STRUCTURE OF MULTIPLEXERS ON STARS

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Abstract. In this paper, we will talk about $n-plexers$, an object crucial in the construction of an n to 1 graph. We will talk mostly about $m-plexers$ on $n-stars$ and try to classify what structures are possible.

1. Definitions and Basic Methods

Definition 1.1. A multiplexer is an ordered pair $P = (G, \Pi)$, where G is a graph with boundary and $\Pi = (\Pi_U, \Pi_K)$ is a partition of the set of all distinct unordered pairs of boundary vertices into two sets, called the *unknown set* and the *known set*. We call an element of the unknown set an unknown pair, and an element of the known set a known pair. We require that P has the following properties.

(1) The graph G is recoverable.
(2) For any valid response matrix on G, if we only know the response matrix entries corresponding to known pairs, we cannot determine the value of any entry corresponding to an unknown pair.
(3) For any valid response matrix on G, if we know only the response matrix entries corresponding to known pairs and one unknown pair, we can recover the response matrix.

If $|\Pi_U| = n$, we will call this an $n - plexer$.

Definition 1.2. An $n-$star, denoted $\bigstar_n$, is a graph with boundary with one interior node and n boundary nodes, with one edge connecting each boundary node to the interior node and no other connections

Definition 1.3. The complete graph on n vertices is a graph with boundary with n boundary nodes, no interior nodes, and an edge connecting any two boundary nodes. It will be denoted K_n

Definition 1.4. By the quadrilateral $[v_1, v_2, v_3, v_4]$, we mean the set of edges $[v_1, v_2], [v_2, v_3], [v_3, v_4], [v_1, v_4]$.

In the special case where we are looking for an $m - plexer$ on $\bigstar_n$, we can view the definition of a multiplexer in a more visual way, which will sometimes be useful for us.

First, we note that a choice of a pair of boundary vertices of $\bigstar_n$ is equivalent to the choice of an edge of K_n, which is equivalent to a choice of an entry of the response matrix. As such, we will make references to unknown entries of the response matrix and unknown edges. Second, we note that $\bigstar_n$ has one interior node, so that as in Nick Addington's paper, the problem of recovering a Kirchhoff matrix on $\bigstar_n$ from a response matrix on K_n is equivalent to recovering the Kirchhoff matrix from the R matrix of K_n instead, where the R matrix is defined by $R = \Lambda - A$. The benefit of this is discussed in Nick's paper:

Theorem 1.5. *Any 2×2 sub-matrix of the R matrix of K_n has zero determinant.*

Proof. The proof is in Nick's paper. Basically, it can be shown that the R matrix is rank 1, and therefore has the desired property. □

What this means for us is that when are checking to see what entries are determined in the R matrix by our known pieces of information, we only need to look at 2×2 determinants.

In the special case where we have an off-diagonal 2×2 sub-matrix, this theorem takes the following form:

$$\lambda_{ij}\lambda_{kl} = \lambda_{ik}\lambda_{jl} \quad \text{for all pairwise distinct } i, j, k, l \tag{1}$$

This condition is called the quadrilateral condition because it says that if we form any quadrilateral in K_n, the products of the weights on opposite sides are equal.

We now develop some basic tools for proving properties of multiplexers. We will mainly focus of what sort of structure is forced upon us by definition (1.1). Also, it should be noted that we will be using both matrices and determinants and quadrilaterals on K_n to prove properties of multiplexers, so the reader should realize that anything stated for one of these two settings holds true in the other.

Theorem 1.6. *When attempting to construct an $m-plexer$ on $\bigstar_n$, the following conditions must be met:*

1. *If we choose an unknown entry in the response matrix, then any 2×2 sub-matrix containing that entry must have another unknown entry in it. Equivalently, if we have a choose an unknown edge in K_n, any quadrilateral containing that edge must have another unknown edge.*
2. *If we have a 2×2 sub-matrix with three known entries, the fourth entry is also a known entry.*
3. *If we have an $m-plexer$ $(\bigstar_n, \Pi)$, then there does not exist a $k-plexer$ $(\bigstar_n, \Phi)$, $k < m$, such that $\Phi_U \subset \Pi_U$.*
4. *If we have an $m-plexer$ $(\bigstar_n, \Pi)$, then there does not exist a $k-plexer$ $(\bigstar_n, \Phi)$, $k > m$, such that $\Pi_U \subset \Phi_U$.*

Proof. The proofs of (1) and (2) are basic applications of definition (1.1) and are left to the reader to verify. Also, we only need prove one of (3) and (4), since the two statements are obviously equivalent. We choose to prove (3). Suppose we do have $\Phi_U \subset \Pi_U$. We show that this leads to a contradiction. Since $k < m$, we know there exists a pair $[i, j]$ such that $[i, j] \in \Pi_U$, but $[i, j] \notin \Phi_U$. According to property (3) of definition 1.1, if we specify the value of the pair $[i, j]$, we determine the values of all other elements of Π_U, which in particular specifies all the values of Φ_U. Φ_U is the unknown set of a multiplexer; therefore property (2) of definition 1.1 says that we cannot determine any one of the unknown pairs only from known pairs. This contradiction completes the proof. $\square$

2. Examples of Multiplexers

The goal of this section will be to provide a very large class of multiplexers on n-stars. Work is still being done here, and we hope to get some sort of classification of all possible multiplexers on n-stars. We start this section by deriving a large class of multiplexers on $\bigstar_n$.

We describe a partition of the set of all distinct unordered pairs of boundary vertices of $\bigstar_n$. Let i be an integer. We define the partition $C^{(i)}$ to have $C_U^{(i)} = \{[x, y]$ where either $x \le i$ and $y \le i$ or $x > i$ and $y > i\}$. We take $C_K^{(i)}$ to be the set of all other distinct unordered pairs. Intuitively, this saying that on K_n the edges we are choosing form the disjoint union of K_i and K_{n-i}, using all the vertices of K_n.

Theorem 2.1. *$(\bigstar_n, C^{(i)})$ is a multiplexer.*

Proof. That $\bigstar_n$ is recoverable is obvious. We now show properties (2) and (3) of definition 1.1 hold. We know that the entries of $C_K^{(i)}$ have the form $a \le i$ and $b > i$ or $a > i$ and $b \le i$. This corresponds to a rectangle of known information in the top right corner of our R matrix and in the bottom left corner of our R matrix. For example, if $n = 8, i = 5$ our matrix would have the following form, where bullets mark the diagonal and x marks an unknown entry.

$$\begin{pmatrix} \bullet & x & x & x & x & \lambda_{16} & \lambda_{17} & \lambda_{18} \\ x & \bullet & x & x & x & \lambda_{26} & \lambda_{27} & \lambda_{28} \\ x & x & \bullet & x & (x) & \lambda_{36} & (\lambda_{37}) & \lambda_{38} \\ x & x & x & \bullet & x & \lambda_{46} & \lambda_{47} & \lambda_{48} \\ x & x & x & x & \bullet & \lambda_{56} & \lambda_{57} & \lambda_{58} \\ \lambda_{16} & \lambda_{26} & \lambda_{36} & \lambda_{46} & (\lambda_{56}) & \bullet & (x) & x \\ \lambda_{17} & \lambda_{27} & \lambda_{37} & \lambda_{47} & \lambda_{57} & x & \bullet & x \\ \lambda_{18} & \lambda_{28} & \lambda_{38} & \lambda_{48} & \lambda_{58} & x & x & \bullet \end{pmatrix}$$

We verify property (2). As previously remarked, since we are dealing with $\bigstar_n$, we only need to look at determinants of 2×2 sub-matrices. In particular, we can only recover an unknown entry if it is in a 2×2 sub-matrix with exactly three known entries. But if we have three known entries in one 2×2 sub-matrix, we clearly must also have the fourth known, since our known entries appear in rectangular blocks. This verifies property (2). Property (3) is also easy to verify. Assume

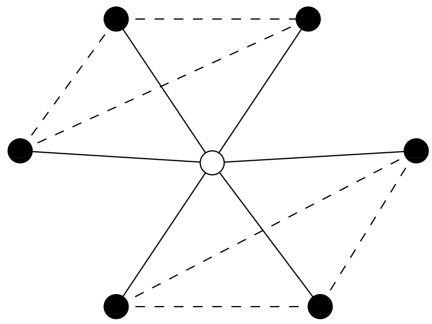

FIGURE 1. Example of Theorem 2.1 when n=6 and i=3

we specify the value of some x, say it is λ_{ab} where without loss of generality we assume $a \leq i$ and $b \leq i$. We use this to recover all entries of the form λ_{jk}, where $j > n-i$ and $k > n-i$. We can do this using the relation $\lambda_{ab}\lambda_{jk} = \lambda_{ak}\lambda_{bj}$ where for any pair j and k so that $j > i$ and $k > i$, λ_{ak} and λ_{bj} are known entries. A similar argument shows that specifying λ_{ab} where $a > i$ and $b > i$ gives all entries of the form λ_{jk} where $j \leq i$ and $k \leq i$. This shows that property (3) is satisfied, and hence we have that $(\bigstar_n, C^{(i)})$ is a multiplexer. □

Corollary 2.2. *$\bigstar_n$ always contains a multiplexer of size $\frac{(2m^2+n^2-2mn-n)}{2}$, where $1 \leq m < n$.*

Proof. By Theorem 2.1, $(\bigstar_n, C^{(m)})$ is a multiplexer, for all $1 \leq m < n$ where $|C_U^{(i)}|$ is easily computed as $\frac{(2m^2+n^2-2mn-n)}{2}$ □

For this class of multiplexers, we can consider their size as a function of m, say f. Clearly f is differentiable, since f is a polynomial, and its derivative is easily computed as $2m-n$, and the second derivative is a positive constant. Therefore, the graph of f is concave up and attains minimum at $m = n/2$, and the maximum occurs at $m = 1$. Therefore, we have a lower and upper bound for plexers constructed as in theorem 2.1. Therefore, we have successfully proved the following theorem.

Theorem 2.3. *When constructing multiplexers on $\bigstar_n$ using theorem 2.1, we have that the maximum size possible is $\frac{(n-1)(n-2)}{2}$ and the minimum size possible is $\lceil \frac{(n^2-2n)}{4} \rceil$*

We also have the following neat result.

Theorem 2.4. *Defining a partition $D^{(i)}$ as $(D_U^{(i)}, D_K^{(i)}) = (C_K^{(i)}, C_U^{(i)})$, we have that $(\bigstar_n, D^{(i)})$ is a multiplexer if and only if $i = 1, 2$*

Proof. We have a proof of this, but we are still working on the details. The motivation behind why it only works for $i = 1, 2$ is that for higher numbers we can use the square root trick to violate property (2) of definition 1.1. □

3. A Maximum and Minimum Theorem

The goal of this section is stated in the following theorem:

Theorem 3.1. *Assume there exists an $m - plexer$ on $\bigstar_n$, $n \geq 5$. Then we necessarily have the two inequalities $n - 1 \leq m \leq \frac{(n-1)(n-2)}{2}$.*

This theorem will be proved in parts throughout the entire section. We start out by establishing a lower bound.

Theorem 3.2. *For $n \geq 5$, there is no $k - plexer$ on $\bigstar_n$ if $k < n - 1$.*

Proof. The motivation for this proof will be part (1) of Theorem 1.6, which says that if we have a quadrilateral with one unknown edge, it must contain another unknown edge. We begin by selecting an unknown edge, which without loss of generality we assume to be $[1, 2]$. Consider the quadrilaterals $Q_k = [1, 2, k, k + 1]$ for $3 \leq k \leq n - 1$. There are $n - 3$ of these quadrilaterals whose pairwise intersection is the edge $[1, 2]$. Now consider the quadrilateral $Q = [1, 3, n, 2]$. We show $Q \cap Q_k = [1, 2]$ for all $3 \leq k \leq n - 1$. To do this, we must check that none of the quadrilaterals Q_k contain the edges $[1, 3], [3, n]$ or $[n, 2]$. That is, we must check that none of $[1, k + 1], [2, k]$, and $[k, k + 1]$ are ever equal to $[1, 3], [3, n]$ or $[2, n]$. Since $k \geq 3$, $[1, k + 1] \neq [1, 3]$ for any k. Similarly, since $k \leq n - 1$, $[2, k] \neq [2, n]$ for any k. Also, since $n \geq 5$, $[k, k + 1] \neq [3, n]$ for any k, since $n - 1 \neq 3$. We conclude from this that $Q \cap Q_k = [1, 2]$, for all k. Therefore, we have a set of $n - 2$ quadrilaterals, $\{Q, Q_3, \ldots, Q_{n-1}\}$, such that their pairwise intersection is $[1, 2]$. Therefore, by part (1) of Theorem 1.6, each of these $n - 2$ quadrilaterals must have another unknown edge. Also, since these quadrilaterals have pairwise intersection $[1, 2]$, we know that this will necessarily add $n - 2$ unknown edges. We therefore conclude that if we have a $k - plexer$ $(\bigstar_n, \Pi)$, $k = |\Pi_U|$ is at least $n - 1$, as required. □

Note, we do need that $n \geq 5$ for this to work, and in fact, there exists a $2 - plexer$ on $\bigstar_4$.

Now that we have established a lower bound, we will show that there is no better lower bound.

Theorem 3.3. *If $n \geq 4$, there exists an $(n - 1) - plexer$.*

Proof. As in Theorem 2.4, we know that $(\bigstar_n, D^{(1)})$ is a multiplexer for all $n \geq 4$. But we know that $|D^{(1)}_U| = |C^{(1)}_K| = \frac{n(n-1)-(n-1)(n-2)}{2} = \frac{2n-2}{2} = n - 1$. Therefore, we have that $(\bigstar_n, D^{(1)})$ is an $(n - 1) - plexer$ for all $n \geq 4$. □

We now prove the existence of an upper bound. We begin with a lemma

Choose k matrix indices $(i_1, j_1), \ldots, (i_k, j_k)$, $i_k > j_k$, where k is an integer. Define the map L' as the map which takes a conductivity function γ to the vector $(\lambda_{i_1 j_1}, \ldots, \lambda_{i_k, j_k})$, where λ_{i_l, j_l} is the (i_l, j_l) entry of Λ_γ. we have the following lemma

Lemma 3.4. *If $k < n$, L' is not injective.*

Proof. We know the map L which takes γ to Λ_γ is a homeomorphism, since we are working over $\bigstar_n$. In particular, L is continuous, which gives clearly that L' is continuous. Also, we see clearly that the domain of the map L' can be viewed as an open subset of $\mathbb{R}^n$, while the range can be viewed as a subset of $\mathbb{R}^k$. Therefore, if we had that L' were injective, we would have by the invariance of domain theorem that $k \geq n$ and that the range of L' is an open set of $\mathbb{R}^k$. In particular, this means that if $k < n$, L' is not injective $\square$

We use this lemma to establish the existence of an upper bound on the size of an $m - plexer$ on $\bigstar_n$

Theorem 3.5. *Assume there exists an $m - plexer$ on $\bigstar_n$, $n \geq 4$. Then we necessarily have $m \leq \frac{(n-1)(n-2)}{2}$.*

Proof. Assume we have an ordered pair $(\bigstar_n, \Pi)$ such that $|\Pi_U| > \frac{(n-1)(n-2)}{2}$. But then $|\Pi_K| = \frac{n(n-1)}{2} - |\Pi_U| < \frac{n(n-1)-(n-1)(n-2)}{2} = n - 1$. Therefore, when we try to verify property (3) of definition 1.1, we would be trying to recover the R matrix with only $|\Pi_K| + 1 = k < n$ known entries. But by lemma 3.4, no matter what the position of these k entries in the matrix are, it will always be possible to find values of these k entries that do not uniquely determine the R matrix. If this were not true, the map L' would be injective, which it cannot be, since $k < n$. Therefore, we have that $(\bigstar_n, \Pi)$ is not a multiplexer, as required. $\square$

Finally, we show that there is no better upper bound.

Theorem 3.6. *For all $n \geq 4$, we have an $\frac{(n-1)(n-2)}{2} - plexer$.*

Proof. We showed in Theorem 2.1 that $(\bigstar_n, C^{(1)})$ is a multiplexer, and a simple computation shows that $|C_U^{(1)}| = \frac{(n-1)(n-2)}{2}$, as required. $\square$

Thus, we have proven Theorem 3.1, and in fact we proved a little better, since our minimum works for $n \geq 5$ where our maximum works for $n \geq 4$.