Using Special Functions to Determine Injectivity on Cubic Networks

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August 2006

Abstract

This paper looks at the map from γ to Λ in its bilinear form and utilizes it to determine injectivity of the differential of two cubic networks. A definition of special functions is offered and it is proved that if special functions exist for a given network, then that network is locally one-to-one. An algorithm for the construction of special functions on any cubic network is also provided. The unit cube and the 2-cube are both studied in detail and it is concluded that they are both locally one-to-one.

Contents

1 Introduction

In [1], Curtis and Morrow proved the existence of special functions on circular planar resistor networks and used them as part of a recovery algorithm for wellconnected critical graphs. Special functions were also used by Giansiracusa in [2] to partially recover non-critical circular planar networks. This paper takes a different approach, constructing special functions for highly non-planar cubic networks and using them to prove injectivity on the map of the differential. The motivation behind this approach is that an injective differential map corresponds to a graph that is locally one-to-one. If it turns out that a cubic network is locally one-to-one, then it will be worthwhile to look for global one-to-oneness and to try to prove recoverability.

1.1 Introduction to Cubic Networks

We consider a cubic network (also termed a three-dimensional square lattice) as in Figure 1. Each point of intersection between two or more line segments, or at the end of a line segment, is called a *node*. The set of nodes is denoted Ω_0. The *interior* of Ω_0, called $int\Omega_0$ consists of those nodes which are the vertices of the cube. Each interior node p has six neighboring nodes. The *boundary* of Ω_0, called $\partial\Omega_0$, is $\Omega_0 - int\Omega_0$. Each boundary node p has only one neighboring node, which is an interior node. An *edge pq* is a line segment which connects a pair of neighboring nodes p and q. The set of edges is denoted Ω_1. An edge pq where p is a boundary node and q is its neighboring interior node will be aclled a boundary spike. A cubic network of this type consists of a network $\Gamma = (G, \gamma)$ where $G = (\Omega_0, \Omega_1)$ and γ is a postive real-valued conductivity function on Ω_1. Further information on the properties of cubic networks can be found in [3] and [4].

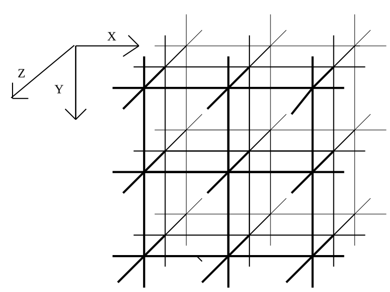

Figure 1: A 2 x 2 x 2 cubic network

1.2 The Response Matrix of a Cubic Network

The following conventions will be used to describe cubic networks:

- The boundary nodes on the right-hand face are called East nodes, or simply E.
- The bounary nodes on the bottom face are called South nodes, or S.

- The boundary nodes on the left-hand face are called West nodes, or W.
- The boundary nodes on the top face are called North nodes, or N.
- The boundary nodes on the face coming out of the xy-plane in the positive z direction, are called Front nodes, or F.
- The bounadry nodes on the face leaving the xy-plane in the negative z direction, are called Back nodes, or B.

Given this labeling system, the response matrix of a cubic network has the block form shown below:

$$\Lambda_\gamma = \begin{array}{c|c|c|c|c|} & E & S & W & N \\ \hline E & \Lambda(E;E) & \Lambda(E;S) & \Lambda(E;W) & \Lambda(E;N) \\ \hline S & \Lambda(S;E) & \Lambda(S;S) & \Lambda(S;W) & \Lambda(S;N) \\ \hline W & \Lambda(W;E) & \Lambda(W;S) & \Lambda(W;W) & \Lambda(W;N) \\ \hline N & \Lambda(N;E) & \Lambda(N;S) & \Lambda(N;W) & \Lambda(N;N) \\ \hline \end{array}$$

In the case of the unit cube, E stands for the four indices corresponding to the nodes on the E face, S stands for the four indices corresponding to the nodes on the S face, etc. Thus, the block $\Lambda(W;N)$ at position (W,N) is a 4 x 4 matrix which gives current on the W face due to boundary values imposed on the N face. A similar block structure will be used for vectors of voltages and vectors of currents. For example, u_N stands for a vector of four values which are boundary voltages on the N face. The block form of Λ_γ leads to the following theorem, which will be used in §3.1 to construct special functions.

Theorem 1.1 *Let A and B be two sets of boundary nodes corresponding to opposite faces on an n x n x n network Γ, i.e., let (A,B) be equal to (N,S), (E,W), or (F,B). Then the submatrix $\Lambda(A;B)$ of Λ_γ is non-singular.*

Proof: Each face A,B on Γ has $(n+1)^2$ boundary nodes. Let $k=(n+1)^2$. Since there is a k-connection from A to B, $det\Lambda(A;B) \neq 0$. So $\Lambda(A;B)$ is non-singular. □

2 The Differential as a Bilinear Form

Consider an electical network $\Gamma=(G,\gamma)$, where G is a graph with a boundary and γ is a conductivity function defined on the edges of G. For each conductivity function γ on G, let Λ_γ be the response matrix. Let $T:(\mathbb{R}^+)^N \to \mathbb{R}^{n^2}$, where N is the number of edges in G and n is the number of boundary nodes, be the map which sends γ to Λ_γ. §4.6 of [1] computes the differential of T and shows that since a matrix can be identified with a bilinear form, the dT can be thought of as a map that takes vectors to the space of bilinear forms. That is,

$$dT(K)(x,y)=\sum_{p\sim q} K_{pq}(f(p)-f(q))(g(p)-g(q)), \tag{1}$$

where f and g are solutions to the Dirichlet problem with boundary conditions x and y and satisfy the equations

$$f = \begin{bmatrix} I \\ -C^{-1}B^T \end{bmatrix} x$$

and

$$g = \begin{bmatrix} I \\ -C^{-1}B^T \end{bmatrix} y$$

and K is some direction. Here, f depends linearly on x and g depends linearly on y. The bilinear form of the differential makes injectivity easy to detect. It is clear that dT is injective if and only if

$$\sum_{p\sim q} K_{pq}(f(p) - f(q))(g(p) - g(q)) = 0 \qquad (2)$$

$\forall$ f,g implies that

$$K_{pq} = 0. \qquad (3)$$

We will use these equations to prove injectivity of the differential after we construct some special functions.

3 Special Functions

Let$\Gamma = (G, \gamma)$ be an electrical notwork and let u be a γ-harmonic function on Γ. *Special functions* are sets of γ-harmonic functions f and g that are obtained by imposing conditions on u, some of which are boundary values and some of which are boundary currents. In [2], Giansiracusa constructs special functions for circular planar networks which satisfy the condition that for each edge with endpoints (p, q),

$$(f(p) - f(q))(g(p) - g(q) = 0 \text{ if } pq \neq e$$

$$(f(p) - f(q))(g(p) - g(q)) \neq 0 \text{ if } pq = e,$$

where e is either a boundary spike or a boundary edge. Using these functions he showed that if $\sum_{p\sim q} K_{pq}\Delta f \Delta g = 0$, then $K_{pq} = 0$ on the boundary spikes and edges of a critical circular planar network. After deleting boundary edges and contracting boundary spikes, he found new sets of special functions for the network, and was able to repeat the procedure until all of the edges of the network were accounted for.

In the case of the cubic network, we want to find functions that will account for all edges pq without any contraction or deletion. Thus, we will use a slightly more general definition for special functions.

Definition 3.1 Let $\Gamma = (G, \gamma)$ be a three-dimensional square lattice network. *Special functions* are any functions f and g which are γ-harmonic and satisfy the condition that for only one edge with endpoints $(p, q) = (p_0, q_0)$

$$(f(p) - (f(q))(g(p) - g(q)) \neq 0 \tag{4}$$

and that for all other edges with endponts $(p, q) \neq (p_0, q_0)$

$$(f(p) - f(q))(g(p) - g(q)) = 0. \tag{5}$$

Note: It might happen that for some pair of functions f_1 and g_1 $(f(p) - f(q))(g(p) - g(q)) \neq 0$ for more than one edge pq of G. If all but one of these edges have already been isolated by other sets fo special functions f and g, then f_1 and g_1 are still special functions. In other words, we can use several sets of special functions in conjuction with each other to isolate and determine injectivity on all the edges of a graph.

3.1 Building Special Functions on Cubic Networks

We construct special functions using the method set forth for rectangular graphs in §4.2 of [1]. The general algorithm for n x n x n network is as follows:

1. First, select three faces on the cube. Two faces, call them A and B, must be opposite, that is, there must exist a k-connection from the bounadry nodes of one to the boundary nodes of another, where $k = (n+1)^2$. The third face, call it C, is selected arbitarily.
2. Put some non-zero voltage on a single boundary node on C, call that node C_1.
3. Put zero current flow on A.
4. Put zero voltage on all the remaining boundary nodes, excluding B and C_1.
5. Use harmonic continuation to determine current flow.

Lemma 3.1 *If a particular pattern is obtained by following the preceding algorithm, then there is a γ-harmonic function f which produces that current pattern.*

Proof: Let the value of the current at all nodes on the network be $\psi = \Lambda y$, where y is the function that gives the boundary values imposed on the network. If the above algorithm is followed, then the current on the A face is $\psi_A = \Lambda(A; C)y_C$. Since Theorem 1.1 implies that $\Lambda(A; B)$ is non-singular, there is a unique solution $x_B = [x_1, x_2, ..., x_k]$ to the k x k linear system $\Lambda(A; B)x_B + \psi_A = 0$. $\square$

In other words, we can choose boundary voltages on one face to produce zero current on the opposite face, so current patterns obtained by way of the algorithm are legitimate flows and they must be produced by real γ-harmonic functions.

We still need another function g to make a pair of special functions. We construct g by repeating the algorithm until it produces a current pattern that overlaps the first current pattern f on a single edge pq of the network. Therefore, f and g satisfy (4) on this single edge and (5) on all other edges and they are special functions.

Theorem 3.1 *If there exist special functions f and g such that $(f(p)-f(q))(g(p)-g(q)) \neq 0$ on some edge with endpoints $(p,q) = (p_0, q_0)$, then that edge is locally one-to-one.*

Proof: Let $\sum_{p\sim q} K_{pq}(f(p)-f(q))(g(p)-g(q)) = 0$. Since $(f(p_0)-f(q_0))(g(p_0)-g(q0)) \neq 0$, it must be that $K_{p_0q_0} = 0$. Therefore, the derivative of the map from γ to Λ is injective when $K = K_{p_0q_0} = 0$ and p_0q_0 is locally one-to-one. □

It is clear that if a network is locally one-to-one, there will be a set of special functions for every edge pq. With these sets of special functions one can determine injectivity of the differential at every conductance γ in every direction K.

4 The Unit Cube

We begin by building special functions for the unit cube. We hope that by beginning with the simplest example of a three-dimensional square lattice, we will find patterns that can be extended to larger cubic networks. The special function, call it f, was constructed by selecting the N, E, and W faces. Non-zero voltage was placed on a boundary spike in N and zero voltage everywhere else except E. Zero current flow was imposed on W. The resulting current pattern is shown on the cube on the left in Figure 2. Notice that non-zero voltage was imposed on a spike near the W face and the zero current flow. If the non-zero voltage is placed instead on a spike closer to the E face, then the resultant current flow is the much simpler pattern shown on the cube on the right in Figure 2. It turns out that this is a trend which holds for larger cubic networks: As we place non-zero voltage nearer to the face with zero current flow, the pattern of the consequent current becomes larger and increasingly complex.

Now we need another γ-harmonic function g to make a set of special functions for an edge in the unit cube. Instead of constructing another function by way of the algorithm in §3.1, we observe that any rotation of f is also a special function, due to the symmetry of the cube. For example, the current pattern on righthand cube in Figure 3 can be achieved on any pair of corner boundary spikes, simply by altering the boundary data. Figure 4 shows how changing the boundary data produces the same current pattern in a different location.

Since any rotation of f is also a special function, it becomes a simple task to look for a rotation that, when superimposed on f, only ovverlaps on a single edge pq. We find two suitable rotations of f on the unit cube and call them

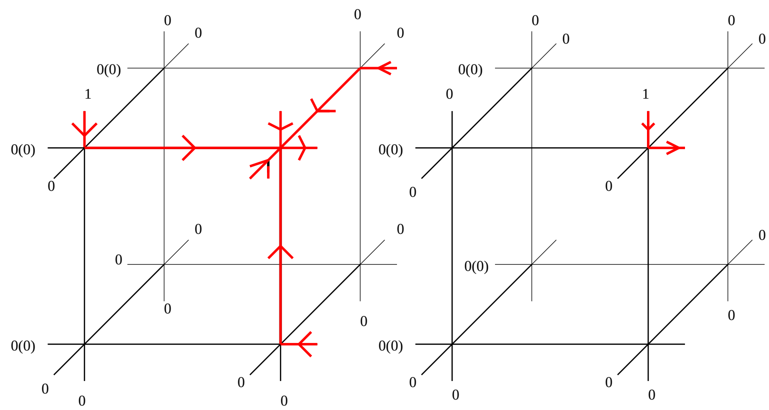

Figure 2: Two different current patterns caused by special functions on the unit cube. The pattern on the left is more complicated because the non-zero voltage is imposed on a spike nearer to the zero-current flow.

g and h (See Figure 4). It is clear that when pq is a boundary spike $(f(p) - f(q))(g(p) - g(q)) \neq 0$ and when pq is an edge between interior nodes $(g(p) - g(q))(h(p) - h(q)) \neq 0$. Again due to the symmetry of the cube, all of the boundary spikes are equivalent in that if one is accounted for by a set of special functions, then a rotation of those special functions will take care of the other boundary spikes. The same is true for the edges between interior nodes. Thus, the two sets of special functions f,g and g,h, together with Theorem 3.1, are sufficient to prove that the unit cube is locally one-to-one.

Note: Although our examples show a unit cube with three boundary spikes on each corner, one spike can be removed from each corner without affecting injectivity of the differential. In fact, the same special functions found for the cube with three corner spikes work for the cube with only two boundary spikes.

It is of note to say that, when drawn in the plane, the 1 x 1 x 1 cubic network is equivalent to the annular network consisting of two circles and four rays (Figure 5). It has been shown that this network is recoverable, however, we have had trouble demonstrating that the differential is injective. If we attach an additional boundary spike to each of the boundary nodes, so that it is analogous to the cube with sixteen boundary nodes, it becomes possible to use the same special functions f,g, and h to prove that the two circle four ray network is locally one-to-one.

5 The 2-Cube

Our next step was to look at the 2 x 2 x 2 square lattice. In the case of the unit cube, there were two types of edges pq, so only two sets of special functions were necessary to determine injectivity of the differential. The two cube, on the

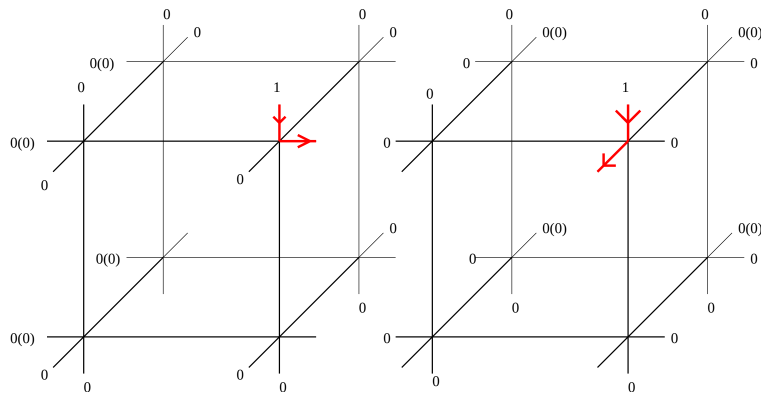

Figure 3: Two current patterns on the unit cube. The pattern on the right is a simple rotation of the pattern on the left is obtained by placing zero current flow on B instead of W.

other hand, has six types, of edges, which are listed below and labeled as in Figure 6:

- Boundary spikes on the corner of the cube are labeled a and are called corner spikes.
- Boundary spikes in the middle of an edge on the cube are labeled b and are called middle spikes.
- Boundary spikes in the very center of a face on the cube are labeled c and are called center spikes.
- Edges between two interior nodes and adjacent to corner spikes are labeled d and are called exterior edges.
- Edges between two interior nodes and adjacent to middle spikes are labeled e and are called middle edges.
- Edges between two interior nodes and adjacent to center spikes are labeled i and are called interior edges.

We find six pairs of special functions to prove that this network is locally one-to-one, one pair for each type of edge. These pairs consist of various rotations of four basic types of special functions. The first type (f) is obtained by placing the non-zero voltage on a corner spike as far away as possible from the face with zero current flow. The second type (g) is obtained by placing the non-zero voltage on a middle spike far away from the face with zero current flow. The current pattern resulting from the third type (h) of special function is a bit more complex, because the non-zero voltage is placed on a center spike, and is thus nearer to the zero current flow. The pattern caused by the fourth type (j)

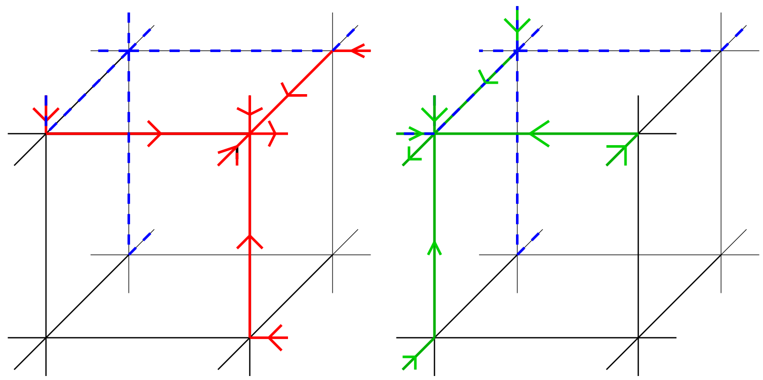

Figure 4: The figure on the left is a unit cube with the current flows due to special functions f and g overlapping on a boundary spike. The figure on the right is a unit cube with the current flows due to special functions g and h overlapping on an interior edge.

of special function is very complex and spreads over nearly half of the graph. It is obtained by placing the non-zero voltage at a middle spike on the face with zero current flow. These four flow types and their associated boundary data are shown in Figure 7. Rotations of f and h overlap on a, two rotations of h overlap on c, and rotations of g and j overlap on b, thus taking care of all the boundary spikes. Rotations of h and j overlap on both d and e and two rotations of j overlap on i. Figures 8 and 9 show different rotations of these current flows working as special functions on every edge pq of the 2-cube. Since there are special functions for every edge pq on the 2-cube, Theorem 3.1 tells us that the network is locally one-to-one.

6 Future Work

The work presented in this paper can be extended in several directions. It seems natural that, since we have proved the 2-cube to be locally one-to-one, we try to recover it in its entirety. It is likely that the special functions found here will aid in its recovery. If the 2-cube does prove to be recoverable, it will likely be difficult to generalize the method of recovery to larger cubics. We simply do not have enough information about larger networks. However, it is possible that, in recovering larger networks, patterns will be found which enlighten us about the properties of non-planar networks.

Another interesting idea for future research is to look at networks of higher dimensions. We suspect that the four-dimensional square lattice is locally one-to-one and found a special function for the network, unfortunately, it was difficult to visualize and we realized that our method of finding more special functions by rotation would not be very useful. One suggestion for solving the

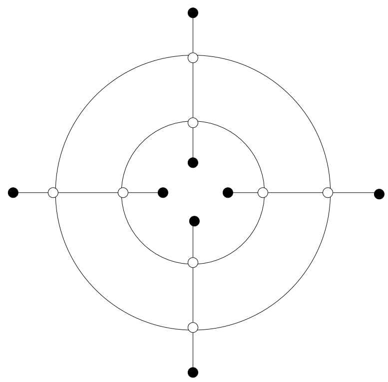

Figure 5: The 2 circle 4 ray annular network

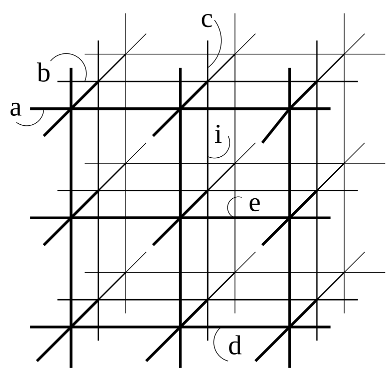

Figure 6: The different edges pq on a 2-cube

problem of visualization is to implement a bookkeeping system that would involve labeling of all of the nodes on a graph with coordinates and then keeping track of the current as it flows between neighboring nodes. Transitioning between a three-dimensional lattice and a four-dimensional lattice would not be too difficult: each node would have a fourth coordinate and a new neighbor. Of course, with objects of higher dimensions, the number of nodes quickly becomes enormous. It seems that a program might be needed to keep track of all the nodes. A program could also be written to determine all possible current flows on a network using the algorithm given in this paper. By looking at all the possible flow patterns, one could more easily see where two overlap. If we had the appropriate code, we would be able to construct special functions for networks of n-dimensions and determine whether or not they are locally one-to-one.

Furthermore, while constructing special functions is useful for understanding the differential of a network and determining edges to be locally one-to-one, it

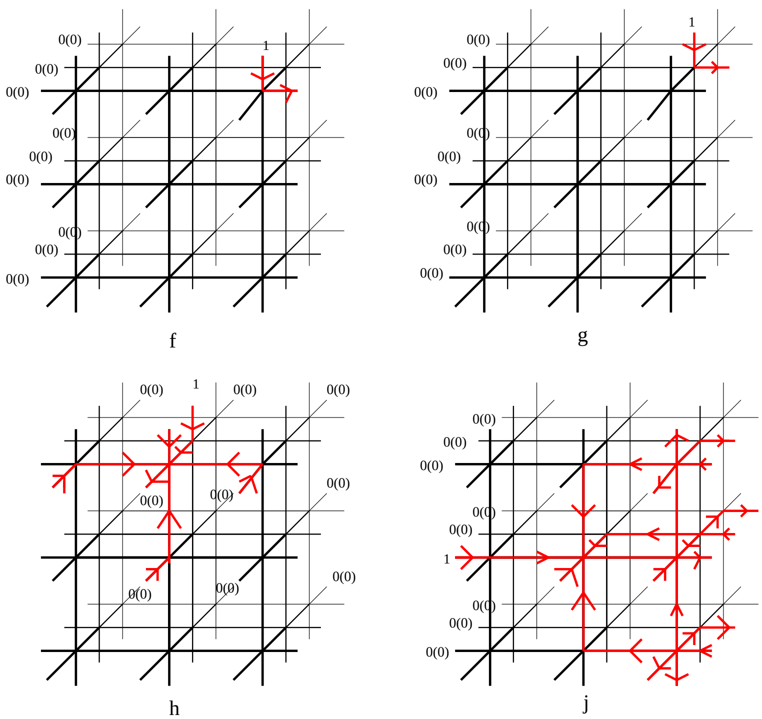

Figure 7: Flows resulting from special functions of type f, g, h, and j

is not necessarily enlightening about the best way to recover a network. There is still much research to be done on the properties non-planar networks before we can fully understand the conditions necessary for recoverability.

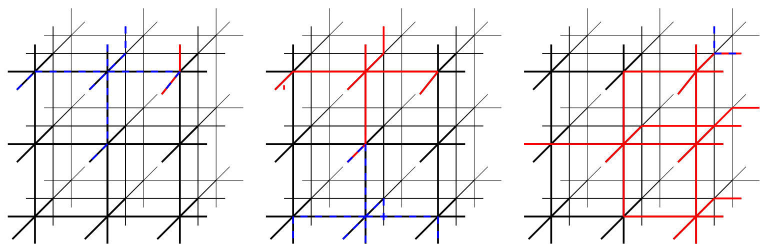

Figure 8: Special functions on edges a,b, and c

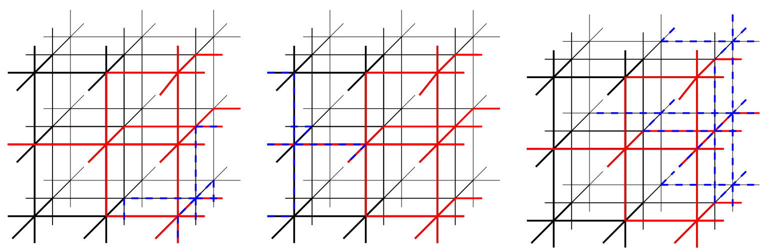

Figure 9: Special functions on edges d,e, and i

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