Discrete Cauchy Integrals

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Abstract

This paper discusses and develops discrete analogues to concepts from complex analysis. These concepts are interpreted on discrete electrical networks and include the Cauchy integral theorem and the Cauchy integral formula. To develop these, we begin by defining a diamond complex on a graph and its dual, and then explain how to do discrete calculus on such.

Contents

1 Definitions

Before we begin our discussion of discrete Cauchy integrals on discrete electrical networks, what is meant by a discrete electrical network must be established. Consider a graph G with edges E and vertices V so that we may write $G = (V, E)$. Note that the edges of G form faces. To make the graph into an electrical network Γ, assign a positive conductivity γ to each edge. Now, we may construct the dual graph of Γ, denoted Γ^*, by placing vertices of Γ^* on the faces of Γ and drawing the edges of Γ^* through the edges of Γ. In this way, there will be one edge in Γ^* for each edge in Γ. Finally, define the conductivities of the edges in Γ^* as the inverse of the conductivities in Γ, or $\frac{1}{\gamma}$. For more information about graphs and their duals, in particular, relating to inverse problems of discrete electrical networks, see [1].

Definition 1.1 (The Diamond)**.** Given a graph Γ, embedded on a surface, for each pair of edges in Γ and Γ^*, four new edges may be drawn enclosing the original pair of edges by connecting each vertex on Γ to each vertex on Γ^*. Doing this for every pair of dual edges creates a diamond complex $\diamond$ [2]. A single diamond from such a structure looks as is seen below:

Figure 1: Diamond of the Graph and its Dual

For those familiar with [1], the diamond structure can be thought of as the dual of the medial graph.

Now, we need to define functions on our structures. We have three distinct types of functions: vertex functions, edge functions, and face functions.

Definition 1.2 (Types of functions)**.** Vertex functions are defined nodally and generally represent electric potential.

Edge functions, which are essentially 1-forms, are defined on an edge. They are defined in such a way that they can be integrated along a curve or path, and they are oriented. Please note that these edges may be on the graph, its dual, or the diamond.

Correspondingly, face functions, which are essentially 2-forms, are defined on faces.

2 Discrete Calculus

The typical calculus operations must be redefined on the graph and its dual. It is important to note that these operations can be performed on the graph alone, the dual alone, or on the diamond.

2.1 Differentiation

Differentiating a vertex function f yields an edge function df. One can also take the derivative of an edge function α to obtain a face function $d\alpha$, which will be discussed in Section 2.3. Differentiating an exact edge function df yields a face function ddf; however, such a face function $ddf = 0$ because all exact forms are closed. This is essentially Kirchhoff's Voltage Law in disguise (the sum of voltage drops around any closed circuit is zero).

Now we must make sense of multiplying two functions f and g together, which we will represent by $f \odot g$.

$$(f \odot g)(p) = f(p)g(p) \tag{1}$$

Lemma 2.1. *The operation $\odot$, as defined above, yields a linear function.*

Since $f \odot g$ is a linear function, we can derive the following product rule for taking the derivative of a product of functions.

$$\begin{aligned} d(f \odot g)(e = pq) &= f(q)g(q) - f(p)g(p) \\ &= f(q)g(q) - f(q)g(p) + f(q)g(p) - f(p)g(p) \\ &= f(q)dg + g(p)df \\ &= f(q)g(q) - f(p)g(q) + f(p)g(q) - f(p)g(p) \\ &= g(p)df + f(p)dg \\ &= \frac{f(q)dg + g(p)df}{2} + \frac{g(p)df + f(p)dg}{2} \\ &= \frac{[f(p) + f(q)]}{2}dg + \frac{[g(p) + g(q)]}{2}df \end{aligned} \tag{2}$$

Motivated by this form of the product rule, we wish to define the product of a vertex function f and an edge function *alpha*, to obtain another edge function. Note that if we define:

$$(f \odot \alpha)(pq) = \frac{f(p) + f(q)}{2}\alpha(pq)$$

Then we may write Equation 2 as:

$$d(f \odot g) = f \odot dg + g \odot df,$$

which is an altogether nice way of writing the relation.

2.2 Integration

Implicitly, we have already been practicing integration. A more technical, though clumsy, way to write an edge function α evaluated at a single edge e is:

$$\alpha(e) \equiv \int_e \alpha$$

We will tend to use the version on the left when evaluating an edge function on only one edge, but when considering integration along a path, we will use the latter notation, or possibly $\oint \alpha$ when the integration is over a closed loop.

2.3 Stokes' Theorem

We mentioned in Section 2.1 that one can differentiate not only vertex functions to obtain edge functions, but also d acts on edge functions to produce face functions. The face function $d\alpha$ is defined so that it obeys the general Stokes' Theorem:

$$d\alpha(D) = \oint_{\partial D} \alpha$$

This explains our former statement that $ddf = 0$, since $ddf(D) = \int_{\partial D} df = \int \int_{\partial\partial D} f$ which is integration over an empty set.

2.4 Analytic Functions

In traditional complex analysis, an analytic function on an open set Ω is one that is differentiable everywhere in Ω. In our case, we define an analytic function on the graph and its dual with the following equation [2]:

$$i\gamma(e)df(e) = df(e^{\perp}) \tag{3}$$

As with continuous complex analysis, the analyticity of a function implies the existence of Cauchy-Riemann equations. The above equation can be broken up into the Discrete Cauchy-Riemann equation as defined by Karen Perry [3]: $\gamma u_x(e) = v_y(e^{\perp})$. Begin by letting $f = u + iv$. Furthermore, let $df(e)$ be the derivative in the "x"-direction and $df(e^{\perp})$ be the derivative in the "y"-direction. This simple exercise is left to the reader.

3 The Cauchy Integral Theorem

Now that we know how to define differentiation and integration on the diamond complex $\diamond$, we are able to state the discrete analogue of the Cauchy Integral Theorem:

Theorem 3.1 (The Cauchy Integral Theorem)**.** *If f and g are analytic functions on a domain Ω in the diamond complex, then for all region bounding curves*

$C \in \Omega$, we have:

$$\oint_C f \; dg = 0 \tag{4}$$

Proof. We will first show that the theorem holds for a single face D in the diamond complex, as in Figure 1. Letting the values of f and g on the vertices of the diamond be $f_x, f_y, f_{x'}, f_{y'}, g_x, g_y, g_{x'}$, and $g_{y'}$, the above integral is written:

$$\begin{aligned}
\oint_{\partial D} f \; dg &= \sum_{pq \in \partial D} (f \odot dg)(pq) \\
&= \sum_{pq \in \partial D} \left(\frac{f_p + f_q}{2} \right) (g_q - g_p) \\
&= \frac{1}{2} \left[(f_{y'} + f_x)(g_x - g_{y'}) + (f_x + f_y)(g_y - g_x) + (f_y + f_{x'})(g_{x'} - g_y) + (f_{x'} + f_{y'})(g_{y'} - g_{x'}) \right] \\
&= \frac{1}{2} \left[f_x(g_y - g_{y'}) + f_y(g_{x'} - g_x) + f_{x'}(g_{y'} - g_y) + f_{y'}(g_x - g_{x'}) \right] \\
&= \frac{1}{2} \left[(f_{x'} - f_x)(g_{y'} - g_y) + (f_{y'} - f_y)(g_x - g_{x'}) \right]
\end{aligned}$$

Now, since g is analytic on the diamond, we should have that $dg(yy') = i \; \gamma_{xx'} \; dg(xx')$. Therefore we have:

$$\begin{aligned}
\oint_{\partial D} f \; dg &= \frac{1}{2} \left[i\gamma_{xx'}(f_{x'} - f_x)(g_{x'} - g_x) + (f_{y'} - f_y)(g_x - g_{x'}) \right] \\
&= \frac{1}{2} \big[i \; \gamma_{xx'}(f_{x'} - f_x) + (f_y - f_{y'}) \big] (g_{x'} - g_x)
\end{aligned}$$

But f is also analytic, so we know that $i\gamma_{xx'} \; (f_{x'} - f_x) = (f_{y'} - f_y)$. Therefore:

$$\oint_{\partial D} f dg = \frac{1}{2} [(f_{y'} - f_y) + (f_y - f_{y'})](g_{y'} - g_y) = 0$$

Now we will show the general case, letting D be the set of faces bounded by C. Then, as Figure 2 shows, we may write the oriented integral $\oint f dg$ as a sum of oriented integrals around each face in D. But since the integral around any face in D is 0, we know that their sum is also zero, and hence Theorem 3.1 holds in general. □

Examining the proof of Theorem 3.1 yields an interesting corollary:

Corollary 3.2 (Morera's Theorem)**.** *If $\oint_{\partial D} f dg = 0$ for all diamond faces D in some domain Ω and all analytic functions g on Ω, then f must be analytic on Ω too.*

Proof. The proof of the Cauchy Integral Theorem for a single face does not use the fact that f is analytic up to the line:

$$\oint_{\partial D} f \; dg = \frac{1}{2} \big[(f_{x'} - f_x) + i \; \gamma_{yy'}(f_{y'} - f_y) \big] (g_{y'} - g_y)$$

Figure 2: An illustration of how a single integral around a large domain may be written as a sum of integrals around individual faces.

But if this is to vanish for all analytic functions g, it must hold that $(f_{x'} - f_x) = -i\gamma_{yy'}(f_{y'} - f_y)$. Therefore f satisfies the Cauchy-Riemann Equations on D. The proof, however, did not depend on which face in Ω was chosen, so f satisfies the Cauchy-Riemann Equations everywhere in Ω. Therefore f is analytic on Ω by definition. $\square$

4 Before Attacking the Cauchy Integral Formula...

Note that the discrete Cauchy Integral Theorem utilizes strictly analytic functions; to extend this theorem to a discrete Cauchy Integral Formula there is necessity for functions with poles. For our purposes we can understand a pole to be a vertex at which a function is not γ-harmonic, or has a nonzero net current.

Definition 4.1 (Green's function)**.** A function g_x is a Green's Function on a graph Γ if it is harmonic on interior nodes of Γ excepting one interior node x where g_x has a net current of magnitude 1. On the boundary of Γ, g_x has zero potential. We say that g_x has a pole at node x. Further notes on such functions can be found in Karen Perry's paper on Discrete Complex Analysis [3].

A natural question to ask is how to extend g_x in such a way so that it is defined on not only Γ but also Γ^*. The way to do this is to consider the edge function $dg_x(e)$ and use the equation for analyticity to define the edge function $\widetilde{dg_x}(e^\perp)$ on the dual. It is noteworthy that this edge function on the dual is a multi-valued vertex function. Hence, although it seems an obvious first approach, integrating an edge function fdg_x around a simple closed curve along diamond edges is not a way to develop the Cauchy Integral Formula. This leads to the idea that we must integrate an edge function of the form $f\nu$ where ν is an edge function which includes a pole on both Γ and Γ^*.

Recall that integrating around any simple closed curve can be broken up into a summation of integrations around smaller curves. Using this, we can simplify calculations by assuming that g_x is zero on the boundary of the subgraph of Γ contained by the smallest simple closed curve surrounding both poles. We also assume that g_y, with a pole y on Γ^*, is zero on the boundary of the subgraph of Γ^*. The reason that we can do this is because whatever the Green's functions g_x, g_y were initially, the difference between our redefined functions and the actual functions from our definition, which are analytic, is an analytic function.

Finally, before constructing the Cauchy Integral Formula, we observe a useful property of edge functions.

Lemma 4.2. *If edges e_1, e_2 are orthogonal edges and edge functions α, β satisfy the Cauchy Riemann equations, then*

$$\alpha(e_1)\beta(e_2) = \alpha(e_2)\beta(e_1)$$

Proof. Since e_1, e_2 being orthogonal edges implies that one is on the graph and the other is on the dual. By inspection, either e_1 is the dual of e_2 or e_2 is the dual of e_1. Assume the latter. Since α, β satisfy the Cauchy Riemann equations, $i\gamma\alpha(e_1) = \alpha(e_2)$ and $i\gamma\beta(e_1) = \beta(e_2)$. Hence,

$$\begin{aligned} \alpha(e_1)\beta(e_2) &= \tfrac{\alpha(e_2)}{i\gamma} i\gamma\beta(e_1) \\ &= \alpha(e_2)\beta(e_1) \end{aligned}$$

□

5 The Cauchy Integral Formula

We now wish to find a discrete analogue of the Cauchy Integral Formula:

$$\frac{1}{2\pi i}\oint_{\partial D}\frac{f(z)}{z-p}dz = f(p),$$

where $p \in D$ and f is analytic on $\overline{D}$. For this to have an analogue on a discrete network, we must first find the discrete version of $\nu = \dfrac{dz}{z-p}$. Since $f\nu$ is to be integrated, while f is a vertex function, we must require that ν be an edge function. But just as with the Cauchy Integral Theorem, the edge function in question must be defined on the edges of the diamond, rather than merely on

Figure 3: A diagram showing the contour used in constructing edge function ν.

either the graph or the dual. Note also that the discrete version of $f(p)$ is f evaluated both on the graph and on its dual, or some form like $f(x) + f(y)$, where x is a point in G and y a point in $G_\perp$. At first, we will assume that (x, y) is an edge in $\diamond$.

This edge function ν must also have the property that it satisfies the discrete Cauchy-Riemann equations on each face of the diamond complex not adjacent to either x or y, so that $\oint f\nu = 0$ if the curve does not surround (x, y). Also, we wish for ν to be globally well-defined, so that we may form the product $f\nu$. We shall now construct such an edge function. We will then find the integral of $f\nu$ around a closed curve surrounding both x and y, as shown in Figure 3. To do this, we first construct an edge function μ which is meromorphic with poles at x and y defined on both the graph and the dual graph. To do this, consider the Green's edge function dg_x with pole at x and the Green's edge function dg_y. Now consider the γ-harmonic extension of dg_x to $\widetilde{dg}_x$ defined on $G_\perp$, so that the pair of edge functions $dg_x + i\widetilde{dg}_x$ is meromorphic on the primal and dual graphs, with a pole at x. Similarly, take the $-1/\gamma$-harmonic conjugate of dg_y to obtain $\widetilde{dg}_y$ defined on the primal graph G, so that $i\widetilde{dg}_y - dg_y$ is meromorphic on the graph and its dual, with a pole at y. Now define the edge function μ to be $(-dg_x + i\widetilde{dg}_y) + (-i\widetilde{dg}_x - dg_y)$, which is a meromorphic edge function with poles at x and y.

We can now construct a diamond edge function ν out of this graph edge function μ. We desire that for any two incident diamond edges (x_1, y_1) and (y_1, x_2), $\nu(x_1, y_1) + \nu(y_1, x_2)$ should be equal to the integral of μ along any path λ from x_1 to x_2, such that the closed loop formed by $\lambda \cup (x_1, y_1) \cup (y_1, x_2)$ does not completely surround either pole (although the reader may easily verify that λ may be allowed to intersect them). In this way we see that

$$\begin{aligned}
\nu(12) + \nu(23) &= \mu(1x) + \mu(x3), \text{ or} \\
\nu(23) &= -\nu(12) + \mu(1x) + \mu(x3). \\
\text{Similarly, we have } \nu(34) &= -\nu(23) + \mu(2y) + \mu(y4) \\
&= \nu(12) - \mu(1x) - \mu(x3) + \mu(2y) + \mu(y4).
\end{aligned}$$

Continuing around the egg-shaped contour in Figure 3, we find that:

$$\nu(12) = \nu(12) + [\mu(2y) + \mu(y6) + \mu(62)] - [\mu(x3) + \mu(35) + \mu(5x)]$$

But because we have defined μ in terms of the Green's functions at x and y, the expressions in brackets are both equal to i, so the above condition shows that ν is single-valued on the contour surrounding x and y. Therefore, since ν is analytic outside of this contour, we can extend ν to be single-valued everywhere outside this contour.

Using this as the definition of ν, we can calculate the integral $\oint_C f\nu$ for the curve shown in Figure 3. We have:

$$\begin{aligned}
2\oint_C f\nu &= \sum_C (f_i + f_{i+1})\nu(i, i+1) \\
&= (f_1 + f_2)\nu(12) + (f_2 + f_3)\nu(23) + (f_3 + f_4)\nu(34) \\
&+ (f_4 + f_5)\nu(45) + (f_5 + f_6)\nu(56) + (f_6 + f_1)\nu(61) \\
&= f_1(\nu(61) + \nu(12)) + f_3(\nu(23) + \nu(34)) + f_5(\nu(45) + \nu(56)) \\
&+ f_2(\nu(12) + \nu(23)) + f_4(\nu(34) + \nu(45)) + f_6(\nu(56) + \nu(61))
\end{aligned}$$

But we defined these sums in parentheses to be expressible as path integrals of μ, so we have:

$$\begin{aligned}
2\oint_C f\nu &= f_1(\mu(62)) + f_3(\mu(2y) + \mu(y4)) + f_5(\mu(4y) + \mu(y6)) \\
&+ f_2(\mu(1x) + \mu(x3)) + f_4(\mu(35)) + f_6(\mu(5x) + \mu(x1))
\end{aligned}$$

Now notice the following two terms in the sum: $f_3\mu(y4)+f_5\mu(4y)$. We may write these as:

$$\begin{aligned} f_3\mu(y4)+f_5\mu(4y) &= (f_5-f_3)\mu(4y) \\ &= df(35)\mu(4y) \\ &= df(4y)\mu(35)\text{by Lemma 4.2} \\ &= f_y\mu(35)-f_4\mu(35) \end{aligned}$$

Similarly, we have $f_2\mu(1x)+f_6\mu(x1)=f_x\mu(62)-f_1\mu(62)$. Thus we may rewrite the integral as:

$$\begin{aligned} 2\oint_C f\nu &= f_1(\mu(62))+f_3(\mu(2y))+f_5(\mu(y6))+f_y\mu(35)-f_4\mu(35) \\ &+ f_2(\mu(x3))+f_4(\mu(35))+f_6(\mu(5x))+f_x\mu(62)-f_1\mu(62) \\ &= f_3(\mu(2y))+f_5(\mu(y6))+f_x\mu(62) \\ &+ f_2(\mu(x3))+f_6(\mu(5x))+f_y\mu(35) \end{aligned}$$

Now we may rewrite $\mu(35)$ as $i-\mu(x3)-\mu(5x)$ and $\mu(62)$ as $i-\mu(y6)-\mu(2y)$, and we get:

$$\begin{aligned} 2\oint_C f\nu &= f_3\mu(2y)+f_5\mu(y6)+if_x-f_x\mu(y6)-f_x(2y) \\ &+ f_2\mu(x3)+f_6\mu(5x)+if_y-f_y\mu(x3)-f_y\mu(5x) \\ &= (f_3-f_x)\mu(2y)+(f_5-f_x)\mu(y6)+if_x+(f_2-f_y)\mu(x3)+(f_6-f_y)\mu(5x) \\ &= if_x+if_y+df(x3)\mu(2y)-df(2y)\mu(x3)+df(x5)\mu(y6)-df(y6)\mu(x5) \\ &= if_x+if_y \\ \oint_C f\nu &= i\frac{f_x+f_y}{2} \end{aligned}$$

This is the first stage of proving our version of the Cauchy Integral Formula. Now for any other simple closed curve surrounding both x and y, we can collapse it to a similar egg-shaped contour through a homology, which will not change the integral due to the analyticity of f. Therefore, for any curve C which surrounds x and y, $\oint_C f\nu = i\frac{f(x)+f(y)}{2}$. We may now lift the restriction on the adjacency of x and y to find the Cauchy Integral Formula:

Theorem 5.1 (Cauchy Integral Formula)**.** *For all simple closed curves C which completely surround two points $x\in G$ and $y\in G_\perp$, as well as a path through $\diamond$ connecting them, there exists an edge function ν as defined above such that:*

$$\oint_C f\nu = i\frac{f(x)+f(y)}{2}$$

Proof. Let the points in the path connecting x and y be $\{x_0=x, y_0, x_1, y_1, \ldots, y_n = y\}$ so that (y_i, x_{i+1}) and (x_i, y_i) are edges in $\diamond$. Then for each of these edges, we have:

$$\begin{aligned} i\frac{f(x_i)+f(y_i)}{2} &= \oint_C f\nu_{x_i,y_i} \text{ and} \\ i\frac{f(x_{i+1})+f(y_i)}{2} &= \oint_C f\nu_{x_{i+1},y_i} \text{for suitable choice of } \nu_{x,y} \end{aligned}$$

Now let $\nu = \sum_j(\nu_{x_j,y_j} - \nu_{x_{j+1},y_j})$. Then:

$$\begin{aligned} \oint_C f\nu &= \oint_C f\sum_j(\nu_{x_j,y_j} - \nu_{x_{j+1},y_j}) \\ &= \sum_j\left(\oint_C f\nu_{x_j,y_j} - \oint_C f\nu_{x_{j+1},y_j}\right) \\ &= \sum_j\left(i\frac{f(x_j)+f(y_j)}{2} - i\frac{f(x_{i+1})+f(y_i)}{2}\right) \end{aligned}$$

Which is a telescoping series, collapsing to $i\dfrac{f(x)+f(y)}{2}$. □

6 Future Research

1. To continue with complex analysis, one formula for which a discrete analogue might be able to be developed is the Riemann Roch Theorem. One form of this theorem is a statement about the dimension of the space of meromorphic functions. Unfortunately, before this theorem can be addressed, we need to have a better understanding of zeros and poles, which are not currently well-defined.

2. A more basic element of complex analysis, for which there may or may not be a discrete analogue, which also requires an understanding of zeros and poles is the Argument Principle. The Argument Principle, written in the continuous form below, counts the number of zeros N and poles P contained in a simple close contour C for a meromorphic function f.

   $$\int_C \frac{df}{f} = 2\pi i[N-P]$$

   One problem here is that it is difficult define division in such a way that even an analytic function works. Furthermore, if one can get an analytic function to work so that it yields $N - P = 0$, one still needs to find a way to deal with zeros.

3. In an attempt to deal with zeros, we defined a region of zeros as one with no current flowing in its edges, or one with constant potential. If a region of zeros is imposed onto a graph current bounces off of it creating alternating signs of potentials on the boundary. Such is similar to the

Alternating Property in [1] which is a determinantal relationship that was used in characterizing critical circular planar response matrices. It is not probable that a region of zeros could made to be the hole in an annular graph, but are there similar determinantal relations that could be used to characterize the response matrix for an annular graph? If so, what do the well-connected annular graphs look like? Also, what happens when one integrates around the hole of an annulus? Could such a hole, or in our case, a collection of boundary nodes, be a pole with an order of magnitude greater than one or be a collection of poles?

4. Many of the results arrived at in this paper can be attributed to Mercat in his paper entitled "Discrete Riemann Surfaces and the Ising Model" [2]. He develops graphs, their duals, and the diamond structure, and then discusses complex analysis on such. However, how he relates these things to the Ising Model is unclear. Hence, it would be interesting to research how the Ising Model relates to said structures and to develop a forward problem about the Ising Model on discrete networks. Work on such has already been done by physicists but simplying this information and relaying it to the UW REU program could be worthwhile.

References

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[3] Perry, K. "Discrete Complex Analysis." 2003.