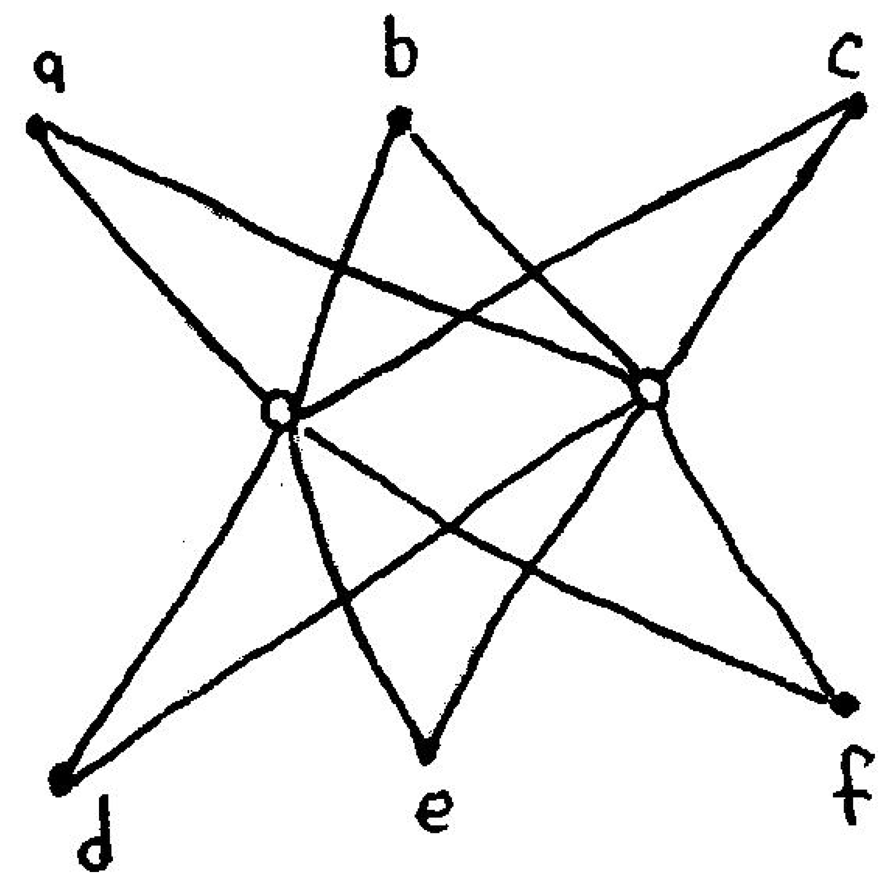
$$\begin{array}{c} ad\ be\ cf \\ ad\ bf\ ce \\ ae\ bd\ cf \\ ae\ bf\ cd \\ af\ bd\ ce \\ af\ be\ cd \end{array}$$

There are k! rows for a k-connection. If we assume that sub-connections $(a);(e)$ and $(bc);(ef)$ don't exist, then we eliminate the rows that contain those paths. These particular sub-connections would directly eliminate four rows. If all rows are eliminated, then there is no contradiction between the broken k-connection and its broken sub-connections. If some rows aren't eliminated, those represent paths that aren't disjoint and we get the following:

Theorem 2.2 *If there is a row (set of paths) of a broken k-connection (with pairs $(a_1, ..., a_k);(b_1, ..., b_k)$) that is NOT eliminated, then the paths of that row were squeezed through k-1 interior vertices.*

Proof: There are $k!$ sets of possible paths for a k-connection, and if a k-connection doesn't exist, this implies that each row is either eliminated by the k-connection's sub-connections or the k paths together are not disjoint. Assume a row is not eliminated. Thus each of the 1-connections, 2-connections,...,$k-1$-connections exists. None of the paths in the row are connected by a boundary-to-boundary edge because if one was, then a broken $k-1$-connection would be implied, which is a contradiction. Since there are no boundary-to-boundary edges connecting nodes on opposite sides, there must be at least $k-1$ interior vertices for a $k-1$-connection to exist. If there are more than $k-1$ interior vertices, a k-connection would exist if all k boundary nodes in both pairs had paths to at least k interior vertices. □

Example 2.1 .

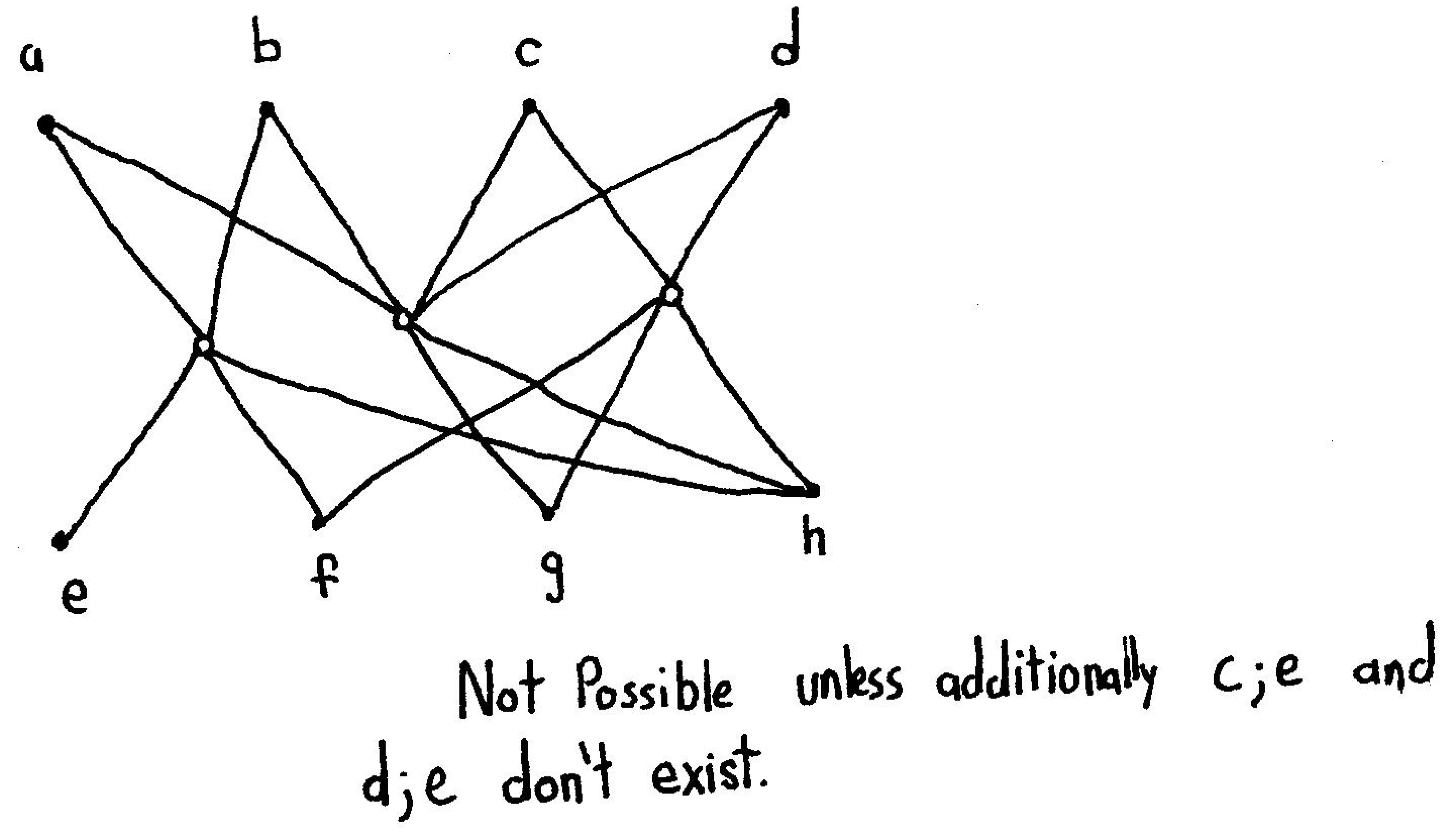


The proof shows that there can be more than $k-1$ interior vertices in the particular subgraph involving the selected boundary nodes, but the extras will be unused in determining paths from one pair $(a_1, ..., a_k)$ to another $(b_1, ..., b_k)$. The term "forced through" in the theorem means just that. In other papers the set of necessary interior vertices is called a *cutset*.

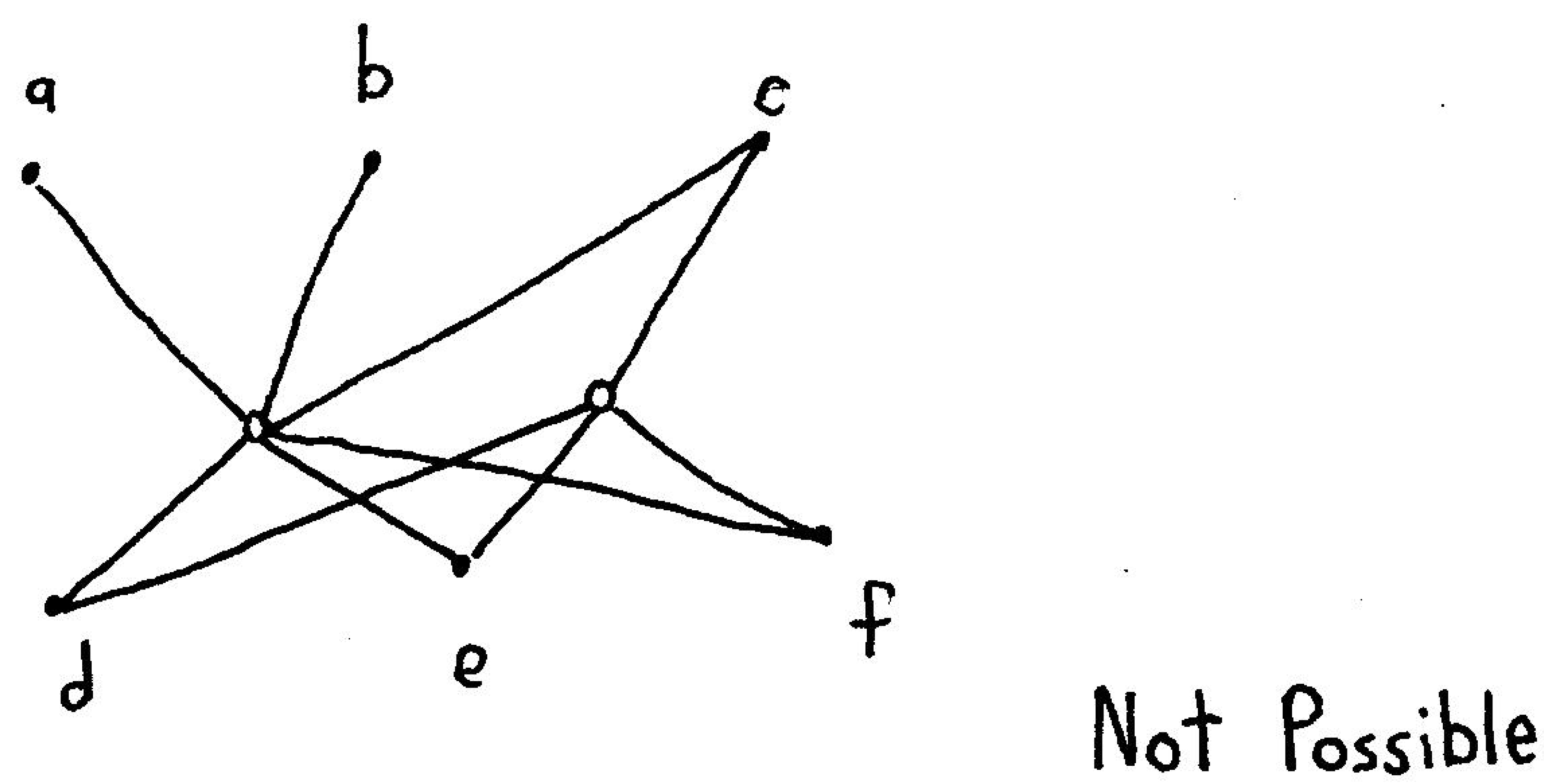
Corollary 2.1 *A broken k-connection with a boundary-to-boundary edge between nodes on opposite pairs is reducible to a broken k-1 connection. Namely, the broken k-1 connection exists on all the nodes except the two that were connected by the boundary-to-boundary edge.*

Given a broken k-connection I began to look at combinations of sub-connections that would render the graph impossible to draw without directly eliminating all the rows of possible paths. Here are three examples:

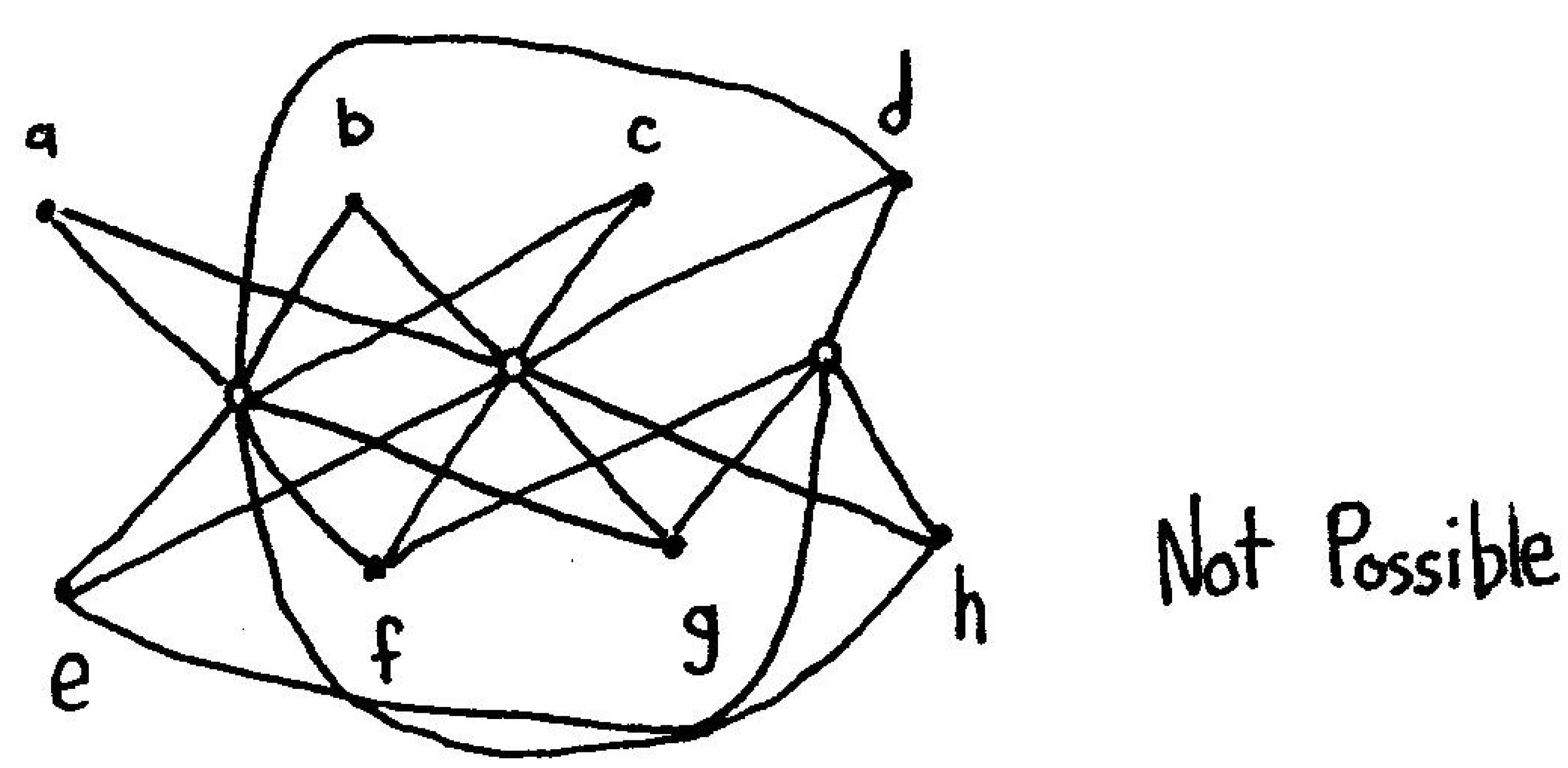
Example 2.2 *Suppose that the 4-connection* $(abcd);(efgh)$ *and the 2-connections* $(ab);(ef)$ *and* $(cd);(eg)$ *don't exist.*



Example 2.3 *Suppose that the 3-connection* $(abc);(def)$ *and the 2-connection* $(ab);(de)$ *doesn't exist.*



Example 2.4 *Suppose that the 4-connection* $(abcd);(efgh)$ *and the 3-connection* $(abc);(efg)$ *doesn't exist.*

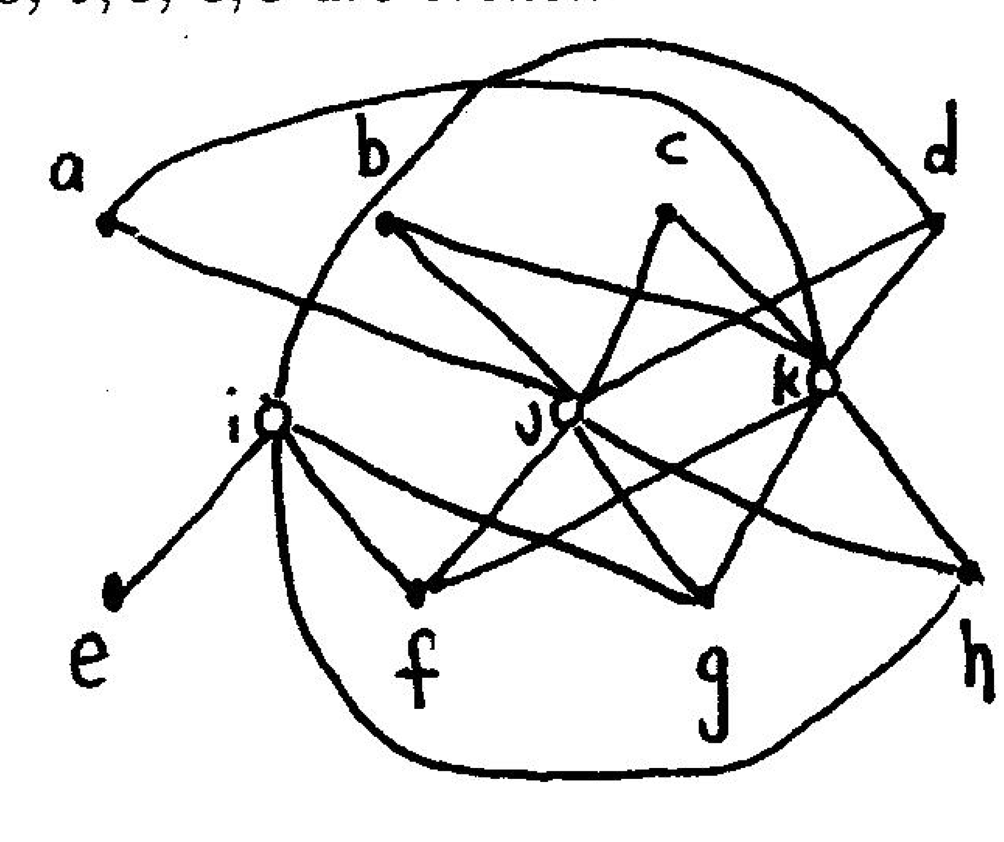


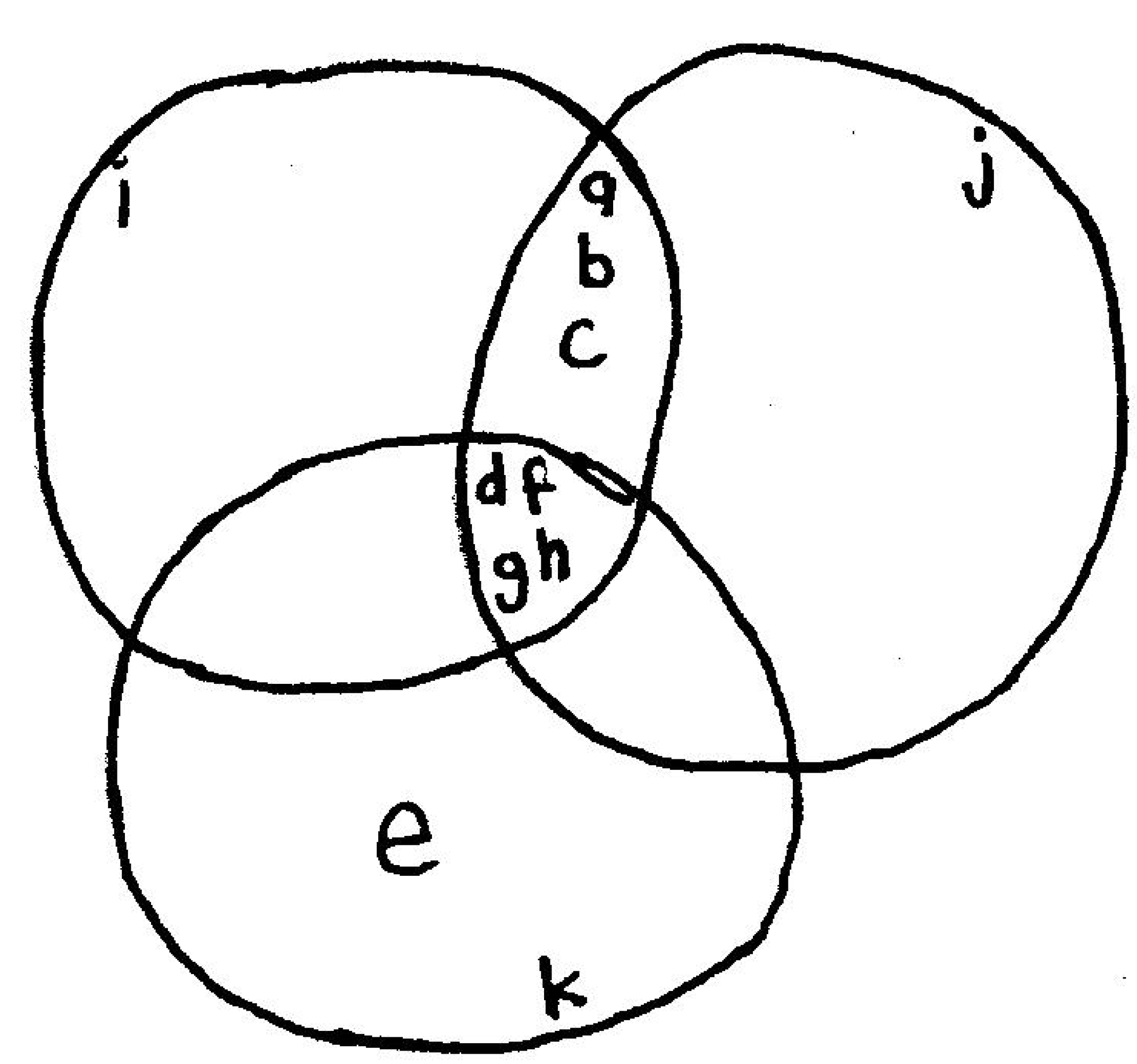
Theorem 2.3 *Suppose a k-connection doesn't exist with not all of its rows eliminated. Then at least k $k-1$-connections don't exist, or all of them do.*

Proof: Assume $P = (a_1, ..., a_k)$ is not connected to $Q = (b_1, ..., b_k)$ and that the connection wasn't broken by all rows of paths being crossed out. Then $(a_1, ..., a_k)$ is forced through $k-1$ interior nodes to $(b_1, ..., b_k)$ as shown in previous theorem. Now WLOG suppose $(a_1, ..., a_{k-1})$ is not connected to $(b_1, ..., b_{k-1})$. If among any $k-1$ boundary nodes all of the $k-1$ are reached then all $k-1$ connections exist. If one doesn't exist, since we are assuming no boundary-to-boundary edges between nodes of opposite pairs by previous corollary, then we have $k-1$ boundary nodes forced through $k-2$ interior nodes. Thus none of $a_1, ..., a_{k-1}$ can have a path through the last interior vertex if any of $b_1, ..., b_[k-1]$ has a path to it (likewise if $a_1, ..., a_{k-1}$ does have a path to the remaining interior vertex then none of $b_1, ..., b_[k-1]$ can have a path to it and the argument works the same). If $a_1, ..., a_{k-1}$ can't have a path through it, then a $k-1$-connection doesn't exist for any of the k groups of $k-1$ boundary nodes in Q. □

For example, if $P = (abcd)$ and $Q = (efgh)$ then since (abc) is not connected to (efg) and abc is forced through 2 interior vertices (since "among any three boundary nodes (on both sides) all 3 interior vertices are reached" implies all 3-connections exist) then we know that none of a,b, or c can have a path to the third interior vertex if any of e,f,or g has a path to it and thus the 3-connections (abc) to (efh), (abc) to (ehg), and (abc) to (fhg) are additionally broken.

Example 2.5 *Suppose that the 4-connection $(abcd)$; $(egfh)$ and the 1-connections $a; e$, $b; e$, $c; e$ are broken.*





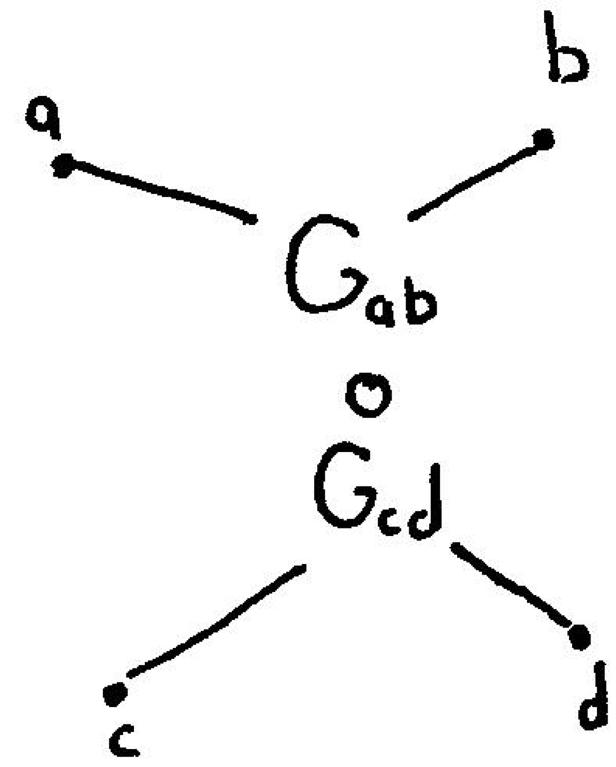
The three 1-connections imply a broken 3-connection. With observation, they imply four broken 3-connections.

The existence problem has many more cases than the ones I have considered. For example, if there is a broken 5-connection, how many 3-sub-connections can be broken for the set to correspond to a graph? Furthermore, which sets of 3-sub-connections can be broken? The number of broken sub-connections relies on which nodes were chosen. A helpful way to think of the problem is to draw a Venn Diagram where the circles represent interior vertices. A node drawn in a completely disjoint circle means that it connects only to that interior vertex, and similarly a node drawn where two circles overlap means it connects to both of those. If we are given the information a is not connected to b then they will have to be drawn in disjoint sets. Beyond broken 5-connections the Venn Diagrams no longer serve as good visual tools. Something else not yet considered is the relationships among k-connections. For example, the set of connections (ab) is not connected to (cd), (ac) is not connected to (bd), and (ad) is connected to (bc) does not correspond to a graph. A more subtle example will pop up in the next section.

2.2 Construction

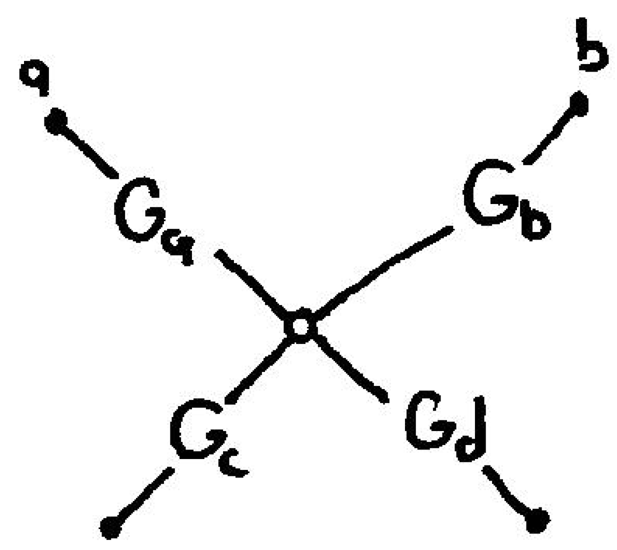
The methods used here are shy of an algorithm for producing graphs. The problem is not as complicated as characterizing existence, though. One difference between the graphs drawn here and those drawn in the previous section is that before we overlooked the fact that the edges going to the interior vertices were actually disjoint paths. Simply put, each edge is generally some structure of interior vertices, with the best case scenario being no interior vertices (a true edge). If, as in a previous theorem, we have a non-existent k-connection with not all of its rows crossed out, and thus the cutset is k-1 interior vertices, then all the relevant paths emanating from each boundary node is substituted by a structure of interior vertices that terminates on the k-1 interior nodes. The notation I used here is $G_{a_1,\ldots,a_k}$. Here's an example of a graph that is constructed in the same way that we searched for existence, with this generality included:

(ab); (cd) not connected, all 1-connections exist



Some theorems make the drawings simpler.

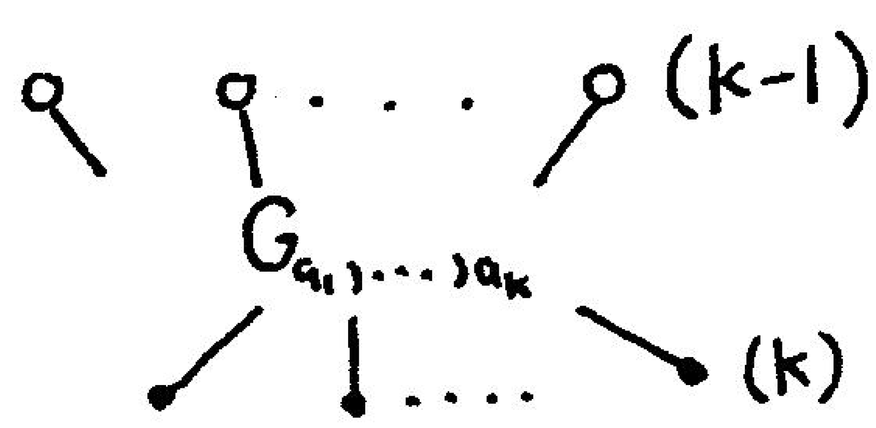
Theorem 2.4 *Suppose there are broken 2-connections* $(a, b); (c, d)$ *and* $(a, c); (b, d)$, *and these 1-sub-connections exist. Then* G_{ab} *and* G_{cd} *split into* G_a *,*G_b *and* G_c*,* G_d*, respectively.*



Proof: .$(a, b); (c, d)$ broken implies G_{ab} is disjoint from G_{cd}. The additional condition (a, c) not connected to (b, d) implies that node a can't have any path through the interior except for the cutset, and in particular can't have a path through G_{ab}. □

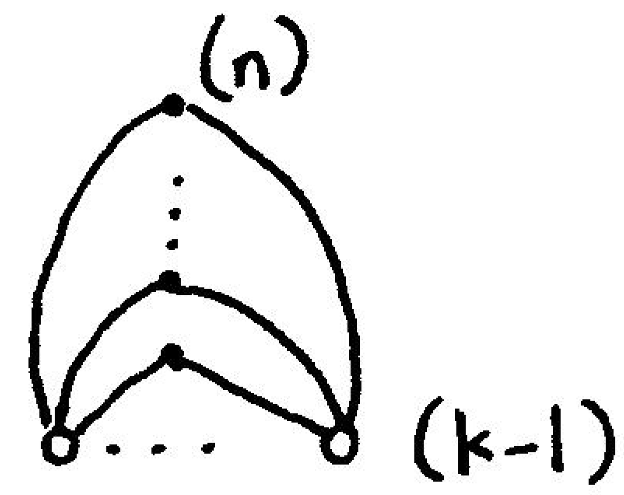
Theorem 2.5 *If a set of k nodes has no k-connections, its paths to other boundary nodes happen through exactly k-1 interior vertices.*

This lets the $G_{a_1,\ldots,a_k}$'s be simplified to:



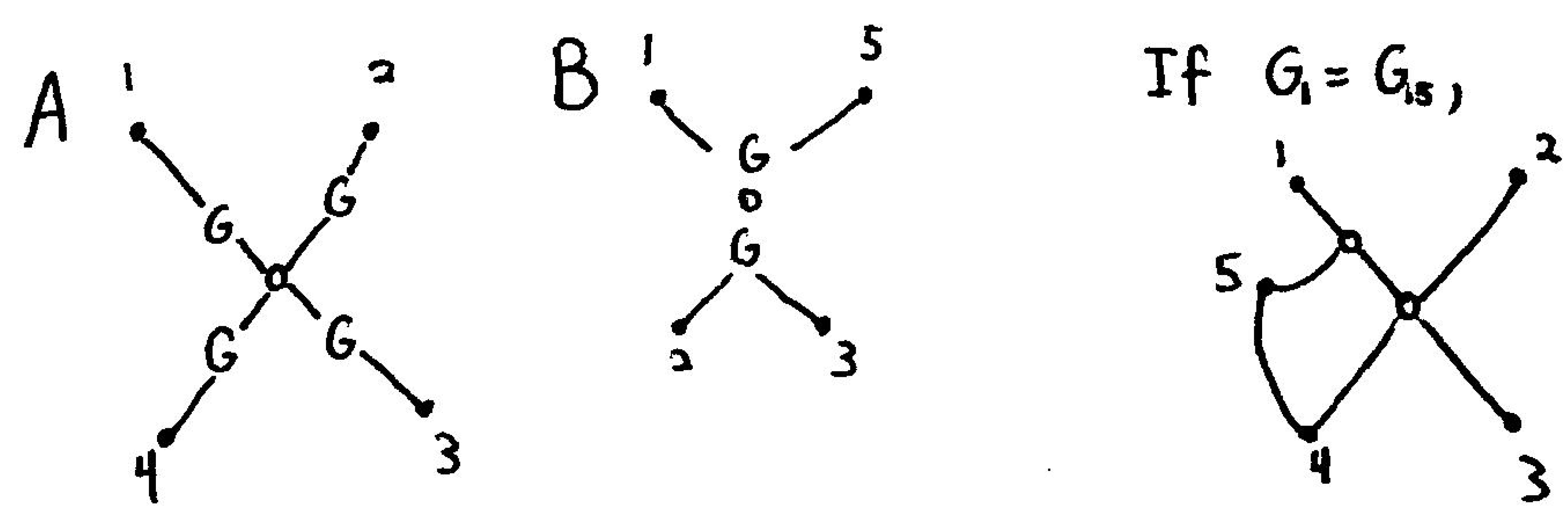
Where $G_{a_1,\ldots,a_k}$ is disjoint from the rest of the graph.

Theorem 2.6 *Suppose there are no k-connections but all* $k-1$*,*$k-2$*,...,1 connections exist. The corresponding graph's geometry is:*

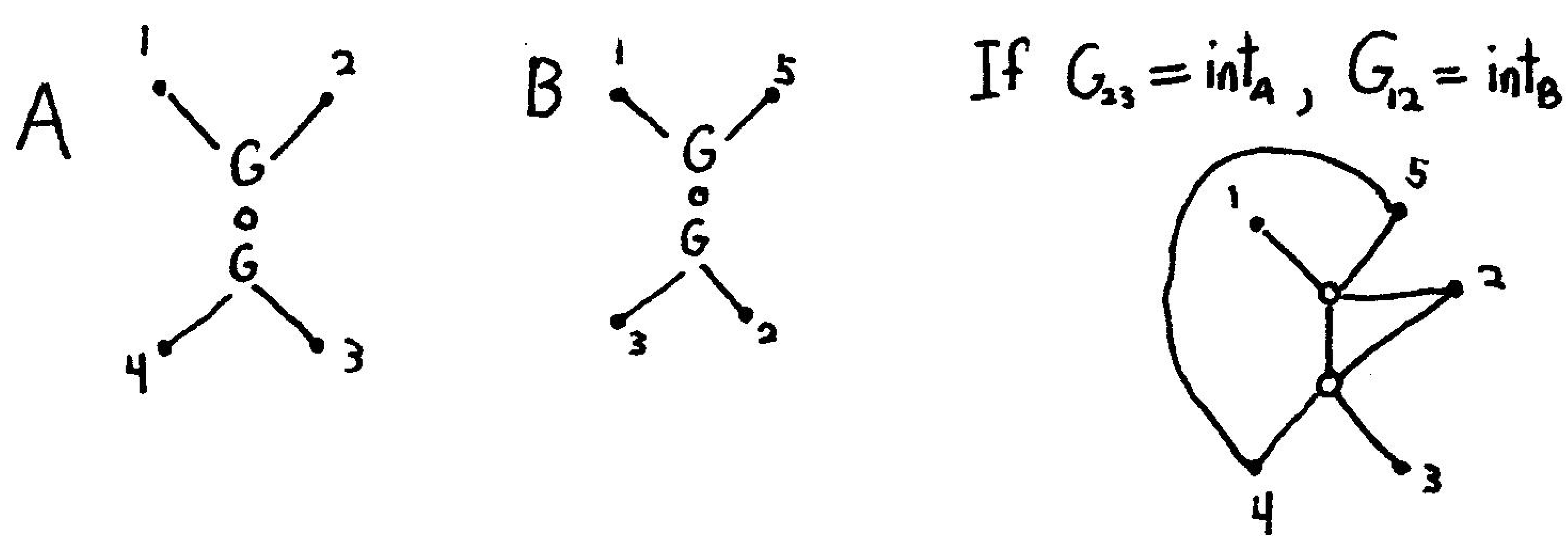


Deducing the graph from the $G_{a_1,\ldots,a_k}$'s is a problem unto itself. Here are two examples on only five boundary nodes:

1. Suppose that all 1-connections exist and the 2-connections $12;34$, $13;24$, $14;23$, and $15;23$ are broken.

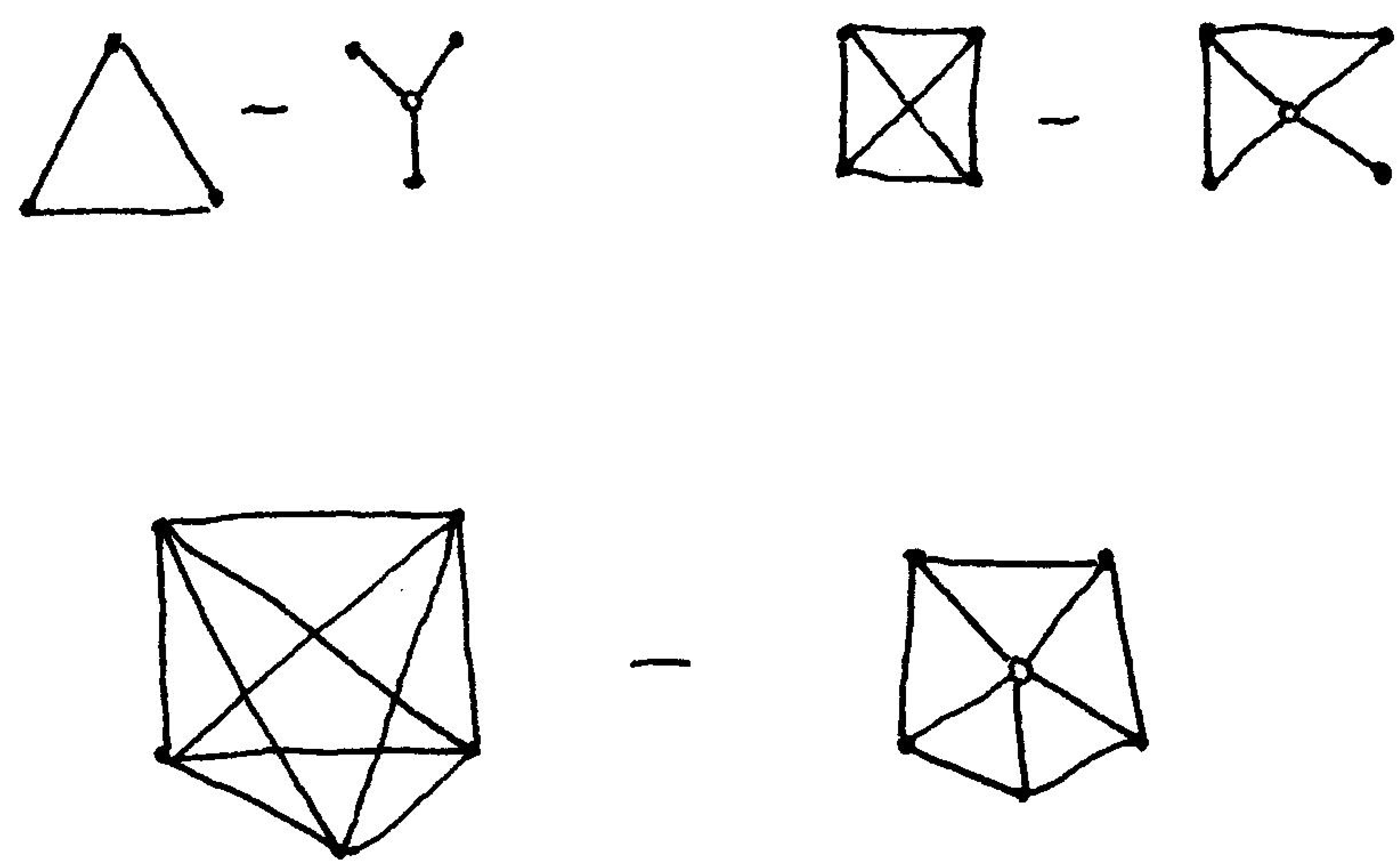


2. Suppose that all 1-connections exist and the 2-connections $12;34$ and $15;23$ are broken.



With this much analysis required on only 2-connections, one can imagine how difficult construction can become on larger graphs.

There is one final tool in graph construction. Suppose we were able to construct a graph with the given connections. Are there are any other graphs that correspond to that set of connections? In most cases, there are. A $Y - \Delta$ transformation is an example. Do all sets of connections correspond to graphs in a unique genus? No. A $Y - \Delta$ transformation can change the genus of a graph as well. If there is a Y that traverses a handle, changing it to a Δ will make it not traverse the handle. Essentially a Δ is a complete graph on 3 vertices. We can generalize $Y - \Delta$ transformations:



We can extend these beyond well-connected transformations to same-connected transformations, and then the possibilities are endless. This allows us to talk about graphs that have the same connections, and how they might transform into one another. We can also think about the minimum genus of a set of connections. See the *Open Questions*

3 Open Questions

- Is there enough information to build graphs from connections in a circular ordering only?
- Will looking at specific connections- which ones exist AND how they exist- build a more accurate graph?
- Is there information hidden in the response matrix that finds general con-

nections and non-circular planar connections? Is there information in the response matrix that determines the genus of the graph?

- Modify Karen Perry's theorem- *a graph is critical iff its z-sequence is 1,...,n,1,...,n*- for the noncircular planar case.
- Use non-orientable embeddings to get the correct z-sequence.
- Develop an algorithm for choosing a circular ordering of boundary nodes so as to maximize connections.
- Continue the work on constructing graphs.
- Determine the genus of a set of connections, taking genus to be the smallest genus over same-connection transformations.

References

[1] Edward B. Curtis and James A. Morrow. "Inverse Problems for Electrical Networks." Series on Applied Mathematics – Volume 13. World Scientific, 2000.

[2] Karen Perry. "Discrete Complex Analysis." REU 2003.

[3] Nick Reichert."Generalized Circular Medial Graphs." REU 2004.

I would like to thank Owen Beisel for the methods on reducing the genus of a graph, and Jennifer French for the minimum number of k-connections, if they exist.