3-TO-1 GRAPH PAPER OUTLINE

ILYA GRIGORIEV

Contents

Abstract. The discrete inverse boundary problem is the problem of recovering the values of the conductors of a resistor network from voltages and currents measured at the boundary vertices of the network. The inverse boundary problem for most networks has either one solution or infinitely many solutions. Some networks had also been known to have 2^n solutions for some n.

We study the question of whether these are the only possible values. To this end, we develop various methods of constructing resistor networks with specific relationships between the boundary information and the conductances. Using these methods, we construct a resistor network that has exactly three solutions to the inverse boundary problem, providing a counterexample and resolving the question.

The methods developed are also used to resolve another question: if negative values are allowed for the (formal) conductances, then how many solutions are possible? We show how to construct a resistor network for any given number of such solutions. Finally, we indicate how the solution of this problem might lead to additional insights on what other numbers of non-negative solutions might be admitted by networks, and discuss several open questions in the area.

1. Introduction

The discrete inverse boundary problem is the problem of recovering the values of the conductors of a resistor network from voltages and currents measured at the boundary vertices of the network. Whether the problem has a general solution depends on the graph that represents the way conductors are linked together in the network. Curtis and Morrow [1] heave solved the problem for a large class of "circular planar" graphs. They found, that for a circular planar graph, a geometric property of "criticality" determines whether the inverse problem has a unique solution for a set of boundary data or if the problem has infinitely many solutions.

The problem becomes far more difficult for graphs that are not circular planar. For instance, there is an example of a graph for which a set of boundary data determines exactly 2 possible solutions for the conductances of the edges. There are similar examples of graphs that have exactly 2^n solutions to the inverse boundary problem for any natural number n. If a general method of solving the inverse boundary problem for any graph exists, it must account for such graphs.

Rather than attempting to solve the difficult problem for an arbitrary graph, we have considered when it is possible to construct a graph such that its solution to the inverse boundary problem would have some properties we desire. A question that motivated this investigation was whether there is a graph with exactly 3 solutions to the inverse boundary problem. We developed methods that allowed us to construct such a graph. We have also made an attempt to construct a graph with n solutions to the inverse boundary problem, for any natural n.

2. Conductances and Responses

This will take two-three pages. I'll probably follow Russel's paper more or less.

Here will be the description of graphs, conductances, responses, Schur complements, the ugly eqautions that come from them.

Also, just enough of K-star stuff to make the next section understandable.

3. Ernie's triangle-in-triangle graph

I am not sure how much detail goes into here. This probably should follow Ernie's paper, or maybe Jenny's. I'd use the simplest explanation possible that convinces people that this is, in fact, a 2-1 graph and, ideally, that its solutions are nice and positive for reasonable boundary data.

4. How to construct a 3-1 graph

I think this can be pretty much the first four pages of my final REU paper (final.pdf) with minor modification. It explains how we can try to understand the triangle-in-triangle graph through multiplexers, and how we can use the multiplexers to construct a potential 3-1 graph. It would end at the cubic equation we have for the 3-1 graph and avoid discussion of what responses are valid, whether the conductances are real or complex, etc.

5. How to translate the solution of the equation to conductances

Here, I would discuss how to make precise what we somewhat hand-wavy did in the previous section. I am not sure whether this section should go here or a couple sections down (so that a reader does not have to be bogged down in details unless he wants to).

We have not checked that a solution to the equation for any particular set of boundary data actually corresponds to a valid conductance. So, some of the "solutions" we found may be fake. However, any actual conductances must correspond to a solution of the equation.

For "reasonable" sets of boundary data, we can make any solution of the equation all correspond to some conductances if we allow the conductances to be negative, complex, or infinite. We can call such weird conductances "generalized solutions to the inverse boundary problem".

All sets of boundary data obtained from conductances are "reasonable". I am not sure whether we want to go into more detail - this is possible but messy. Perhaps there can be an appendix about how to construct "reasonable" sets of boundary values from scratch.

6. Working with generalized solution to the inverse boundary problem

This would cover the same things as the second half of my REU final paper, but rewritten. It would talk about considering the equation we have as a function and using analysis to show that my first graph always has three real solutions.

I am still not quite sure whether I can prove that my first graph never has more than one positive solution. However, when I finally do that, it should go here.

I would say how to use inversions to construct graphs that could possibly have three real positive solutions. I'd say how the methods we have so far are not enough to conclude whether or not they actually do have three real, positive solutions.

7. The Region of All-Positiveness

This section would contain the gist of my second paper. We have an actual set of real, positive conductances, and we write our function $f(a)$ from the previous section. Let's say that the value $a = a_0$ corresponds to the conductances we started with. That is, a_0 is the solution to the equation $f(a) = \lambda$.

There must be a neighborhood U of a_0 such that any $a \in U$ corresponds to a set of real, positive conductances. If we make sure that the three solutions to $f(a) = \lambda$ are all in U, we have our three-to-one graph.

Then, I'll discuss the tricks to change the responses in such a way that they don't affect either a_0 or U, but nevertheless change $f(a)$ enough to make it have solutions to $f(a) = \lambda$ where we want.

So far, I do not know if there are always such tricks, I only know how to do this for the graph I was dealing with.

8. Conclusion

Here, it would be amazing to put something about how the methods of the paper are useful for something.

9. Credits

I'm not at all sure how the credits should work - there are just way too many people I'm relying on, and it's very difficult to separate what comes from whom.

References

[1] Curtis, Edward B., and James A. Morrow. "Inverse Problems for Electrical Networks." Series on applied mathematics – Vol. 13. World Scientific, ©2000.
[2] Jennifer French and Shen Pan. "2^n to 1 graphs." 2004
[3] Addington, Nicolas. "A Method for Recovering Arbitrary Graphs." 2003
[4] Esser, Ernie. "On Solving the Inverse Conductivity Problem for Annular Networks." 2000
[5] Russel, Jeffrey. "Star and K Solve the Inverse Problem." 2003.

E-mail address: `ilyag@uchicago.edu`