★★★★★ MULTIPLEXERS!

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Abstract. In this paper, we will talk about n-plexers, an object crucial in the construction of an n to 1 graph. We will talk mostly about m-plexers on n-stars and try to classify what structures are possible.

1. Definitions and Basic Methods

Definition 1.1. A *type I multiplexer* is an ordered pair $P = (G, \Pi)$, where G is a recoverable graph with boundary and $\Pi = (\Pi^U, \Pi^K)$ is a partition of the set of all distinct unordered pairs of boundary vertices into two sets, called the *unknown set* and the *known set*. We call an element of the unknown set an unknown pair, and an element of the known set a known pair. We require that P has the following properties.

(1) For any valid response matrix on G, if we only know the response matrix entries corresponding to known pairs, we cannot determine the value of any entry corresponding to an unknown pair.
(2) For any valid response matrix on G, if we know only the response matrix entries corresponding to known pairs and one unknown pair, we can recover the response matrix for all choices of the unknown pair.

If $|\Pi^U| = n$, we will call this a type I n-plexer.

Definition 1.2. An n-star, denoted $\bigstar_n$, is a graph with boundary with one interior node and n boundary nodes, with one edge connecting each boundary node to the interior node and no other connections

Definition 1.3. The complete graph on n vertices is a graph with boundary with n boundary nodes, no interior nodes, and an edge connecting any two boundary nodes. It will be denoted K_n

Definition 1.4. By the quadrilateral $[v_1, v_2, v_3, v_4]$, we mean the set of edges $[v_1, v_2], [v_2, v_3], [v_3, v_4], [v_1, v_4]$. We will also sometimes refer to this in matrix language, referring to a quadrilateral as a 2×2 submatrix. It will always be clear from context which of these two we mean.

In the special case where we are looking for an m-plexer on $\bigstar_n$, we can view the definition of a multiplexer in a more visual way, which will sometimes be useful for us.

First, we note that a choice of a pair of boundary vertices of $\bigstar_n$ is equivalent to the choice of an edge of K_n, which is equivalent to a choice of an entry of the response matrix. As such, we will make references to unknown entries of the response matrix and unknown edges. Second, we note that $\bigstar_n$ has one interior node, so that as in Nick Addington's paper, the problem of recovering a Kirchhoff matrix on $\bigstar_n$ from a response matrix on K_n is equivalent to recovering the Kirchhoff matrix from the R matrix of K_n instead, where the R matrix is defined by $R = \Lambda - A$. The benefit of this is discussed in Nick's paper:

Theorem 1.5. *Any 2×2 sub-matrix of the R matrix of K_n has zero determinant.*

Proof. The proof is in Nick's paper. Basically, it can be shown that the R matrix is rank 1, and therefore has the desired property. $\square$

What this means for us is that when are checking to see what entries are determined in the R matrix by our known pieces of information, we only need to look at 2×2 determinants. This is a very useful simplification which we use very frequently, so it should be noted that it is only true if Nick Addington's algorithm is a complete characterization of recovering graphs with one interior node. For the duration of this paper, this result will be assumed.

In the special case where we have an off-diagonal 2×2 sub-matrix, Theorem 1.5 takes the following form:

$$\lambda_{ij}\lambda_{kl} = \lambda_{ik}\lambda_{jl} \quad \text{for all pairwise distinct } i, j, k, l \tag{1}$$

This condition is called the quadrilateral condition because it says that if we form any quadrilateral in K_n, the products of the weights on opposite sides are equal.

Another special case that is important is called the square root trick. The square root trick makes use of the symmetry of the R matrix to recover an entry only from two diagonal entries. We have the following equation, which follows directly from the above determinant relations and the symmetry of the R matrix:

$$\lambda_{ii}\lambda_{jj} = \lambda_{ji}\lambda_{ij} = \lambda_{ij}^2 \tag{2}$$

Taking the square root, we get an expression for λ_{ij}:

$$\lambda_{ij} = \sqrt{\lambda_{ii}\lambda_{jj}} \tag{3}$$

This equation gives us the following useful theorem.

Theorem 1.6. *If we have an $n \times n$ R matrix, knowing the diagonal elements is sufficient to recover the entire matrix.*

Proof. This result is directly proved by equation (3) $\square$

We now develop some basic tools for proving properties of multiplexers. We will mainly focus of what sort of structure is forced upon us by Definition 1.1. Also, it should be noted that we will be using both matrices and determinants and quadrilaterals on K_n to prove properties of multiplexers, so the reader should realize that anything stated for one of these two settings holds true in the other.

Theorem 1.7. *When attempting to construct an m-plexer on $\bigstar_n$, the following conditions must be met:*

(1) *If we choose an unknown entry in the response matrix, then any 2×2 sub-matrix containing that entry must have another unknown entry in it. Equivalently, if we have a choose an unknown edge in K_n, any quadrilateral containing that edge must have another unknown edge.*
(2) *If we have a 2×2 sub-matrix with three known entries, the fourth entry is also a known entry.*
(3) *If we have a type I m-plexer $(\bigstar_n, \Pi)$, then there does not exist a k-plexer $(\bigstar_n, \Phi)$, $k < m$, such that $\Phi^U \subset \Pi^U$.*
(4) *If we have an m-plexer $(\bigstar_n, \Pi)$, then there does not exist a type I k-plexer $(\bigstar_n, \Phi)$, $k > m$, such that $\Pi^U \subset \Phi^U$.*

Proof. The proofs of (1) and (2) are basic applications of Definition 1.1 and are left to the reader to verify. Also, we only need prove one of (3) and (4), since the two statements are obviously equivalent. We choose to prove (3). Suppose we do have $\Phi^U \subset \Pi^U$. We show that this leads to a contradiction. Since $k < m$, we know there exists a pair $[i, j]$ such that $[i, j] \in \Pi^U$, but $[i, j] \notin \Phi^U$. According to condition (2) of Definition 1.1, if we specify the value of the pair $[i, j]$, we determine the values of all other elements of Π^U, which in particular specifies all the values of Φ^U. We know this because $(\bigstar_n, \Pi)$ is a type I plexer. But Φ^U is the unknown set of a multiplexer; therefore condition (1) of Definition 1.1 says that we cannot determine any one of the unknown pairs only from known pairs. This contradiction completes the proof. $\square$

Later in the paper, we will give some general constructions for type I multiplexers. However, it will often be convenient for us to assume that $n \geq 5$ in these constructions. As such, we take a few moments here to show how these constructions work on $\bigstar_3$ and $\bigstar_4$.

We first do the case $\bigstar_3$. This case turns out to be very easy, since we have that picking any unknown element forms a type I multiplexer. This is easy to see from the R matrix:

$$\begin{pmatrix} \bullet & \times & \checkmark \\ \times & \bullet & \checkmark \\ \checkmark & \checkmark & \times \end{pmatrix}$$

We clearly see that our single unknown entry is not recoverable, and incidentally, that no diagonal entries are recoverable either. We will call this partition $\Pi_{1\oplus 2}$ so that it agrees with later notation. Also, we know that there are no other type I multiplexers on $\bigstar_3$, since any other partition Π would have $\Pi^U_{1\oplus 2} \subset \Pi^U$, which means by Theorem 1.7 that $\Pi = \Pi_{1\oplus 2}$.

We now do the case $\bigstar_4$. We will see by Theorem 3.5 that we only need to look at partitions with $|\Pi^U| \leq 3$. We now show the R matrices of all of the type I multiplexers on $\bigstar_4$:

$$\begin{pmatrix} \bullet & \times & \checkmark & \checkmark \\ \times & \bullet & \checkmark & \checkmark \\ \checkmark & \checkmark & \bullet & \times \\ \checkmark & \checkmark & \times & \bullet \end{pmatrix} \quad \begin{pmatrix} \bullet & \times & \times & \times \\ \times & \bullet & \checkmark & \checkmark \\ \times & \checkmark & \bullet & \checkmark \\ \times & \checkmark & \checkmark & \bullet \end{pmatrix} \quad \begin{pmatrix} \bullet & \times & \times & \checkmark \\ \times & \bullet & \times & \checkmark \\ \times & \times & \bullet & \checkmark \\ \checkmark & \checkmark & \checkmark & \bullet \end{pmatrix}$$

We will call the partitions of these $\Pi_{2\oplus 2}$, $\Pi_{1,3}$, and $\Pi_{1\oplus 3}$ respectively. We now give a brief argument that these are the only type I multiplexers on $\bigstar_4$. We can assume without loss of generality that λ_{12} is an unknown entry. There are three cases we have to check. The first case is λ_{13} and λ_{14} are known, the second case is λ_{13} is unknown and λ_{14} is known, and the last case is both λ_{13} and λ_{14} are unknown. Clearly, the last case can only give us $\Pi_{1,3}$. We check the case where both λ_{13} and λ_{14} are known. Our matrix looks like this:

$$\begin{pmatrix} \bullet & \times & \checkmark & \checkmark \\ \times & \bullet & & \\ \checkmark & & \bullet & \\ \checkmark & & & \bullet \end{pmatrix}$$

Clearly, we have either λ_{23} and λ_{24} are either both known or both unknown, from Theorem 1.7. Therefore the only way to add exactly one unknown entry is to add λ_{34}, which gives us $\Pi_{2\oplus 2}$. Also, we see that if we make both λ_{23} and λ_{24} unknown, then we can switch vertex 1 with vertex 2 in order to get $\Pi_{1,3}$.

Now the only case remaining is the case where λ_{13} is unknown and λ_{14} is known. Now we have that our matrix looks as follows:

$$\begin{pmatrix} \bullet & \times & \times & \checkmark \\ \times & \bullet & & \\ \times & & \bullet & \\ \checkmark & & & \bullet \end{pmatrix}$$

Due to our upper bound, we can only add one other unknown element if we want to get a multiplexer. But because of Theorem 1.7 and from looking at quadrilaterals in the matrix that we must have one of λ_{23} or λ_{24} as unknown, as well as one of λ_{23} and λ_{43} as unknown. The only way to only add one unknown entry and satisfy these is to make λ_{23} unknown, which gives us $\Pi_{1\oplus 3}$, as required. Therefore, on $\bigstar_4$, we can only have $\Pi_{1,3}$, $\Pi_{1\oplus 3}$, or $\Pi_{2\oplus 2}$.

Definition 1.8. We say a type I multiplexer is *type Ia* if it initially has no diagonal entries recoverable in the R matrix, and that it is *type Ib* if it initially has some diagonal entries recoverable in the R matrix.

Using these definitions, we see clearly that $(\bigstar_3, \Pi_{1\oplus 2}$, $(\bigstar_4, \Pi_{1\oplus 3})$, and $(\bigstar_4, \Pi_{2\oplus 2})$ are type Ia multiplexers, whereas $(\bigstar_4, \Pi_{1,3}$ is a type Ib multiplexer.

2. Examples of Multiplexers

The goal of this section will be to provide a very large class of type I multiplexers on n-stars. In fact, we will construct all of the type I multiplexers, although we will not prove this characterization until later. We start this section by deriving a large class of type I multiplexers on $\bigstar_n$.

We describe a partition of the set of all distinct unordered pairs of boundary vertices of $\bigstar_n$. Let k be an integer. We take the partition $\Pi_{k\oplus(n-k)}$ to have $\Pi^U_{k\oplus(n-k)} = K_k \sqcup K_{n-k}$. We then have $\Pi^K_{k\oplus(n-k)}$ as all other pairs of boundary vertices, forming a complete bipartite graph $K_{k,n-k}$. This can be phrased formally as $\Pi^U_{k\oplus(n-k)} = \{[x, y]$, where either $x \le k$ and $y \le k$ or $x > k$ and $y > k\}$. We have a very nice representation of this in block matrix form as follows, where an X corresponds to an unknown block and a ✓ corresponds to a known block:

$$\begin{pmatrix} \times & \checkmark \\ \checkmark & \times \end{pmatrix}$$

By convention, the diagonals are understood to be always unknown, regardless of the block they appear in.

Theorem 2.1. *$\left(\bigstar_n, \Pi_{k\oplus(n-k)}\right)$ is a type Ia multiplexer.*

Proof. That $\bigstar_n$ is recoverable is obvious. We now show properties (1) and (2) of Definition 1.1 hold. We know that the entries of $\Pi^K_{k\oplus(n-k)}$ have the form $a \le k$ and $b > k$ or $a > k$ and $b \le k$. This corresponds to a rectangle of known information in the top right corner of our R matrix and in the bottom left corner of our R matrix. For example, if $n = 8, k = 5$ our matrix would have the following form, where bullets mark the diagonal and x marks an unknown entry.

$$\left(\begin{array}{ccccc|ccc}
\bullet & \times & \times & \times & \times & \lambda_{16} & \lambda_{17} & \lambda_{18} \\
\times & \bullet & \times & \times & \times & \lambda_{26} & \lambda_{27} & \lambda_{28} \\
\times & \times & \bullet & \times & (\times) & \lambda_{36} & (\lambda_{37}) & \lambda_{38} \\
\times & \times & \times & \bullet & \times & \lambda_{46} & \lambda_{47} & \lambda_{48} \\
\times & \times & \times & \times & \bullet & \lambda_{56} & \lambda_{57} & \lambda_{58} \\
\hline
\lambda_{16} & \lambda_{26} & \lambda_{36} & \lambda_{46} & (\lambda_{56}) & \bullet & (\times) & \times \\
\lambda_{17} & \lambda_{27} & \lambda_{37} & \lambda_{47} & \lambda_{57} & \times & \bullet & \times \\
\lambda_{18} & \lambda_{28} & \lambda_{38} & \lambda_{48} & \lambda_{58} & \times & \times & \bullet
\end{array}\right)$$

We verify condition (1). As previously remarked, since we are dealing with $\bigstar_n$, we only need to look at determinants of 2×2 sub-matrices. In particular, we can only recover an unknown entry if it is in a 2×2 sub-matrix with exactly three known entries. But if we have three known entries in one 2×2 sub-matrix, we clearly must also have the fourth known, since our known entries appear in rectangular blocks. This verifies condition (1). Additionally, the same argument shows that none of the diagonals can initially be recovered either. condition (2) is also easy to verify. Assume we specify the value of some unknown entry, λ_{ab}, where without loss of generality we assume $a \leq k$ and $b \leq k$. We use this to recover all entries of the form λ_{ij}, where $i > n-k$ and $j > n-k$. We can do this using the relation $\lambda_{ab}\lambda_{ij} = \lambda_{aj}\lambda_{ib}$ where for any pair i and j so that $i > k$ and $j > k$, λ_{aj} and λ_{ib} are known entries. A similar argument shows that specifying λ_{ab} where $a > k$ and $b > k$ gives all entries of the form λ_{ij} where $i \leq k$ and $j \leq k$. This shows that condition (2) is satisfied, and hence we have that $\left(\bigstar_n, \Pi_{k\oplus(n-k)}\right)$ is a type Ia multiplexer. □

The diagonal elements of this multiplexer satisfy another useful property, which we will now show.

Theorem 2.2. *If we have $(\bigstar_n, \Pi_{k\oplus(n-k)})$, then specifying the value of any diagonal element specifies the entire R matrix.*

Proof. We will prove this by showing that specifying the value of any diagonal element specifies the value of some unknown entry, which by condition (2) of being a multiplexer gives that the entire matrix is specified. First let $1 \leq i \leq k-1$. Then we can use the relation $\lambda_{ii}\lambda_{(i+1)n} = \lambda_{in}\lambda_{(i+1)i}$ to recover $\lambda_{(i+1)i}$, where since $i \leq k-1$, $i+1 \leq k$ so that λ_{in} and $\lambda_{(i+1)n}$ are both known. Next, we note that if $i = k$, the relation $\lambda_{kk}\lambda_{(k-1)n} = \lambda_{kn}\lambda_{(k-1)k}$ to recover $\lambda_{(k-1)k}$. Also, if $k+1 \leq i \leq n-1$, we can use the relation $\lambda_{ii}\lambda_{(i+1)1} = \lambda_{i1}\lambda_{(i+1)i}$ to recover $\lambda_{(i+1)i}$. Finally, if $i = n$, we can use the relation $\lambda_{nn}\lambda_{(n-1)1} = \lambda_{(n-1)n}\lambda_{n1}$ to recover $\lambda_{(n-1)n}$. Therefore, specifying any diagonal element specifies the entire matrix. □

Corollary 2.3. *$\bigstar_n$ always contains a multiplexer of size $\frac{(2k^2+n^2-2kn-n)}{2}$, where $1 \leq k < n$.*

Proof. By Theorem 2.1, $(\bigstar_n, \Pi_{k\oplus(n-k)})$ is a multiplexer, for all $1 \leq k < n$ where $|\Pi^U_{k\oplus(n-k)}|$ is easily computed as $\frac{(2k^2+n^2-2kn-n)}{2}$ □

For this class of multiplexers, we can consider their size as a function of k, say f. Clearly f is differentiable, since f is a polynomial, and its derivative is easily computed as $2k-n$, and the second derivative is a positive constant. Therefore, the graph of f is concave up and attains minimum at $k = n/2$, and the maximum occurs at $k = 1$. Therefore, we have a lower and upper bound for plexers constructed as in Theorem 2.1. Therefore, we have successfully proved the following theorem.

Theorem 2.4. *When constructing multiplexers on $\bigstar_n$ using Theorem 2.1, we have that the maximum size possible is $\frac{(n-1)(n-2)}{2}$ and the minimum size possible is $\lceil \frac{(n^2-2n)}{4} \rceil$*

We have the following claim:

Claim 2.5. *Every type Ia multiplexer on $\bigstar_n$, $n \geq 5$, is of the form $\left(\bigstar_n, \Pi_{k\oplus(n-k)}\right)$ for some k.*

Proof. This will be proven later in the paper in Section 4 □

We will now discuss the derivation of some type Ib multiplexers. We recall that a type Ib multiplexer is a multiplexer with some recoverable diagonal elements. We begin the derivation by describing a partition on the set of unordered pairs of boundary vertices. We define the partition $\Pi_{k,n-k}$ to have $\Pi^U_{k,n-k}$ equal to a complete bipartite graph $K_{k,n-k}$. We then have that $\Pi^K_{k,n-k}$ is the set of all other unordered pairs of boundary vertices, which is equal to $K_k \sqcup K_{n-k}$. More formally, we have $\Pi^U_{k,n-k} = \{[a,b]$ such that either $a \leq k$ and $b > k$ or $a > k$ and $b \leq k\}$.

Again, we have a block matrix representation of this as follows, where by convention the diagonals are unknown:

$$\begin{pmatrix} \checkmark & \times \\ \times & \checkmark \end{pmatrix}$$

We will use this partition to build up another class of multiplexers, but first we prove a short lemma.

Lemma 2.6. *If we take the R matrix of $\bigstar_n$ to have every off-diagonal entry known, then the diagonal entries are recoverable if and only if $n \geq 3$*

Proof. If $n = 1, 2$ the claim is obvious, since for $n = 1$ there is only one diagonal entry in the whole matrix, and for $n = 2$ there is only one 2×2 determinant, and it has both diagonal entries. Now, let $n \geq 3$. Consider the relation $\lambda_{11}\lambda_{23} = \lambda_{13}\lambda_{21}$. Clearly, this determinant recovers λ_{11}. We can then use the relations $\lambda_{11}\lambda_{ii} = \lambda_{1i}\lambda_{i1}$ to recover λ_{ii} for all other i. Thus, if $n \geq 3$, all diagonal elements are recoverable. $\square$

We now use this to prove the following theorem.

Theorem 2.7. *If $n \geq 5$, $(\bigstar_n, \Pi_{k,n-k})$ is a type Ib multiplexer if $k = 1, 2, n-2$, and $n-1$, and it is not a multiplexer if $3 \leq k \leq n-3$.*

Proof. We can assume without loss of generality that $k \leq \lfloor \frac{n}{2} \rfloor$. We introduce the following notation regarding our block matrix, which will be convenient for us:

$$\begin{pmatrix} \checkmark & \times \\ \times & \checkmark \end{pmatrix} = \begin{pmatrix} I & II \\ III & IV \end{pmatrix}$$

More formally, we see that if we have an entry of the R matrix λ_{ij}, then $\lambda_{ij} \in I$ if $i, j \leq k$, and $\lambda_{ij} \in II$ if $i \leq k$, $j > k$, and so on.

We check that nothing in II or III is recoverable if $k = 1, 2$ and that the matrix is recoverable if $k \geq 3$. We first show that the only possibility for recovering an unknown entry is through the square root trick, for any k. If we don't use the square root trick, the only way to recover an entry of II or III is to have a 2×2 sub-matrix of R such that we have exactly one entry in II or III, and the other three entries in I and IV. We show this cannot happen. Since we have rectangular blocks, if we have three entries in I the fourth is also in I, and similarly for IV. Also, if we have two entries in I, or equivalently in IV, and a third in II or III, then the fourth is also in II or III, so that we have two entries from II and III. If we only have one entry in I and an entry in IV, then we have an entry in II and an entry in III. Therefore, if we do not use the square root trick, we cannot recover entries from II or III.

Now assume $k \geq 3$. We show $(\bigstar_n, \Pi_{k,n-k})$ is not a multiplexer. Since $3 \leq k \leq \lfloor \frac{n}{2} \rfloor$, we have that I is a $k \times k$ and IV is a $(n-k) \times (n-k)$ matrix, where we have $k \geq 3$ and $n - k \geq 3$. Also, both I and IV have all off-diagonal entries marked as known. Therefore, by Lemma 2.6, we know that all of the diagonals of both I and IV are known, which in turn means that all of the diagonals of our R matrix are known. But then we have by Theorem 1.6 that the R matrix is recoverable, so that $(\bigstar_n, \Pi_{k,n-k})$ is not a type I multiplexer when $k \geq 3$.

We now check that we cannot use the square root trick if $k = 1, 2$. Since $k = 1, 2$ and I is a $k \times k$ matrix with all off-diagonal entries as known entries, we have by Lemma 2.6 that the diagonal elements of I are not recoverable. Also, since $n \geq 5$, $n - k \geq 3$, so that the same lemma gives that IV has all diagonal entries recoverable. Therefore, if we are going to use the square root trick, it must be with both diagonal entries in IV, which would imply that the other two entries are also in IV. Therefore, $(\bigstar_n, \Pi_{k,n-k})$ satisfies condition (1) of a multiplexer. Also, we showed it has some diagonal entries recoverable, so that if we show it is a type I multiplexer, we will have it is a type Ib multiplexer, as required. Therefore, all that remains is to show $(\bigstar_n, \Pi_{k,n-k})$ satisfies condition (2) if $k = 1, 2$.

First, let $k = 1$. Then picking any entry λ_{1j} of II specifies the entry λ_{11} by the relation $\lambda_{1j}^2 = \lambda_{11}\lambda_{jj}$, since we know all the diagonals in IV. But then we know all the diagonals of the matrix, and hence the entire matrix, by Theorem 1.6. Similarly, if we have $k = 2$, picking an unknown entry of II or III will give the value of either λ_{11} or λ_{22}, from which we can immediately determine the other via the relation $\lambda_{12}^2 = \lambda_{11}\lambda_{22}$. But then we would know all the diagonals, and hence the whole matrix, by Theorem 1.6. Therefore, any unknown entry we reveal specifies the entire matrix, and thus $(\bigstar_n, \Pi_{k,n-k})$ is a type Ib multiplexer, as required. □

We can use this theorem as a base to produce type Ib multiplexers. The general idea is to add more unknown entries to the set $\Pi^U_{k,n-k}$ in such a way that we can no longer use the square root trick to recover any unknown entries, and we can still recover the entire matrix from all of the unknown entries. We give an intuition for how this works with an example.

Example. We look at the case of $(\bigstar_6, \Pi_{3,3})$. We have the following picture corresponding to this partition:

$$\left(\begin{array}{ccc|ccc} \bullet & \checkmark & \checkmark & \times & \times & \times \\ \checkmark & \bullet & \checkmark & \times & \times & \times \\ \checkmark & \checkmark & \bullet & \times & \times & \times \\ \hline \times & \times & \times & \bullet & \checkmark & \checkmark \\ \times & \times & \times & \checkmark & \bullet & \checkmark \\ \times & \times & \times & \checkmark & \checkmark & \bullet \end{array}\right)$$

As discussed in Theorem 2.7, this is not a plexer, since we can first recover the diagonal elements, and then use the square root trick. Also, we showed that this recovery cannot be accomplished without the square root trick. Therefore, it is possible to add unknown entries to the unknown set in such a way that the entire matrix now has no recoverable unknown entries, simply by blocking the recovery of any diagonal in either the upper left block or the bottom right block. We choose the upper left block, and we get the following picture instead:

$$\left(\begin{array}{ccc|ccc} \bullet & \times & \checkmark & \times & \times & \times \\ \times & \bullet & \checkmark & \times & \times & \times \\ \checkmark & \checkmark & \bullet & \times & \times & \times \\ \hline \times & \times & \times & \bullet & \checkmark & \checkmark \\ \times & \times & \times & \checkmark & \bullet & \checkmark \\ \times & \times & \times & \checkmark & \checkmark & \bullet \end{array}\right)$$

With the addition of this unknown entry, we can no longer recover any diagonal elements in the top left corner. In fact, if we just look at the top left block, we see it is a type Ia multiplexer on $\bigstar_3$, $(\bigstar_3, \Pi_{1\oplus 2})$:

$$\left(\begin{array}{cc|c} \bullet & \times & \checkmark \\ \times & \bullet & \checkmark \\ \hline \checkmark & \checkmark & \bullet \end{array}\right)$$

We claim that this is a multiplexer, and rather than prove that directly in this case, we will state and prove the theorem in generality, and leave this example as an intuition for that generalization.

We define another partition of the set of unordered pairs of boundary vertices, $\Pi_{l\oplus(k-l),n-k}$. We will say that $\Pi^U_{l\oplus(k-l),n-k}$ will be the complete bipartite graph $K_{k,n-k}$, with additional unknown edges forming a $K_l \sqcup K_{k-l}$. We then have that $\Pi^K_{l\oplus(k-l),n-k}$ is all other unordered pairs of boundary vertices. More formally, we can say $\Pi^U_{l\oplus(k-l),n-k} = \{[x,y]$ such that one of the following four properties is satisfied: $x \leq l$ and $y \leq l$, $l+1 \leq a \leq k$ and $l+1 \leq b \leq k$, $x \leq k$ and $y > k$, or $x > k$ and $y \leq k\}$. We again have a nice way of viewing this in block matrix form:

$$\left(\begin{array}{cc|c} \times & \checkmark & \\ \checkmark & \times & \times \\ \hline \multicolumn{2}{c|}{\times} & \checkmark \end{array}\right)$$

Using all of this new notation, we can restate our example by saying that we used the partition $\Pi_{1\oplus 2,3}$.

Theorem 2.8. *If* $3 \leq k \leq n-3$, $n \geq 6$ *and* $1 \leq l \leq k-1$, *then we have* $\left(\bigstar_n, \Pi_{l\oplus(k-l),n-k}\right)$ *is a type Ib multiplexer.*

Proof. We assume without loss of generality that $k \leq \lfloor \frac{n}{2} \rfloor$. We first check condition (1) of being a multiplexer. Due to the block nature of the matrix, and the location of the unknown edges, the only way to determine the value of any unknown edge in the upper left corner is using only known edges from the upper left corner. But we know that this is impossible, since the upper left corner forms a type Ia plexer. Also, we know from Theorem 2.7 that the only way to recover an unknown edge from the top right or bottom left corners is using the square root trick. But since the top left corner is a type Ia plexer, we cannot recover any diagonals in the upper left corner, so that all the diagonals we know are in one block of all checks on the off-diagonals, so that we cannot use the square root trick. This shows that $(\bigstar_n, \Pi_{l\oplus(k-l),n-k})$ satisfies condition (1) of being a multiplexer.

We now show that there are some recoverable diagonal entries. We know that since $3 \le k \le \lfloor \frac{n}{2} \rfloor$, we have that $n-k \ge 3$, and that the bottom right corner is an $n-k \times n-k$ matrix with all known edges on the off-diagonal, so that we can recover all of those entries by Lemma 2.6. In particular, we know all λ_{jj} where $j > k$. Therefore, if we succeed in showing that $(\bigstar_n, \Pi_{l\oplus(k-l),n-k})$ satisfies condition (2), we will have that it is a type Ib multiplexer, as required.

We now show that $(\bigstar_n, \Pi_{l\oplus(k-l),n-k})$ satisfies condition (2) of being a multiplexer. We first show that specifying an unknown edge in the upper left corner specifies the matrix. Since the upper left corner is a type Ia plexer, we have by property (2) of a multiplexer that this entry specifies the entire upper left corner, and in particular, all of the diagonals. This in turn gives that we know all of the diagonals, which means we can recover the matrix, by Theorem 1.6. Now say specify the value of some λ_{ij} with $i \le k, j > k$. Then we know that we can use the relation $\lambda_{ij}^2 = \lambda_{ii}\lambda_{jj}$ to recover λ_{ii}. But since the upper left corner is actually a $K_l \sqcup K_{k-l}$, we have by Theorem 2.2 that this diagonal specifies the values of the entire top left corner, and in particular the values of all the diagonals. This gives us again that we know all of the diagonals of the R matrix, and therefore we know the entire matrix by theorem 1.6. Therefore, we have that any unknown entry specifies the value of the entire R matrix, so that $(\bigstar_n, \Pi_{l\oplus(k-l),n-k})$ satisfies condition (2) of being a multiplexer, which gives us that $(\bigstar_n, \Pi_{l\oplus(k-l),n-k})$ is a type Ib multiplexer, as required. □

We end this section with the following claim.

Claim 2.9. *Every type Ib plexer on $\bigstar_n$, $n \ge 5$, is either of the form $(\bigstar_n, \Pi_{k,n-k})$, where $k = 1, 2, n-2$, or $n-1$, or of the form $(\bigstar_n, \Pi_{l\oplus(k-l),n-k})$ for some $3 \le k \le n-3$, $1 \le l \le k-1$.*

Proof. This will be proven later in the paper, in Section 4. □

3. A Maximum and Minimum Theorem

The goal of this section is stated in the following theorem:

Theorem 3.1. *Assume there exists an m-plexer on $\bigstar_n$, $n \ge 5$. Then we necessarily have the two inequalities $n-1 \le m \le \frac{(n-1)(n-2)}{2}$.*

This theorem will be proved in parts throughout the entire section. We start out by establishing a lower bound.

Theorem 3.2. *For $n \ge 5$, there is no k-plexer on $\bigstar_n$ if $k < n-1$.*

Proof. The motivation for this proof will be part (1) of Theorem 1.7, which says that if we have a quadrilateral with one unknown edge, it must contain another unknown edge. We begin by selecting an unknown edge, which without loss of generality we assume to be $[1,2]$. Consider the quadrilaterals $Q^K = [1,2,k,k+1]$ for $3 \leq k \leq n-1$. There are $n-3$ of these quadrilaterals whose pairwise intersection is the edge $[1,2]$. Now consider the quadrilateral $Q = [1,3,n,2]$. We show $Q \cap Q^K = [1,2]$ for all $3 \leq k \leq n-1$. To do this, we must check that none of the quadrilaterals Q^K contain the edges $[1,3],[3,n]$ or $[n,2]$. That is, we must check that none of $[1,k+1],[2,k]$, and $[k,k+1]$ are ever equal to $[1,3],[3,n]$ or $[2,n]$. Since $k \geq 3$, $[1,k+1] \neq [1,3]$ for any k. Similarly, since $k \leq n-1$, $[2,k] \neq [2,n]$ for any k. Also, since $n \geq 5$, $[k,k+1] \neq [3,n]$ for any k, since $n-1 \neq 3$. We conclude from this that $Q \cap Q^K = [1,2]$, for all k. Therefore, we have a set of $n-2$ quadrilaterals, $\{Q, Q_3, \ldots, Q_{n-1}\}$, such that their pairwise intersection is $[1,2]$. Therefore, by part (1) of Theorem 1.7, each of these $n-2$ quadrilaterals must have another unknown edge. Also, since these quadrilaterals have pairwise intersection $[1,2]$, we know that this will necessarily add $n-2$ unknown edges. We therefore conclude that if we have a k-plexer $(\bigstar_n, \Pi)$, $k = |\Pi^U| \geq n-1$, so that there is no k-plexer if $k < n-1$, as required. $\square$

Note, we do need that $n \geq 5$ for this to work, and in fact, there exists a 2-plexer on $\bigstar_4$. We also note that the above actually shows that there is no choice of partition Π such that if $|\Pi^U| < n-1$, $(\bigstar_n, \Pi)$ satisfies condition (1) of being a multiplexer.

Now that we have established a lower bound, we will show that there is no better lower bound.

Theorem 3.3. *If $n \geq 4$, there exists a type I $(n-1)$-plexer.*

Proof. If $n = 4$, we know this from our catalogue at the beginning(eventually this will make sense). If $n \geq 5$, we know from Theorem 2.7 that $(\bigstar_n, \Pi_{1,n-1})$ is a type I multiplexer for all $n \geq 5$. But we know that $|\Pi^U_{1,n-1}| = |\Pi^K_{1\oplus(n-1)}| = \frac{n(n-1)-(n-1)(n-2)}{2} = \frac{2n-2}{2} = n-1$. Therefore, we have that there does exist an $(n-1)$-plexer on $\bigstar_n$ if $n \geq 4$, as required $\square$

We now prove the existence of an upper bound. We begin with a lemma

Choose K matrix indices $(i_1,j_1),\ldots,(i_K,j_K)$, $i_K > j_K$, where K is an integer. Define the map L' as the map which takes a conductivity function γ to the vector $(\lambda_{i_1 j_1},\ldots,\lambda_{i_K,j_K})$, where λ_{i_L,j_L} is the (i_L,j_L) entry of Λ_γ. we have the following lemma

Lemma 3.4. *If $K < n$, L' is not injective.*

Proof. We know the map L which takes γ to Λ_γ is a homeomorphism, since we are working over $\bigstar_n$. In particular, L is continuous, which gives clearly that L' is continuous. Also, we see clearly that the domain of the map L' can be viewed as an open subset of $\mathbb{R}^n$, while the range can be viewed as a subset of $\mathbb{R}^K$. Therefore, if we had that L' were injective, we would have by the invariance of domain theorem that $K \geq n$ and that the range of L' is an open set of $\mathbb{R}^K$. In particular, this means that if $K < n$, L' is not injective □

We use this lemma to establish the existence of an upper bound on the size of an m-plexer on $\bigstar_n$

Theorem 3.5. *Assume there exists an m-plexer on $\bigstar_n$, $n \geq 3$. Then we necessarily have $m \leq \frac{(n-1)(n-2)}{2}$.*

Proof. Assume we have an ordered pair $(\bigstar_n, \Pi)$ such that $|\Pi^U| > \frac{(n-1)(n-2)}{2}$. But then $|\Pi^K| = \frac{n(n-1)}{2} - |\Pi^U| < \frac{n(n-1)-(n-1)(n-2)}{2} = n-1$. Therefore, if we pick any entry of the R matrix with which to verify condition (2) of Definition 1.1, we would be trying to recover the R matrix with only $|\Pi^K| + 1 = k < n$ known entries. But by Lemma 3.4, no matter what the position of these k entries in the matrix are, it will always be possible to find values of these k entries that do not uniquely determine the R matrix. If this were not true, the map L' would be injective, which it cannot be, since $k < n$. Therefore, we have that $(\bigstar_n, \Pi)$ is not a multiplexer, as required. □

Finally, we show that there is no better upper bound.

Theorem 3.6. *For all $n \geq 3$, we have a type I $\frac{(n-1)(n-2)}{2}$-plexer.*

Proof. We showed in Theorem 2.1 that $\left(\bigstar_n, \Pi_{1\oplus(n-1)}\right)$ is a type I multiplexer, and a simple computation shows that $|\Pi^U_{1\oplus(n-1)}| = \frac{(n-1)(n-2)}{2}$, as required. Also, as shown in the catalogue at the beginning, we do have a 3-plexer on $\bigstar_4$ □

Thus, we have proven Theorem 3.1, and in fact we proved a little better, since our minimum works for $n \geq 5$ where our maximum works for $n \geq 4$.

4. Classification of Type I Multiplexers on $\bigstar_n$

The main goal of this section will be to prove Claims 2.5 and 2.9, which when taken together would give a total classification of type I multiplexers on stars. The first step to this proof is a full classification of type Ia multiplexers.

Theorem 4.1. *Assume $n \geq 5$. If a type I multiplexer $(\bigstar_n, \Pi)$ is not of the form $(\bigstar_n, \Pi_{k\oplus(n-k)})$ for some k, then it is not a type Ia multiplexer*

Proof. The proof will proceed by construction of Π. We must start by marking some unknown entry, and we can assume, after switching rows and columns if necessary, that we have $\lambda_{12} \in \Pi^U$. Currently, we have that $\Pi^U \subset \Pi^U_{2\oplus(n-2)}$. However, by Theorem 1.7, we have that the only way we can have $\Pi^U \subset \Pi^U_{2\oplus(n-2)}$ is if we actually have $\Pi^U = \Pi^U_{2\oplus(n-2)}$, which we assumed does not happen. Therefore, we cannot have $\Pi^U \subset \Pi^U_{2\oplus(n-2)}$, and we must have that there exists some $\lambda_{ij} \in \Pi^U$ and $\lambda_{ij} \in \Pi^K_{2\oplus(n-2)}$. But by construction, that means there exists $\lambda_{ij} \in \Pi^U$ such that $i \leq 2$ and $j \geq 3$, and we can assume, switching rows and columns if necessary, that we have $\lambda_{13} \in \Pi^U$. Therefore, we see in general that we must have at least two entries in the top row marked.

The proof proceeds by assuming that we have a $k \geq 3$ such that $\lambda_{1j} \in \Pi^U$ if $j \leq k$, and $\lambda_{1j} \in \Pi^K$ if $j > k$. Below is a picture of this when $n = 8$ and $k = 5$, where an unspecified entry is left blank.

$$\begin{pmatrix}
\bullet & \times & \times & \times & \times & \checkmark & \checkmark & \checkmark \\
\times & \bullet & & & & & & \\
\times & \bullet & & & & & & \\
\times & & & \bullet & & & & \\
\times & & & & \bullet & & & \\
\checkmark & & & & & \bullet & & \\
\checkmark & & & & & & \bullet & \\
\checkmark & & & & & & & \bullet
\end{pmatrix}$$

Currently, we have that $\Pi^U \subset \Pi^U_{k\oplus(n-k)}$, which would imply that $\Pi^U = \Pi^U_{k\oplus(n-k)}$, which we assumed doesn't happen. Therefore, we have that there exists some λ_{ij} so that $\lambda_{ij} \in \Pi^U$ and $\lambda_{ij} \in \Pi^K_{k\oplus(n-k)}$, which by construction implies that we have $\lambda_{ij} \in \Pi^U$ such that $i \leq k$ and $j > k$. We can assume, after switching rows and columns if necessary that we have $\lambda_{2(k+1)} \in \Pi^U$. But then we have by Theorem 1.7 that we must have one of the three entries $\lambda_{1(k+1)}$, λ_{1j}, and λ_{2j} as an unknown entry, for all $j \geq k+2$. But $\lambda_{1j} \in \Pi^K$ for all $j \geq k+1$, which gives us that we must have $\lambda_{2j} \in \Pi^U$ for all $j \geq k+1$. We also have that the same argument would give that if we have $\lambda_{i(k+1)} \in \Pi^U$, where $i \leq k$, then we also have $\lambda_{ij} \in \Pi^U$ for all $j \geq k+2$, so that if $i \leq k$, we either have $\lambda_{ij} \in \Pi^U$ for all $j \geq k+1$ or $\lambda_{ij} \in \Pi^K$ for all $j \geq k+1$, so that we only need to consider marking whole blocks of unknown entries at once.

The proof proceeds by specifying $1 \leq l \leq k$ such that we have $\lambda_{ij} \in \Pi^U$ for all $2 \leq i \leq l+1$, $j \geq k+1$, and $\lambda_{ij} \in \Pi^K$ for all $l+2 \leq i \leq k$ and $j \geq k+1$. Below we have a picture of this where $n = 8$, $k = 5$, and $l = 3$, where again an empty entry corresponds to an unspecified entry.

$$\begin{pmatrix}
\bullet & \times & \times & \times & \times & \checkmark & \checkmark & \checkmark \\
\times & \bullet & & & & \times & \times & \times \\
\times & & \bullet & & & \times & \times & \times \\
\times & & & \bullet & & \times & \times & \times \\
\times & & & & \bullet & \checkmark & \checkmark & \checkmark \\
\checkmark & \times & \times & \times & \checkmark & \bullet & & \\
\checkmark & \times & \times & \times & \checkmark & & \bullet & \\
\checkmark & \times & \times & \times & \checkmark & & & \bullet
\end{pmatrix}$$

We are trying to construct a type Ia multiplexer, which means that we must not be able to recover any of the diagonals. In particular, we must not be able to recover λ_{ii} for any $i > k$. We recall that we specified λ_{1i} as known for all $i > k$. Also, since we want to not be able to recover λ_{ii} for any $i > k$, we can conclude from Theorem 1.7 and looking at quadrilaterals that we have on of λ_{1i}, λ_{1j}, and λ_{ij} as unknown for all $i, j > k$, which in particular gives us that λ_{ij} is unknown for all $i, j > k$. This is pictured below for our case $n = 8$, $k = 5$, $l = 3$

$$\begin{pmatrix}
\bullet & \times & \times & \times & \times & \checkmark & \checkmark & \checkmark \\
\times & \bullet & & & & \times & \times & \times \\
\times & & \bullet & & & \times & \times & \times \\
\times & & & \bullet & & \times & \times & \times \\
\times & & & & \bullet & \checkmark & \checkmark & \checkmark \\
\checkmark & \times & \times & \times & \checkmark & \bullet & \times & \times \\
\checkmark & \times & \times & \times & \checkmark & \times & \bullet & \times \\
\checkmark & \times & \times & \times & \checkmark & \times & \times & \bullet
\end{pmatrix}$$

We now switch the rows 1 and $l+1$ and columns 1 and $l+1$. This will give us that λ_{ij} is unmarked if $i, j \leq l$. It also gives us $\lambda_{ij} \in \Pi_U$ if $i \leq l$ and $j > l$. Additionally, it tells us that $\lambda_{ij} \in \Pi^K$ for all $l + 1 \leq i \leq k$ and $j > k$. Finally, it tells us that $\lambda_{ij} \in \Pi^U$ for all $i, j > k$. For our example, this takes on the following form:

$$\left(\begin{array}{ccc|ccccc}
\bullet & & & \times & \times & \times & \times & \times \\
& \bullet & & \times & \times & \times & \times & \times \\
& & \bullet & \times & \times & \times & \times & \times \\
\hline
\times & \times & \times & \bullet & & \checkmark & \checkmark & \checkmark \\
\times & \times & \times & & \bullet & \checkmark & \checkmark & \checkmark \\
\times & \times & \times & \checkmark & \checkmark & \bullet & \times & \times \\
\times & \times & \times & \checkmark & \checkmark & \times & \bullet & \times \\
\times & \times & \times & \checkmark & \checkmark & \times & \times & \bullet
\end{array}\right)$$

We notice immediately that if $l = 1$ or $l = 2$, then we have $\Pi^U \supset \Pi_{l,n-l}$, which is a type Ib multiplexer if $l = 1, 2$, which means that either we do not have a type I multiplexer, or $\Pi = \Pi_{l,n-l}$ and we have a type Ib multiplexer. Similarly, we cannot have $n - l = 1, 2$ Therefore, we now assume $3 \leq l \leq n - 3$.

We now show that no choice of n, k, and l can lead to a type Ia multiplexer, which shows that the only type Ia multiplexers are of the form $(\bigstar_n, \Pi_{k\oplus(n-k)})$. As we saw above, our matrix has the following form:

$$\left(\begin{array}{c|cc} & \multicolumn{2}{c}{\times} \\ \hline & & \checkmark \\ \times & & \\ & \checkmark & \times \end{array}\right)$$

where in the bottom right corner, the block of unknown entries is a principal $(n-k)\times(n-k)$ sub-matrix. We want to have a type Ia multiplexer, which means we want that no diagonals of our bottom right block are recoverable. In particular, λ_{nn} is not recoverable. But then we see from quadrilaterals and Theorem 1.7 that one of λ_{in}, λ_{nj}, and λ_{ij} is unknown, for all $l+1 \leq i \leq k$ and $l+1 \leq j \leq k$, which gives that λ_{ij} is unknown. This gives our matrix the following form:

$$\left(\begin{array}{c|cc} & \multicolumn{2}{c}{\times} \\ \hline & \times & \checkmark \\ \times & & \\ & \checkmark & \times \end{array}\right)$$

where as shown before, the top left block is $l \times l$, where $l \geq 3$, and the bottom right block matrix is in total $(n-l)\times(n-l)$ where $(n-l) \geq 3$, which means that we clearly have $\Pi^U \supset \Pi_{(n-k)\oplus(k-l),l}$, which means $\Pi^U = \Pi_{(n-k)\oplus(k-l),l}$, so that we have a type Ib multiplexer, and not a type Ia multiplexer. Therefore, the only type Ia multiplexers on $\bigstar_n$ are $(\bigstar_n, \Pi_{k\oplus(n-k)}$. $\square$

We now give a full classification of type Ib multiplexers. We previously showed that anything of the following form was a type Ib multiplexer:

$$\left(\begin{array}{cc|c} \times & \checkmark & \\ & & \times \\ \checkmark & \times & \\ \hline \multicolumn{2}{c|}{\times} & \checkmark \end{array}\right)$$

We note that the top left corner is really just a copy of $K_k \sqcup K_{m-k}$ for some m and k. But we previously showed that this was a type Ia multiplexer. Using this as motivation, we now will prove the following theorem:

Theorem 4.2. *Any type Ib multiplexer on $\bigstar_n$, except for $(\bigstar, \Pi_{k,n-k}$ for $k = 1, 2$ can be expressed in block form as follows:*

$$\left(\begin{array}{c|c} Ia & \times \\ \hline \times & \checkmark \end{array}\right)$$

Proof. We have a type *Ib* multiplexer, which means some of the diagonals are initially recoverable. In fact, we know that at least three diagonals are recoverable, which is easy to see. A quadrilateral which can recover a diagonal entry would consist of entries λ_{ii}, λ_{ik}, λ_{ji}, and λ_{jk}, and once we recovered λ_{ii}, we could clearly use λ_{ii} and λ_{ik} to get λ_{kk} and similarly we can get λ_{jj}. Now switch rows and columns until all the recoverable diagonals are in the bottom right corner. Using the square root trick, we conclude that the entire bottom right corner must be known, giving our matrix the following form:

$$\left(\begin{array}{c|c} A & \\ \hline & \checkmark \end{array}\right)$$

where A has the property that no diagonals are initially recoverable. From this, we can fill in the top right and bottom left as all unknown entries, since any known entry in these blocks would recover a diagonal of A. Therefore, our matrix has the form:

$$\left(\begin{array}{c|c} A & \times \\ \hline \times & \checkmark \end{array}\right)$$

All that remains is to show that the matrix A is a type Ia multiplexer, and to show that all that remains is to check that the matrix A satisfies condition (2) of being a type I multiplexer.

Pick an unknown edge in A and turn it into a known edge. First, we recover as many diagonals of A as possible using only quadrilaterals in A. Then, reorder the matrix A so that all of the recovered diagonals are in the bottom right corner. Then, arguing as before, we see that the matrix A takes the form:

$$\left(\begin{array}{c|c} A' & \times \\ \hline \times & \checkmark \end{array}\right)$$

which means the entire matrix is now of the form:

$$\left(\begin{array}{c|c} \begin{array}{cc} A' & \times \\ \times & \checkmark \end{array} & \times \\ \hline \times & \checkmark \end{array}\right)$$

Using the square root trick and regrouping the matrix, we see that we have finally have the following as our matrix:

$$\left(\begin{array}{c|c} A' & \times \\ \hline \times & \checkmark \end{array}\right)$$

where none of the diagonals of A' are recoverable. We see clearly that if A' is non-empty, then there are some unknown entries appearing in the block in the upper right hand corner. However, if such an unknown entry exists, it is easy to see that it cannot be recovered from the information we currently have. In fact, the only way it might be possible is the square root trick, which we cannot use since none of the diagonals of A' are recoverable. But this is a contradiction, since we started our recovery process by specifying an unknown entry, which by condition (2) of the definition of a multiplexer is enough to specify the entire matrix. This contradiction arose from the assumption that A' was non-empty. Therefore, A' is empty. But that means specifying an unknown entry of A was enough to specify all of the diagonals of A, and hence all of A by Theorem 1.6. Therefore, we have shown that A is a type Ia multiplexer, as required. $\square$

Combining these two results, we get that the only type I multiplexers are $(\bigstar_n, \Pi_{k,n-k})$ for $k = 1, 2$, $(\bigstar_n, \Pi_{k\oplus(n-k)})$ for any k, or $(\bigstar_n, \Pi_{l\oplus(k-l),n-k})$ for $1 \leq l \leq k$, which finally proves Claims 2.5 and 2.9. Also, since we previously classified $\bigstar_3$ and $\bigstar_4$, we have a full classification of all multiplexers on $\bigstar_n$, for all n.