Circular-planar self-dual graphs

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Contents

1 Definitions and Terminology

We will begin by defining a few terms of importance. First we have the simple notion of a graph with boundary taken from Curtis and Morrow [1]:

Definition 1.1. A *graph with boundary* is a triple $G = (V, V_B, E)$, where V is the set of nodes and E is the set of edges, and V_B is a nonempty subset of V called the set of boundary nodes. The set A *Graph with Boundary* is a decomposition $V = \partial V \cup V_I$, where $\partial V \neq \emptyset$ is the set of boundary vertices, and V_I is the set of interior vertices.

Now we consider a cellular embedding of a graph in a surface:

Definition 1.2. An *embedding* of a graph G in a surface S is a one-to-one map $f : G \to S$ such that vertices map to points in S and edges map to simple disjoint curves in S that connect their boundary points. The embedding is *cellular* if the surface may be divided into disjoint faces, each of which is bounded by edges in $f(E)$ and topologically equivalent to a disc. Moreover, a graph with boundary may be *circularly cellularly embedded* if the boundary points $f(\partial V)$ all lie on a simple curve in S, the inside (or outside) of which is a topological disc and does not intersect $f(G)$.

Some graphs are called circular planar graphs:

Definition 1.3. A *circular planar graph* is a graph G with boundary which is embedded in a disc D in the plane so that the boundary nodes lie on the circle C which bounds D, and the rest of G is in the interior of D. The boundary nodes are labeled $v_1, \ldots, v_n$ in counterclockwise order around C. In our figures we represent C by a dashed circle around G.

We must now introduce a topic central to our discussion: that of the dual to an embedded graph.

Definition 1.4. Recall from the definition of a cellular embedding that the surface S is divided into a finite number of cells, called faces. To each face F, assign the dual point $F_\perp$. If two neighboring faces F_1 and F_2 share a common edge e, connect their dual points with an edge $e_\perp$. Also, to each point $v \in V_I$, with neighboring edges $\{e_k\}$, assign the face $v_\perp$, bounded by the dual edges $\{e_{k\perp}\}$. Finally, the points on the boundary circle have no dual faces, but dual boundary points are placed between the original boundary points, and they are connected to the duals of edges which bordered the original boundary. The result is a circular cellular embedding of a new graph: $G_\perp$, the dual to the graph G.

For clarity, when speaking of a graph and its dual, we will often refer to the graph as the *primal graph*, and, of course the dual as the *dual graph*. Now that we know precisely what a dual is, we need to be able to identify when two graphs are, for our purposes, identical:

Definition 1.5. An *isomorphism* between two graphs is a one-to-one and onto vertex map which preserves edges. Specifically, if $f : G_1 \to G_2$ is an isomorphism, then $f(v_1)$ is a neighbor of $f(v_2)$ in G_2 if and only if v_1 is a neighbor of v_2 in G_1. If there exists an isomorphism between G_1 and G_2, then we say G_1 is *isomorphic* to G_2. Also, if G_1 and G_2 are the same graph, then the *identity* or *trivial* isomorphism maps each vertex to itself.

For the remainder of the paper, we concern ourselves with characterizing primal graphs G that are isomorphic to their dual graphs $G_\perp$, or graphs which are *self-dual.*

2 The Medial Graph

The notion of a *medial graph* of a circular planar graph from Curtis and Morrow [1] is integral to the study of self-dualing graphs.

Definition 2.1. See Curtis and Morrow [1, §8.1].

As described in [1] we can two-color the faces of the medial graph such that in each of the gray faces lies a node of the primal graph and in each of the white faces lies a node of the dual graph. We shall refer to the set of gray cells of the medial graph as $\mathcal{M}_P$, for those cells relating to the primal graph and, the set of white cells as $\mathcal{M}_D$, for those cells relating to the dual graph.

Lemma 2.2. *Given a graph G and a two coloring of its medial graph $\mathcal{M}$, the following statements are equivalent:*

a. G is isomorphic to its dual graph $G_\perp$

b. There is a bijective map M from the gray cells of the medial graph $\mathcal{M}_P$ to the white cells of the medial graph $\mathcal{M}_D$ such that given two cells C_1 and C_2 in $\mathcal{M}_P$, $M(C_1)$ and $M(C_2)$ in $\mathcal{M}_D$ are adjacent if and only if C_1 C_2 are adjacent, where two cells are said to be adjacent if they share a vertex.

Proof. This result follows nearly from the definitions. Identifying the vertices of G with the cells in $\mathcal{M}_P$, the vertices of $G_\perp$ with the white cells in $\mathcal{M}_D$, and neighboring vertices with adjacent cells, we find that an edge-preserving isomorphism between G and $G_\perp$ is merely an adjecency-preserving bijective map from $\mathcal{M}_P$ to $\mathcal{M}_D$. $\square$

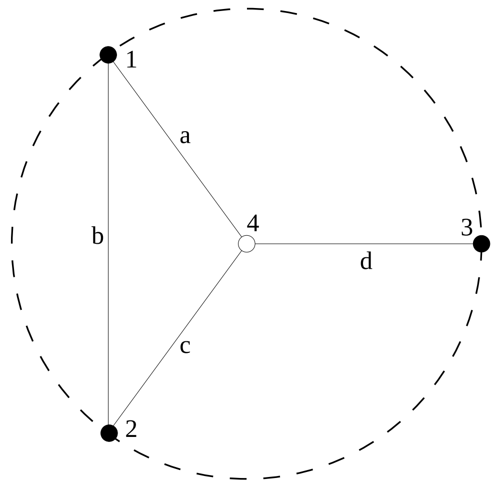

Figure 1: A 3-gon with edges *a,b,c* spliced by node 4 to a 3-star with edges *a,c,d*.

3 *n-gon/n-star* Graphs

There are a special class of self-dual graphs that we stumbled upon early in our investigation of the subject which we have termed the *n-gon/n-star graphs*. The most simple graphs in this class are a single n-gon spliced to an n-star by a common vertex, with all of the nodes boundary except the node that is a part of both the polygon and the star. The simplest example of this is simple a triangle (3-gon) and a 3-star:

It is easily verifiable by inspection that any of these graphs is its own dual (See figure 2). This result will be verified more rigorously for the entire class of n-gon/n-star graphs by theorem 4.1.

Also, because the degree of a node matches the number of edges of the corresponding cell of the medial graph, the medial graphs of n-gon/n-star graphs exhibit distinct patterns (See figures 3 and 4).

N-gon/n-star self-dual graphs can be made more complex by combining multiple n-gon/n-stars together. Each additional n-gon/n-star construction adds one interior node to the graph.

Furthermore, we can attach any number of n-gon/n-star graphs by this method and still have a graph that is self dual (See Figure 6).

We can even attach combinations of n-gon/n-stars of different degrees as in Figure 7.

At first it is not obvious how these fit together to form new graphs which are are self dual, however, the best way to see how these multiple graphs are fit together is by observing how the medial graphs are combined. The key is to choose a geodesic of each of two medial graphs and draw a single medial graphs with these geodesics identified, such that the new medial graph maintains an appropriate symmetry,

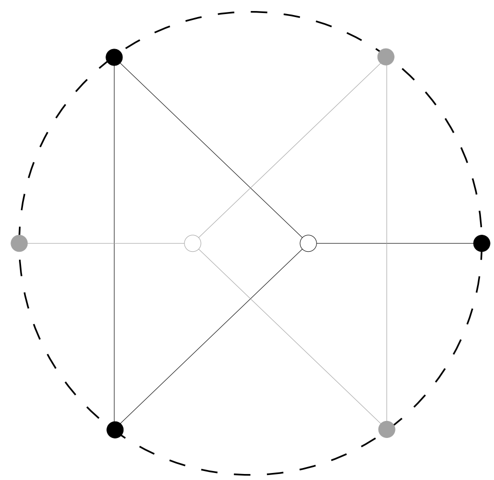

Figure 2: A 3-gon/3-star (black) and its dual graph (gray).

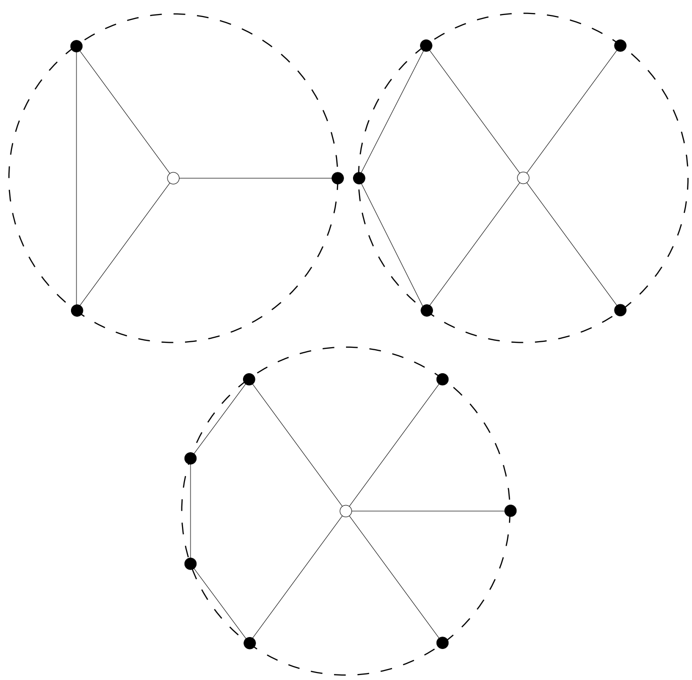

Figure 3: Illustrates the class of *n-gon/n-star* self-dual graphs.

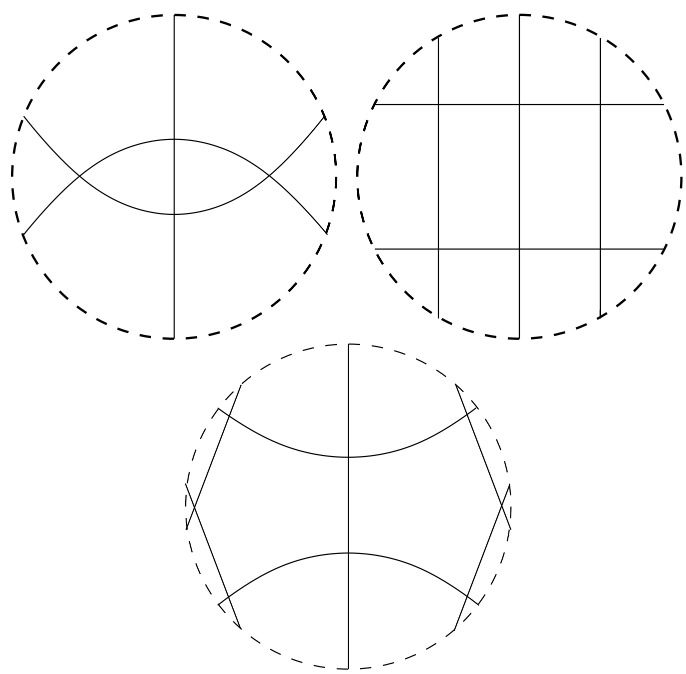

Figure 4: Medial graphs of the n-gon/n-star graphs in figure 3.

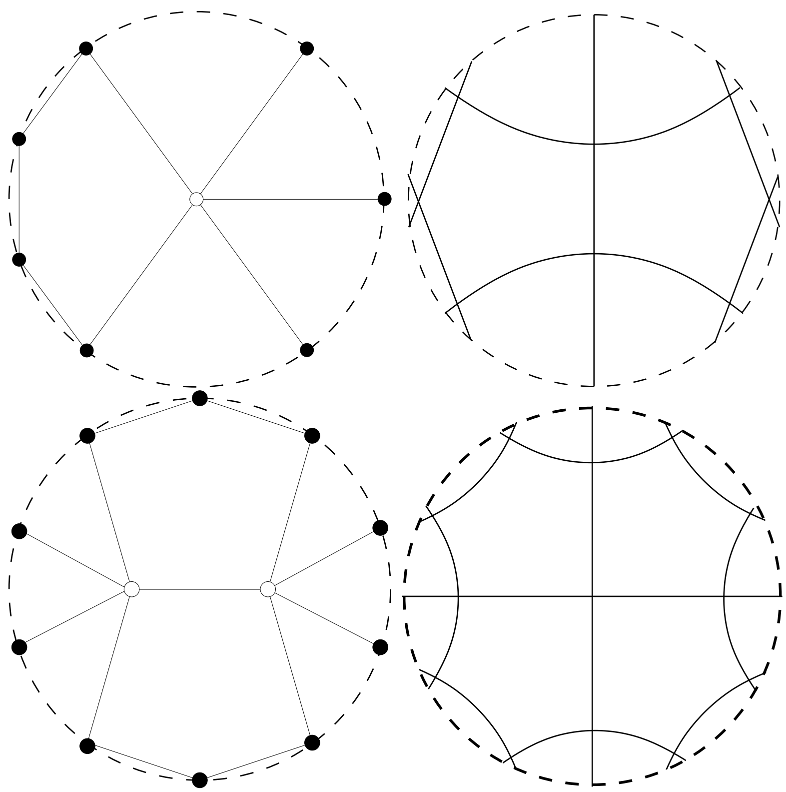

Figure 5: Two 5-gon/5-star graphs combined to form a self-dual double 5-gon/5-star.

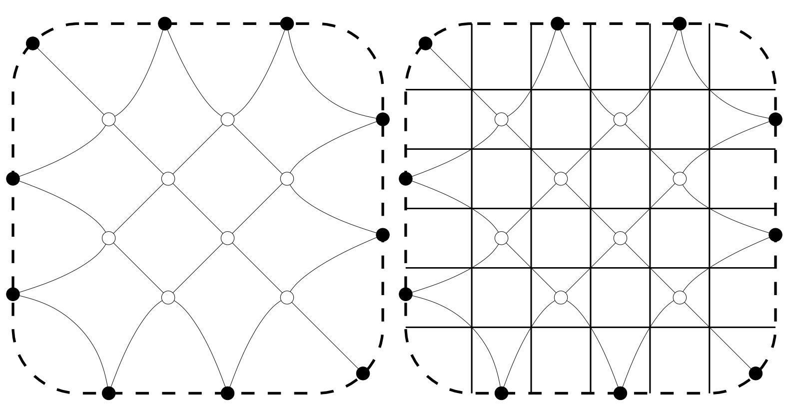

Figure 6: Multiple 4-gon/4-stars joined together.

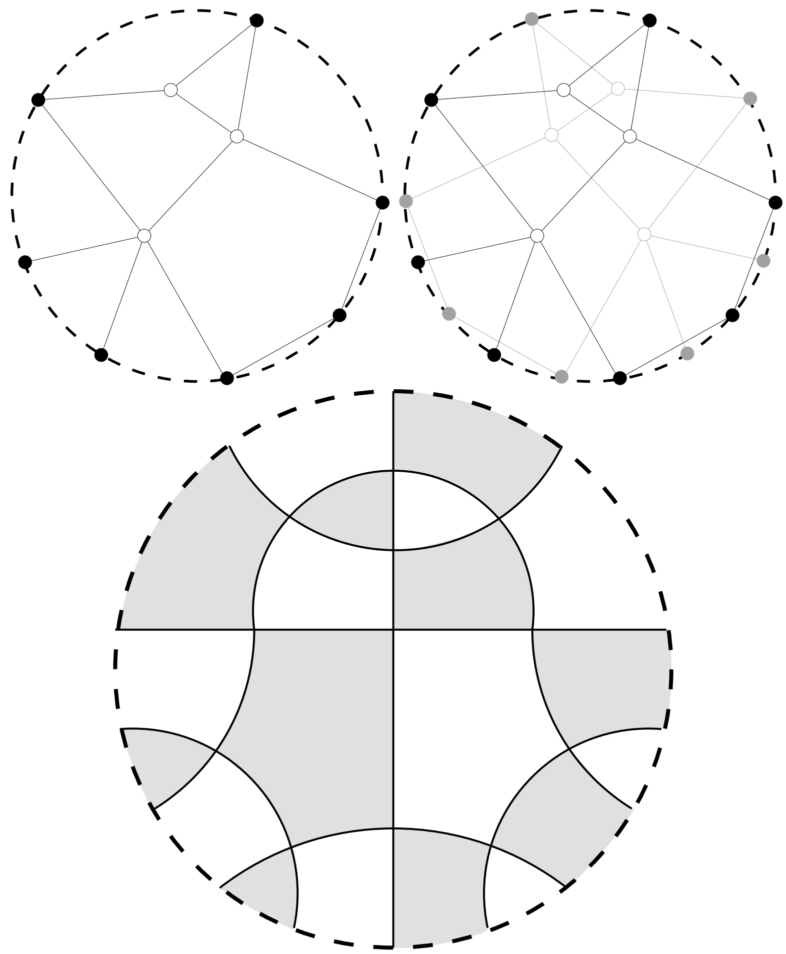

Figure 7: A 3-gon/3-star stacked on top of a 4-gon/4-star, stacked on a 5-gon/5-star, then with its dual drawn in gray, and finally its medial graph.

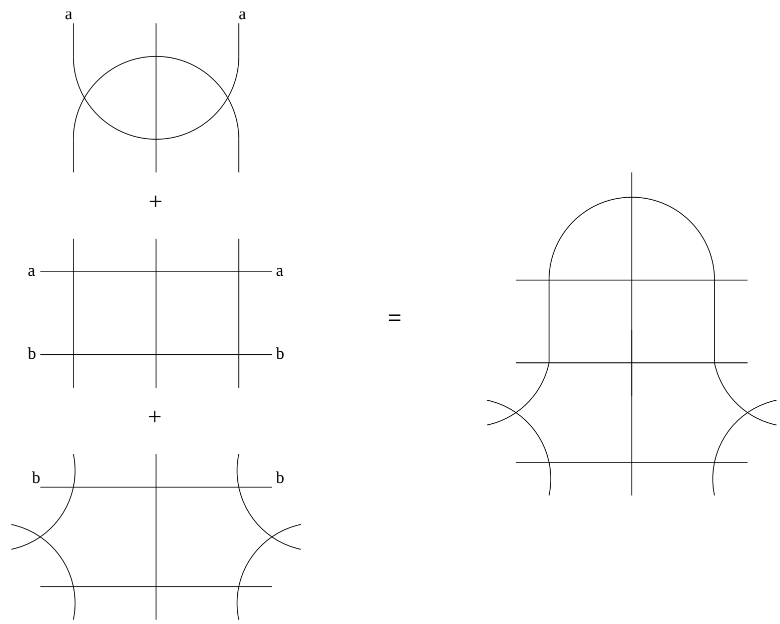

Figure 8: The splicing process of for the 345-gon/star in Figure 7.

a process we call ‘splicing’. (For acceptable symmetries, see Sections 4 and 5.) For example, notice the rotational symmetry maintained in the medial graph of the double 5-gon/5-star in Figure 5 and the multiple 4-gon/4-star in Figure stacked 4gon 4star or the reflective symmetry in the medial graph of the stacked 345-gon/star in Figure 7. Also, to see how the splicing process works, try to see the individual medial graph of each 5-gon/5-star in the double of Figure 5 and locate the spliced edge. The splicing process of the stacked 345-gon/star in Figure 7 is broken down more clearly in Figure 8.

4 Rotationally Isomorphic Duals

Next we consider a larger class of self-dual graphs. For these graphs, the isomorphism which relates the graph to its dual is merely a global rotation through π or $\pi/2$. We begin with a theorem on the characteristics of these graphs, and their medial graphs:

Theorem 4.1. *A circular planar graph is self-dual if it satisfies either of the two sets of conditions below:*

(a)
- *G has an odd number of boundary vertices.*
- *the medial graph of G is π-symmetric.*

(b)
- *G has an even number of boundary vertices, but the number of boundary vertices is not divisible by 4.*
- *the medial graph of G is $\frac{\pi}{2}$-symmetric.*

Proof. Suppose the conditions in (a) are met. Then consider an arbitrary gray cell C in the medial graph and a path Ł from that cell to a gray cell C_B in the boundary, as in Figure 4.1. Now consider the cell C'_B opposite C_B, and a path L_B connecting them through only boundary cells. Now there are an odd number of boundary nodes, so there are an odd number of geodesics which begin and end on the boundary, and L_B must cross each one exactly once. Since crossing each geodesic changes the parity of the cell, C'_B must be white. Now consider the path L', obtained by rotating L through π; it connects C'_B and C', the cell opposite C. Since L did not change the parity of C on its path to C_B, L' cannot change the parity of C'_B on its path to C'. Therefore C' must be white. Thus we have found a mapping from gray cells like C to distinct white cells like C', and the π-symmetric nature of $\mathcal{M}(G)$ guarantees that neighboring black cells will map to neighboring white cells. Therefore by lemma 2.2, we know that G must be self-dual.
The proof for (b) is similar, but the now the map is a rotation through $\pi/2$ rather than π. The path L_B will then cross exactly half of the boundary-to-boundary geodesics; this number is odd since the number of boundary nodes is not divisible by four. Once again, we have found a bijective map from the gray cells to the white cells which preserves adjacency; therefore G is self-dual. ☐

We see, then, that the class of self-dual graphs is much larger than that obtained merely from n-gon/n-star networks. Note that the converse of theorem 4.1 is not true; in section 5 we give another class of self-dual graphs which do not obey the above conditions. However, we do have a slightly weaker version of the converse of this theorem.

Theorem 4.2. *Suppose the graph G is self dual, and that the isomorphism relating the white and black cells of the medial graph is a global rotation. Then G must either satisfy the conditions in (a) or (b) of Theorem 4.1*

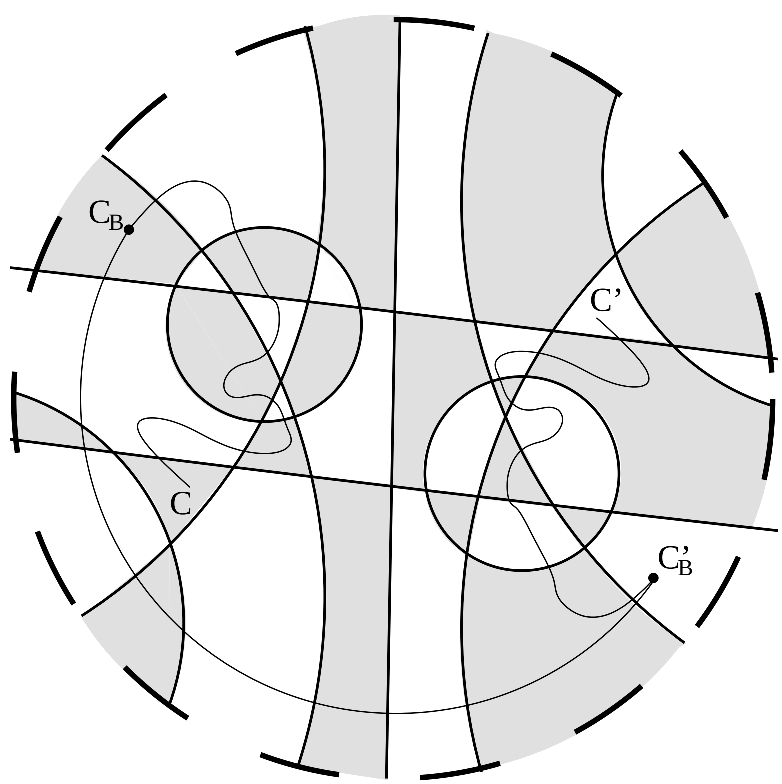

Figure 9: An illustration of the paths used in the proof of part (a) of Theorem 4.1. Part (b) is similar.

Proof. Suppose the medial graph of G is symmetric under a rotation $2\pi/n$, with $n > 1$, and that the rotation maps black cells to white cells and vice versa. First note that the point of rotation will either lie in a cell of the medial graph or on at least one geodesic. If the point of rotation lies in a cell, then by definition this cell must map to itself under a rotation. However, we require that the rotation maps black cells to white cells and vice versa; therefore the point of rotation must lie on at least one geodesic. Letting the number of geodesics intersecting the point of rotation be k, we see immediately that k can only be 1 or 2. We will see that the case $k = 1$ corresponds to conditions (a), and the case $k = 2$ corresponds to conditions (b).

Assume that $k = 1$. Then the only possible rotation symmetry possessed by G's medial graph is a rotation through π, since any other rotation would not map the single geodesic onto itself. Therefore G's medial graph is π-symmetric. Now we know that no other geodesic passes throught the point of rotation, so each geodesic which begins and ends on the boundary must map to a distinct geodesic which begins and ends on the boundary. Therefore, including the geodesic passing through the center, there must be an odd number of geodesics which begin and end on the boundary. Therefore, G has an odd number of boundary nodes. Therefore we have shown that if $k = 1$, the self-dual graph G must belong in category (a).

Now suppose that $k = 2$. Then the medial graph of G must have either $\pi/2$ symmetry, or simply π symmetry. Now the two-coloring the cells near the point of rotation shows that there are two opposite black cells, and two opposite white cells. But the rotation symmetry of the medial graph must map black cells to white cells, and vice versa, so the medial graph cannot be merely π-symmetric, but must also possess $\pi/2$ symmetry. Now any geodesic which begins and ends on the boundary requires that there be exactly three other geodesics which begin and end on the boundary as well. Therefore the total number of such geodesics, including those passing through the center, is of the form $4m + 2$. This means that the number of boundary nodes is even, but not divisible by four. Therefore if $k = 2$, the self-dual graph is of type (b). □

This means that we have a complete characterization of graphs that are related to their duals through rotations. In Section 6, we suggest a possible further extension of these ideas.

5 Reflectively Isomorphic Duals

Now we consider another set of self-dual graphs. Here, the relevant isomorphism is not a global rotation through some angle, but rather a global reflection across some axis. Here, we again find a theorem similar to the combination of theorems 4.1 and 4.2.

Theorem 5.1. *For a self-dual graph, the following two statements are equivalent:*

(a) *The isomorphism relating $\mathcal{M}_P$ and $\mathcal{M}_D$ is a reflection along an axis.*

(b) *The medial graph is reflectively symmetric along the axis, and the axis of symmetry runs along a geodesic.*

Proof. Suppose (b) is true; we will then derive (a). Consider an arbitrary gray cell $C \in \mathcal{M}_P$, and a path L from C to a gray cell C_B, where C_B borders on the geodesic on the axis of reflection, as exemplified in Figure 10. Then C'_B, the cell across from C_B, shares an edge with C_B and hence must be white. Now consider L', the path which runs from C'_B to C'. Since L did not change the parity of C on its path to C_B, L' cannot change the parity of C'_B on its path to C'. Therefore C' must be white. Thus we have found the reflection across the axis to be a bijective map M which takes black cells to white cells, and preserves adjacency. Therefore (a) is true.// Now given (a), we will show that (b) must be true. First, by definition $\mathcal{M}$ must be reflectively symmetric across the axis of symmetry (otherwise such an isomorphism could not exist). Now if any cell occupied both sides of the axis, then it would map to itself under M, and therefore could not be two-colored. Thus segments of geodesics must occupy the entire axis of symmetry. But geodesics may only intersect transversally, so it must be the same geodesic which occupies the whole length of the axis. Therefore (b) must be true. □

Unlike the case of rotations, there is no restriction on the number of boundary nodes required for a valid self-dual reflected graph. All that is needed is for a geodesic to run down the center of a reflectively symmetric graph.
As an example of a graph which is reflectively isomorphic to its own dual, consider the basketball graph shown in figure 11.

Note that for the basketball, the graph and the dual are related by a global reflection through the axis which runs straight down through the center of the graph. Also, the medial graph is reflectively symmetric and has a geodesic running along the axis of symmetry, just as predicted by Theorem 5.1.

Remark 5.2. A rotational isomorphism between G and $G_\perp$ preserves the cyclical ordering of the boundary nodes of G, while the reflective isomorphism reverses them (See Figure 11).

Just as with the reflective isomorphism, we would like to generalize the results of this section to a stronger theorem. This is conjectured in Section 6

6 A Possible Generalization...

Conjecture 6.1. Suppose that a cyclic order of the boundary vertices is chosen. We conjecture that all self-dual graphs, whose duals share this cyclic ordering, may be obtained from theorem 4.1, i.e. they are simply rotations. Furthermore, we conjecture that all self-dual graphs, whose duals have a reversed cyclic ordering ordering may be obtained from theorem 5.1, i.e. they are simply reflections.

For graphs which are uniquely embedded in the unit disc, up to a reordering of the boundary vertices, the above result follows immediately. This is because the only isomorphisms which would preserve the ordering of boundary vertices are rotations,

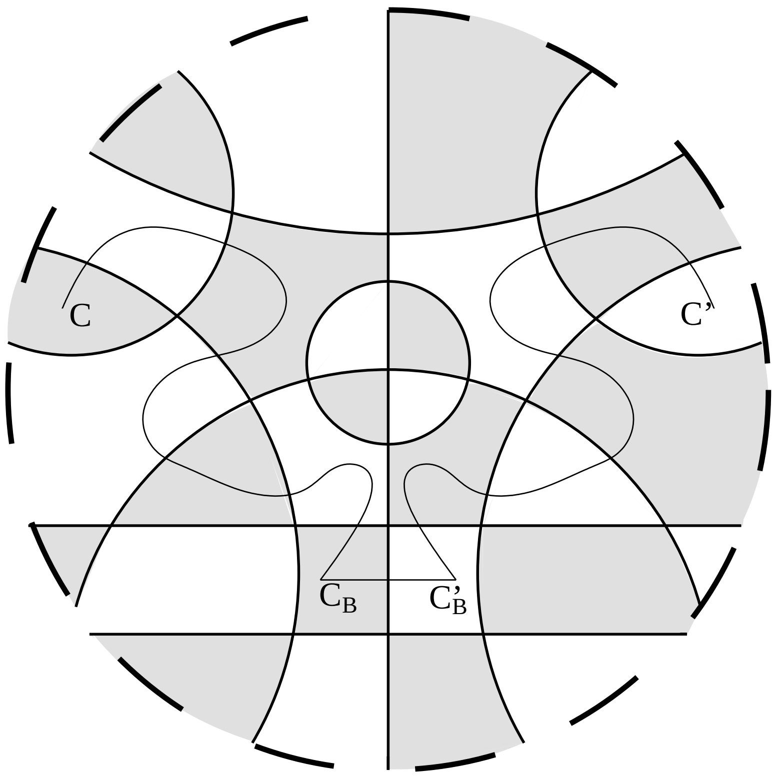

Figure 10: An illustration of the paths used in the proof of Theorem 5.1.

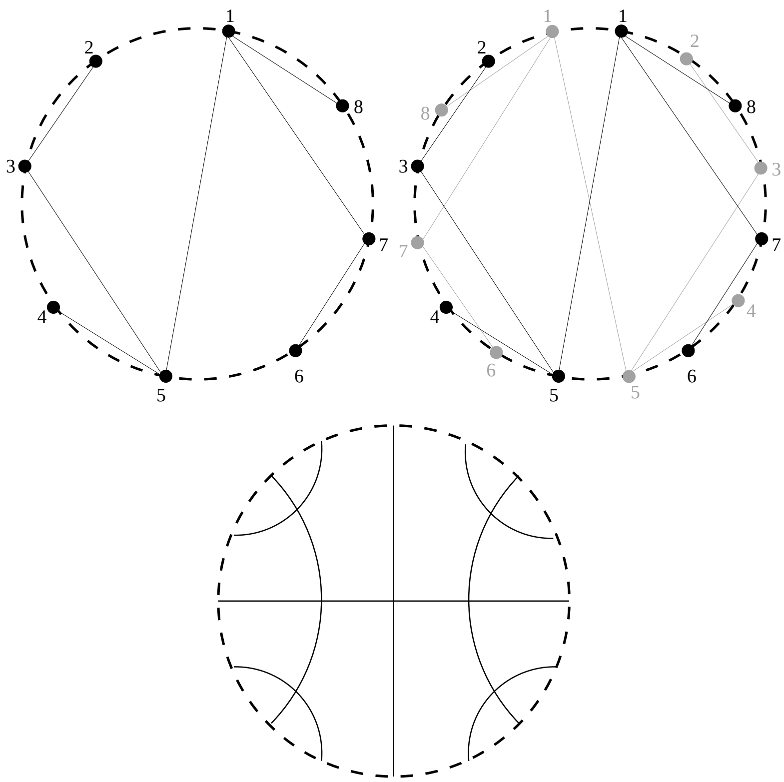

Figure 11: The basketball graph, the superposition of primal and dual graphs, and the associated medial graph.

and the only isomorphisms which would reverse it are reflections. However, we have as of yet not proved that for a general self-dual graph which preserves the order of boundary vertices, only global rotations and global reflections are possible. It has been recently conjectured that all critical graphs are uniquely embedded in the disc, up to a reordering of boundary vertices; if this is true, then all critical self-dual graphs must satisfy our above conjecture. Perhaps a proof could consider first the critical case, then the uniquely embedded case, and then the general case. In any case, a general study of self-dual graphs would likely be extremely beneficial to our knowledge of graph theory.

References

[1] Curtis, B., and James A. Morrow. "Inverse Problems for Electrical Networks." Series on applied mathematics – Vol. 13. World Scientific, ©2000.