THE DIRICHLET PROBLEM ON DIRECTED NETWORKS

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Abstract. The purpose of this paper is to examine the Dirichlet problem on directed current networks. The term *directed networks* refers to networks in which conductances are associated with a direction and the conductance is only used if current agrees with that direction. The difference between these networks and the undirected case is essentially that in the directed current case the Dirichlet to Neumann (response) map is piecewise linear while in the undirected case the same map is linear [2]. Refer to [1] for an examination of the Dirichlet problem on undirected graphs.

Please note that this paper is a continuation of Orion Bawdons work [3] from the 2005 UW REU program. Bawdon provided conjectures but no justification for many of the results in sections 2 and 3.1 and Lemma 3.11, and most of the definitions used are his.

We begin by looking at the case of a star and then progress to arbitrary directed networks. The main results are proofs of the existence of a harmonic extension for all directed networks, and the uniqueness on a symmetric directed network.

Contents

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1. Definitions

Definition 1.1. A *directed graph with boundary* is a triple $(V, \partial V, A)$, where V is a set, ∂V is a nonempty subset of V, and $A \subset V \times V$.

The members of V are called *nodes* .The members of the set int $V = V \setminus \partial V$ will be called the *interior nodes* . The members of the set ∂V will be called *boundary nodes*. $\partial V_i, \partial V_o \subset \partial V$, where $\partial V_i = \{j \in \partial V \mid \exists k \in \text{ int}V, (v, k) \in A\}$ and $\partial V_o = \{j \in \partial V \mid \exists k \in \text{ int}V, (k, v) \in A\}$. Note that ∂V_i and ∂V_o may or may not be disjoint sets.

The members of A are called *arcs* . An arc is the directed-network analogue of an edge.

Definition 1.2. Let $i, j \in V$. i, j are neighbors (denoted $i \sim j$) if either $(i, j) \in A$ or $(j, i) \in A$.

Definition 1.3. Let $i, j \in V$. There is a *path* from i to j (denoted $i \to j$) if there is a sequence $(i_1, ..., i_n)$, with $i_1 = i$, $i_n = j$, and $(i_k, i_{k+1}) \in A$ for all $1 \leq k < n$.

Definition 1.4. A *directed current network* is a pair (Γ, γ) where $\Gamma = (V, \partial V, A)$ is a directed graph with boundary and where $\gamma : V \times V \to \mathbb{R}$ is a nonnegative function with $\gamma_{ij} > 0 \leftrightarrow (i, j) \in A$. This function γ represents the conductance of each arc between two nodes.

Definition 1.5. A *symmetric network* is a network (Γ, γ) where Γ is a symmetric graph; i.e., a graph such that $(i, j) \in A \leftrightarrow (j, i) \in A$.

Definition 1.6. Given $i, j \in V$ and a function $u : V \to \mathbb{R}$, we say that i and j are *u-neighbors* (denoted $i \sim_u j$) if either of the following is true:

- $u(i) > u(j)$ and $\gamma_{ij} > 0$
- $u(i) < u(j)$ and $\gamma_{ji} > 0$

The function u represents the electrical potential at each node. The physical interpretation of this definition is that i and j are u-neighbors when an arc exists between them and current flows along it.

For any node $i \in V$, given a potential function u on V we can define a function representing the current out of i:

$$I_i(u) = \sum_{u(j) \leq u(i)} \gamma_{ij}(u(i) - u(j)) - \sum_{u(j) \geq u(i)} \gamma_{ji}(u(j) - u(i)) \tag{1}$$

The sum is over the currents along the arcs into or out of i. When $u(i) = u(j), \gamma_{ij}(u(i) - u(j)) = \gamma_{ji}(u(j) - u(i)) = 0$, so even though we consider both directions between the nodes with equal voltages, nothing is added to either sum. This function is an application of Ohm's Law: the current along an arc is equal to the voltage difference between the nodes at its endpoints multiplied by the arc's conductance.

Definition 1.7. A function $u : V \to \mathbb{R}$ is called *harmonic, subharmonic* or *superharmonic* if, for all $i \in$ int V, $I_i(u)$ is equal to zero, less than or equal to zero, or greater than or equal to zero, respectively. If for some function $\phi : \partial V \to \mathbb{R}, u = \phi$ on ∂V, u is called an *extension* of ϕ.

The Dirichlet Problem: Given a directed current network and a set of boundary potentials, does there exist a harmonic extension and, if so, is it unique? First we will consider the simplest nontrivial case.

2. The Dirichlet Problem for a Star

Definition 2.1. A *star* is a directed current network with exactly one interior node.

Let Γ be a star with interior node i_0, and let ϕ be a set of boundary potentials.

Lemma 2.2. $\{u(i_0) \mid u$ *is a harmonic extension of* ϕ $\}$ *is a non-empty interval.*

Proof. First, we define two useful values:

$$\delta = \inf\{\phi(j) \mid j \in \partial V\}$$

$$\xi = \sup\{\phi(j) \mid j \in \partial V\}$$

Now we consider three cases:

(1) *A star with no arcs directed inward to* i_0. If $u(i_0) < \delta$, Ohm's law would predict that current flows from ∂V to i_0. But because no arcs are directed inward to i_0, $I_{i_0}(u) = 0$. If $u_{i_0} = \delta$ then $I_{i_0}(u) = 0$. By definition $u(i_0)$ is a harmonic extension of ϕ on $\{u(i_0) \leq \delta\}$. If $u(i_0) > \delta$, then $I_{i_0}(u) > 0$, so there is no harmonic extension of ϕ on $\{u(i_0) > \delta\}$.
We have a harmonic extension u of ϕ on the interval $(-\infty, \delta]$.

(2) *A star with no arcs directed outward from* i_0. Similarly to case 1, if $u(i_0) \geq \xi$, then $I_{i_0}(u) = 0$; therefore, $u(i_0)$ is a harmonic extension of ϕ. No arcs are directed such that current can flow outward from i_0 as predicted by Ohm's law. If $u(i_0) < \xi$, then $I_{i_0}(u) < 0$, so there is no harmonic extension of ϕ on $\{u(i_0) < \xi\}$.
We have a harmonic extension u of ϕ on the interval $[\xi, \infty)$.

(3) *A star with at least one arc directed out from* i_0 *and at least one arc directed into* i_0. Fix boundary voltages ϕ for all $j \in \partial V$, and let $x = u(i_0)$. Let $f(x) = I_{i_0}(u)$, then:

$$f(x) = \sum_{j|\phi(j)\leq x} \gamma_{i_0 j}(x - \phi(j)) - \sum_{j|\phi(j)\geq x} \gamma_{j i_0}(\phi(j) - x)$$

Rearranging:

$$\begin{aligned} f(x) &= \sum_{j|\phi(j)\leq x} (\gamma_{i_0 j}x - \gamma_{i_0 j}\phi(j)) + \sum_{j|\phi(j)\geq x} (\gamma_{j i_0}x - \gamma_{j i_0}\phi(j)) \\ &= \left(\sum_{j|\phi(j)\leq x} \gamma_{i_0 j} + \sum_{j|\phi(j)\geq x} \gamma_{j i_0} \right) x - \left(\sum_{j|\phi(j)\leq x} \gamma_{i_0 j}\phi(j) + \sum_{j|\phi(j)\geq x} \gamma_{j i_0}\phi(j) \right) \end{aligned}$$

Note that this final equation is of the form $f(x) = Mx + B$, with

$$\begin{aligned} M &= \sum_{j|\phi(j)\leq x} \gamma_{i_0 j} + \sum_{j|\phi(j)\geq x} \gamma_{j i_0} \\ B &= -\left(\sum_{j|\phi(j)\leq x} \gamma_{i_0 j}\phi(j) + \sum_{j|\phi(j)\geq x} \gamma_{j i_0}\phi(j) \right) \end{aligned}$$

Furthermore, we know that $M \geq 0$ because every $\gamma_{i_0 j}$ and $\gamma_{j i_0}$ is nonnegative, so a sum of them is also nonnegative. B is a real number.

Take the values $\phi(j)$ for all $j \in \partial V$ (call them ϕ-values) and list them in increasing order. As x, the potential at node i_0, varies between two consecutive ϕ-values, M and B are constants. Therefore the function $f(x)$ is piecewise linear. That is to say, it is linear as it varies between consecutive ϕ-values. M, which is the slope of $f(x)$, can change as x increases, depending on the conductances of the arcs.

$f(x)$ is also continuous. When x equals one of the ϕ-values, some of the $\gamma_{i_0 j}$ or $\gamma_{j i_0}$ disappear from the sum because no current flows along an arc with equal potential at each endpoint. But $f(x)$ approaches the same value from each direction whenever x is close to a ϕ-value.

If $\exists p \in \partial V_i$ and $\exists q \in \partial V_o$ such that $\phi(p) \geq \phi(q)$, then $M > 0$. Because $f(x)$ is continuous, we also know that $f(x)$ is strictly increasing. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$. If there is no $p \in \partial V_i$ and $q \in \partial V_o$ such that $\phi(p) \geq \phi(q)$, then $f(x)$ is not strictly increasing and there is no unique harmonic solution (Figure 2).

We have shown that $f(x)$ is a continuous, strictly increasing, piecewise linear function (Figure 1). By the Intermediate Value Theorem there exists a unique x such that $f(x) = 0$. So there is a unique harmonic extension of ϕ; we have a degenerate non-empty interval on which $x = u(i_0)$ is a harmonic extension of ϕ.

If $\forall p \in \partial V_i$ and $\forall q \in \partial V_o, \phi(p), \phi(q)$, then $M = B = 0$ when $u(i_0) \in [\inf \partial V_i, \sup \partial V_o]$ and $M > 0$ when $u(i_0) \notin [\inf \partial V_i, \sup \partial V_o]$. So $u(i_0)$ is a harmonic extension of ϕ on the nonempty interval $[\inf \partial V_i, \sup \partial V_o]$.

So $\{u(i_0) \mid u$ is a harmonic extension of ϕ $\}$ is a non-empty interval. □

Lemma 2.3. If u is a subharmonic extension of ϕ, then there is a harmonic extension v of ϕ such that $u(i_0) \leq v(i_0)$.

Proof. We consider the same three cases as in the proof of Lemma 2.2:

(1) *A star with no arcs directed inward to* i_0. In this situation, $I_{i_0} \geq 0$. So the only subharmonic extension occurs when $I_{i_0} = 0$. This is also a harmonic extension of ϕ. Given a subharmonic extension u of ϕ, there is a harmonic extension v of ϕ such that $u(i_0) = v(i_0)$.

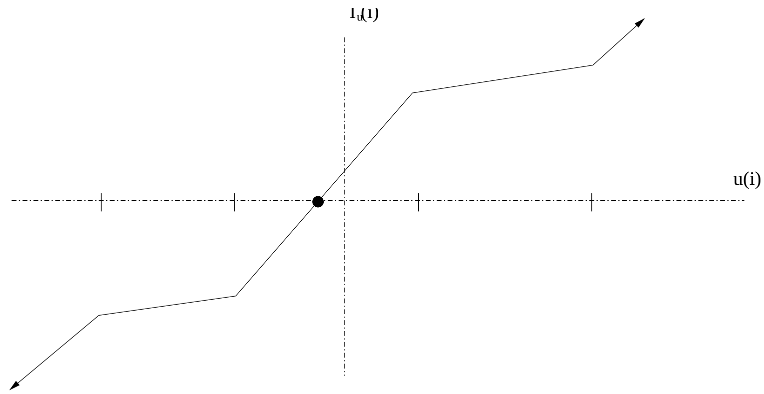

FIGURE 1. This is a graph of the current through the interior node i as a function of the potential at i. Note that the function is continuous, increasing and piecewise linear. There is a unique potential at which the current is zero; this is the harmonic extension.

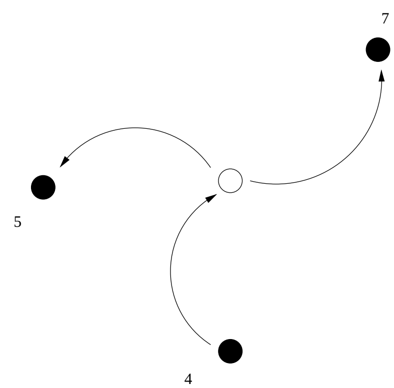

FIGURE 2. In this figure, the label near each boundary node represents the potential at that node. This star has no unique solution, because there is no $p \in \partial V_i, q \in \partial V_o$ such that $\phi(p) \geq \phi(q)$. The set of harmonic extensions of ϕ is the interval [4,5].

(2) *A star with no arcs directed outward from* i_0. Here we have subharmonic extensions that are not necessarily harmonic. Consider the graph of I_{i_0} as a function of $u(i_0)$. It is continuous, because I_{i_0} approaches the same value from each direction whenever x is close to a ϕ-value. It is nondecreasing, because the slopes are always positive or zero.
So we have that I_{i_0} is a continuous and nondecreasing function of $u(i_0)$. Given a subharmonic extension u of ϕ, a harmonic extension v of ϕ can only occur at a higher potential than $u(i_0)$, or an equal potential if the subharmonic extension is already harmonic.

(3) *A star with at least one arc directed outward from i_0 and at least one arc directed inward to i_0.* We have already shown in the proof of Lemma 2.2 that $f(x) = I_{i_0}(u)$, with $x = u(i_0)$, is continuous and strictly increasing. Take a subharmonic extension u of ϕ. If u is also harmonic, then the only harmonic extension v of ϕ occurs at $v(i_0) = u(i_0)$ because there is a unique harmonic extension of ϕ. If u is not also harmonic, then the unique harmonic extension v occurs at a higher potential because $f(x)$ is strictly increasing.

So we have that if u is a subharmonic extension of ϕ, then there is a harmonic extension v of ϕ such that $u(i_0) \leq v(i_0)$. □

If the star is a symmetric network, there is an arc directed in from $p \in \partial V$ and an arc directed out to $q \in \partial V$ such that $\phi(p) \geq \phi(q)$. This gives the following corollary:

Corollary 2.4. *Given symmetric star network with boundary potentials ϕ, there is a unique harmonic extension of ϕ.*

3. The Dirichlet Problem for Arbitrary Directed Networks

Now we will consider the Dirichlet Problem on an arbitrary directed current network with boundary potentials ϕ. First the existence of a solution will be established and then uniqueness will be considered.

3.1. Existence of a Harmonic Extension.

Definition 3.1. A function v has a local maximum at node i if for all j such that $i \sim j$, $v(i) \geq v(j)$, and a local minimum at a node k if for all j such that $k \sim j$, $v(k) \leq v(j)$.

Lemma 3.2. *If for all $i \in \text{int } V$ there exists some $j \in \partial V$ such that $i \to j$ then all subharmonic extensions u of ϕ have no strict local maximum in* int V.

Proof. Let u be a subharmonic extension of ϕ. Suppose u has a strict local maximum at $i_0 \in \text{int } V$, and let $S = \{i \mid i \sim i_0\}$. Since $u(i_0) > u(i)$ for all $i \in S$, $\sum_{j|u(j)>u(i_0)} \gamma_{i_0 j}(u(j) - u(i_0))$ is an empty sum; there are no j to sum over. Since there exists some $j \in \partial V$ such that $i_0 \to k$, there is at least one arc directed out from i_0, so $I_{i_0}(u) = \sum_{j|u(j)<u(i_0)} \gamma_{j i_0}(u(j) - u(i_0)) > 0$. By contradiction u cannot have a local maximum. □

The following lemma can be proved similarly.

Lemma 3.3. *If for all $i \in \text{int } V$ there exists some $j \in \partial V$ such that $j \to i$ then all superharmonic extensions u of ϕ have no strict local minimum in* int V.

A harmonic extension is both subharmonic and superharmonic, so we have the following corollary:

Corollary 3.4 (The Max/Min Principle for Directed Networks)**.** *If for all $i \in \text{int } V$ there exists some $j, k \in \partial V$ such that $j \to i$ and $i \to k$, then all harmonic extensions u of ϕ have no strict local minimum or maximum in* int V.

Corollary 3.5. *If for all $i \in \text{int } V$ there exists some $j \in \partial V$ such that $i \to j$ then for any subharmonic extension u of ϕ and for all $i \in \text{int } V$, $u(i) \leq \sup\{u(j) \mid j \in \partial V\}$.*

Proof. Take a subharmonic extension u of ϕ. Let $M = \sup\{u(i) \mid i \in \text{int } V\}$ and $N = \sup\{u(j) \mid j \in \partial V\}$. Take $i_0 \in \text{int } V$ such that $u(i_0) = N$. Let $S_k = \{j \mid j \in \partial V \text{ and } (k, j) \in A\}$.

Suppose $S_{i_0} \neq \emptyset$. Since u is subharmonic, $I_{i_0}(u) \leq 0$. Since $u(i_0) \geq u(i)$ for all $i \in \text{int } V$, no current can flow into i_0 from any interior node. So $\exists j_0 \in S_{i_0}$ such that $u(i_0) \leq u(j_0) \leq N$. So $M \leq N$.

Else if $S_{i_0} = \emptyset$, then no current can flow into i_0 so $I_{i_0}(u) \geq 0$. Since u is subharmonic it must be the case that $I_{i_0}(u) = 0$. So for all i such that $(i_0, i) \in A$, $u(i) = u(i_0) = M$. There is a $j \in \partial V$ such that $i_0 \to j$. Using induction, for all i_k such that $i_0 \to j$ passes through i_k, $u(i_k) = u(i_0)$. So there exists i_n such that $(i_n, j) \in A$, $u(i_n) = M$ and $S_{i_n} \neq \emptyset$. So $M \leq N$. $\square$

The following corollary results from a similar proof.

Corollary 3.6. *If for all $i \in \text{int } V$ there exists some $j \in \partial V$ such that $j \to i$ then for any superharmonic extension u of ϕ and for all $i \in \text{int } V$, $u(i) \geq \inf\{u(j) \mid j \in \text{int } V\}$.*

Theorem 3.7. *Given any directed network, there is a harmonic extension of ϕ.*

Proof. There are three cases to consider:

(1) We assume the hypothesis of Lemma 3.2. Let $F =\{u \mid u$ is subharmonic with values ϕ on $\partial V\}$. We know this set is non-empty: as an example, set $u(i) = \inf_\phi$ for all $i \in \text{int } V$. This is a subharmonic extension of ϕ.

    Now define $v(i) = \sup_{u(i)|u \in F}$. Suppose this extension v of ϕ is subharmonic and *not* harmonic. Then we should be able to take the harmonic function v' at the interior node i that is determined by the values of v at its neighbors. This will still be a subharmonic extension, but $v'(i) > v(i)$. This is a contradiction; $v(i)$ was defined as the sup over all the subharmonic extensions at i. This contradiction entails that $v(i)$ was already harmonic.

(2) Here we assume the hypothesis of Lemma 3.3. Similarly to Case 1, let $H = \{u \mid u$ is superharmonic with values ϕ on $\partial V\}$. Define $w(i) = \inf_{u(i)|u \in H}$; this is, by a similar indirect proof, a superharmonic *and* harmonic extension of ϕ.

(3) We have considered the cases such that for all i, there is a path to or from some $j \in \partial V$. Now consider what happens if one or both of these conditions is not satisfied. Take networks on $G = (V, \partial V, A)$ containing at least one $i \in \text{int } V$ such that either:
    - For all $j \in \partial V, i \nrightarrow j$ or
    - For all $j \in \partial V, j \nrightarrow i$

Suppose $V' = \{i \mid i \in \text{int } V \text{ and } \forall j \in \partial V,\ i \nrightarrow j\} \neq \emptyset$.
Define the following

$$\begin{aligned}
&\partial V' = \{i \in V' \mid \exists k \in V/V' : k \to i\}\\
&A' = \{(i,k) \in A \mid i,k \in V'\}\\
&G' = (V', \partial V', A')\\
&\tilde{V} = V/V'\\
&\tilde{A} = \{(i,k) \in A \mid i,k \in \tilde{V}\}\\
&\tilde{G} = (\tilde{V}, \partial V, \tilde{A}) \text{ (See Figure 3)}
\end{aligned}$$

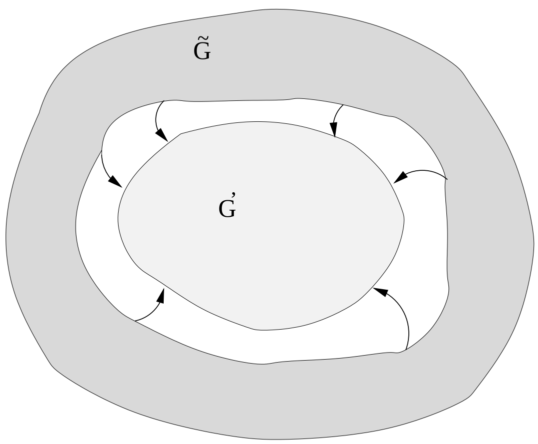

FIGURE 3. For all $i \in G'$ and $j \in \partial V, i \nrightarrow j$. There is a harmonic extension on $\tilde{G}$. By setting all potentials in G' higher than or equal to the sup over the potentials in $\tilde{G}$, we have a harmonic extension over the whole graph that is not unique.

By cases 1 and 2, there exists a harmonic extension u of ϕ on $\tilde{G}$. Let $s = \sup\{u(k) \mid k \in \tilde{V}\}$. u can be extended to G' by setting $u(k) = c$ for all $k \in V'$ where $c \geq s$. This causes u to be a harmonic extension on G since it will prevent current from flowing along any arcs connecting nodes in $\tilde{V}$ to nodes in V' and any arcs in A'.

Note that if $V' = \{i \mid i \in \text{int } V \text{ and } \forall j \in \partial V,\ i \nleftarrow j\} \neq \emptyset$ the result is proved in the same manner by taking $c \leq \inf\{u(k) \mid k \in \tilde{V}\}$ □

Note that in the third case in the proof of this theorem we know that the harmonic solution we get is not unique because c was not a unique value. This gives the following corollary to the proof:

Corollary 3.8. *If there exists $i \in \partial V$ such that either*

- *For all $j \in \partial V, i \nrightarrow j$ or*
- *For all $j \in \partial V, j \nrightarrow i$*

then a harmonic extension of ϕ exists and it is not unique.

We will continue discusing the uniqueness of the harmonic extension in the following section.

3.2. **Uniqueness of the Harmonic Extension.** In this section we assume that every interior node has a path both to and from the boundary.

Definition 3.9. A graph is *connected through the interior* if, for every pair of interior nodes i and k, either $i \to k$ or $k \to i$ and the path does not go through any boundary node.

Definition 3.10. A *connected component* Γ_c is a subgraph of Γ that is connected through the interior. If $\Gamma_c = \Gamma$, the whole graph Γ is connected through the interior. (Figure 5)

When we look for the unique Dirichlet solution we can assume that every network is connected through the interior, because a unique solution on every connected component is a necessary and sufficient condition for a unique solution on the entire graph.

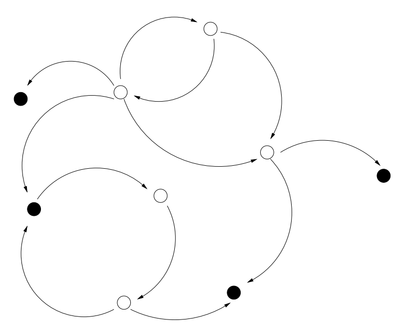

FIGURE 4. This network is not connected through the interior, but it has two connected components.

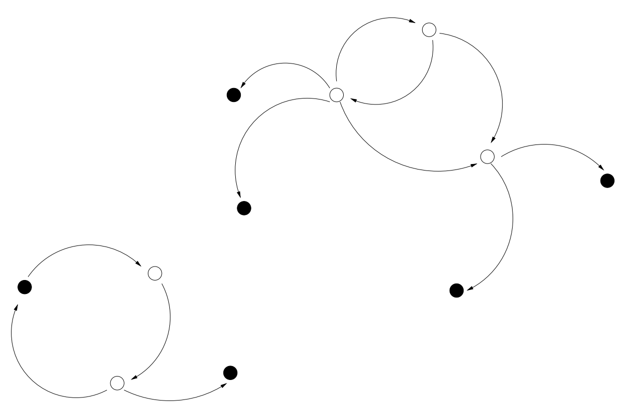

FIGURE 5. These are the two connected components of the network in Figure 4.

Note that in general a symmetric network does not have equal conductances in each direction. The undirected network is a special case of the symmetric network, where the conductances are equal in each direction.

For all $i \in \text{int } V$ let

$$m(i) = \inf\{u(i) \mid u \text{ is a superharmonic extension of } \phi\}$$
$$M(i) = \sup\{v(i) \mid v \text{ is a subharmonic extension of } \phi\}$$

The following result is not used in the rest of this paper, but we provide a proof because the hypothesis appeared in Orion Bawdon's paper.

Lemma 3.11. *Let $i \in \text{int } V$. If $\alpha < M(i)$, then there exists a subharmonic extension u of ϕ with $u(i) = \alpha$. If $\alpha > m(i)$, then there exists a superharmonic extension u of ϕ with $u(i) = \alpha$.*

Proof. Suppose $\alpha < M(i)$. Let $\delta = M(i) - \alpha$ and let $u(k) = M(k) - \delta$ for all $k \in \text{int } V$.

For $l, k \in \text{int } V$, $u(l) - u(k) = (M(l) - \delta) - (M(k) - \delta) = M(l) - M(k)$.

We have that $u(k) < M(k)$ for all $k \in \text{int } V$ and $u(j) = M(j) = \phi(j)$ for all $j \in \partial V$. So for all $k \in \text{int } V$ and $j \in \partial V$, $u(k) - u(j) < M(k) - M(j)$.

So for $k \in \text{int } V$ and $l \in V$, $u(k) - u(l) \leq M(k) - M(l)$.

Therefore for all $k \in \text{int } V$,

$$\begin{aligned} I_k(u) &= \sum_{u(j)<u(k)} \gamma_{kj}(u(k) - u(j)) + \sum_{u(j)>u(k)} \gamma_{jk}(u(k) - u(j)) \\ &\leq \sum_{M(j)<M(k)} \gamma_{kj}(M(k) - M(j)) + \sum_{M(j)>M(k)} \gamma_{jk}(M(k) - M(j)) \\ &= I_k(M) \leq 0 \text{ for all } k \in \text{int } V \end{aligned}$$

so u is a subharmonic extension of ϕ with $u(i) = \alpha$.

A similar proof holds for the existence of a superharmonic extension u with $u(i) = \alpha$ if $\alpha > m(i)$. $\square$

False Conjecture 3.12. Given a directed network with boundary potential function ϕ, if every node in V has a path to and from every other node, there is a unique harmonic extension of ϕ.

Counterexample. Every node in Figure 6 has a path to and from every other node, but there is not, in general, a unique harmonic extension of a boundary potential function. $\square$

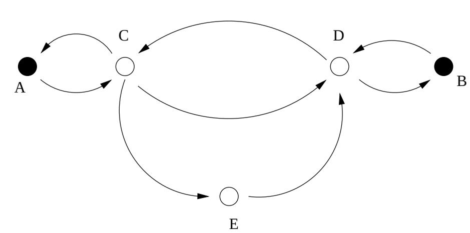

FIGURE 6.

Lemma 3.13. *Given a symmetric network with exactly one boundary node where every interior node has a path to the boundary and a path from the boundary, there is a unique harmonic extension.*

Proof. By Corollary 3.4, the maximum and the minimum of the harmonic extension occur on the boundary. There is only one boundary node j, so the max and min are equal to the potential at j. Therefore the potential at every interior node equals the potential at j. □

Theorem 3.14. *Given a symmetric network with at least 2 boundary nodes that is connected through its interior, all of the following are true:*

(1) *The Dirichlet problem has a unique solution.*
(2) *The interior voltages and all the currents depend continuously on the boundary potential function.*
(3) *For any boundary node p, fixing voltages on $\partial V \setminus p$ implies that the current at p is a continuous, increasing function of the voltage at p.*

Proof. We use induction on the number of interior nodes.

Base case: A graph with no interior nodes. Items 1 and 2 are trivially true. Item 3 is true because, on a star, the current at the interior node is a continuous, increasing function of the voltage at that node (Figure 1). Assume inductively that Theorem 3.14 holds for a graph with n interior nodes.

Now take a graph Γ_0 with $n+1$ interior nodes. Consider a new graph Γ_1 with the same vertices, edges and conductivity function as Γ_0, except one of the interior nodes in Γ_0 corresponds to an boundary node called p in Γ_1. Γ_1 has n interior nodes, so by the inductive assumption and Item 3 there is a unique potential $\phi(p)$ such that $I(p) = 0$. So Γ_0 has a unique Dirichlet solution; this corresponds exactly to the solution on Γ_1 with the uniquely determined $\phi(p)$. Thus Item 1 holds for a graph with $n+1$ interior nodes.

Define $F(\phi) : \phi \mid_{\partial V \setminus p} \mapsto \phi \mid_{\partial V}$. This is the map from the boundary potentials on $\partial V \setminus p$ to the unique potential on p such that the current $I(p) = 0$. Define $\limsup_{\phi \to \phi_0} F(\phi) = M$ and $\liminf_{\phi \to \phi_0} F(\phi) = m$. Now take two sequences:

$$\begin{gathered} \phi_j \mid F(\phi_j) \to M \text{ as } \phi_j \to \phi_0 \\ \psi_j \mid F(\psi_j) \to m \text{ as } \psi_j \to \phi_0 \end{gathered}$$

Define the function $I_p(\phi_j, F(\phi_j))$ as the current at node p due to the unique harmonic extension of the potentials ϕ_j on $\partial V \setminus p$ and $F(\phi_j)$ on p. Take

$$\begin{aligned} I_p(\phi_0, M) &= I_p(\lim_{n\to\infty}(\phi_j, F(\phi_j))) \\ &= \lim_{n\to\infty} I_p(\phi_j, F(\phi_j)), \text{ by continuity from the inductive hypothesis} \\ &= \lim_{n\to\infty} 0 \\ &= 0 \end{aligned}$$

Now take

$$\begin{aligned} I_p(\psi_0, m) &= I_p(\lim_{n\to\infty}(\psi_j, F(\psi_j))) \\ &= \lim_{n\to\infty} I_p(\psi_j, F(\psi_j)) \\ &= \lim_{n\to\infty} 0 \\ &= 0 \end{aligned}$$

From the proof of Item 1, there is a unique harmonic extension. So necessarily $M = m$; otherwise, there would be two different harmonic extensions. Thus F is continuous at ϕ_0. So the potential at p is continuously dependent on the potentials on $\partial V \setminus p$ for Γ_0, and since Item 2 holds for Γ_1, Item 2 holds for Γ_0.

Lastly, we show that Item 3 holds on Γ_0:

(1) Take a harmonic extension on Γ_0 (Figure 7).
(2) Raise the potential at boundary node B (Figure 8), and assume that there is a harmonic extension of these boundary values such that the new potentials at interior nodes E and F are lower than in the harmonic extension of Figure 7.
(3) Now create a subharmonic extension on Γ_0 (Figure 9) by raising the potentials at nodes E and F back to the values they had in the harmonic extension of Figure 7. The current out of each interior node is now negative.

This is a subharmonic extension of the boundary data. Because every subharmonic function is less than or equal to some harmonic function at every interior node, there must be a harmonic extension such that the potentials at all the interior nodes are greater than the potentials in Figure 9. This cannot be the harmonic extension we assumed in Figure 8. Thus we have a contradiction, because there is a unique harmonic extension of the boundary potentials.

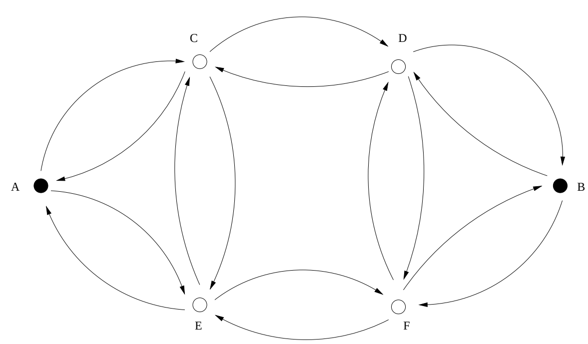

FIGURE 7.

So raising the potential at p will not cause any potentials in the interior to decrease in the new harmonic extension. In fact, all the potentials in the interior will strictly increase. It must be the case that at least one interior node's potential does

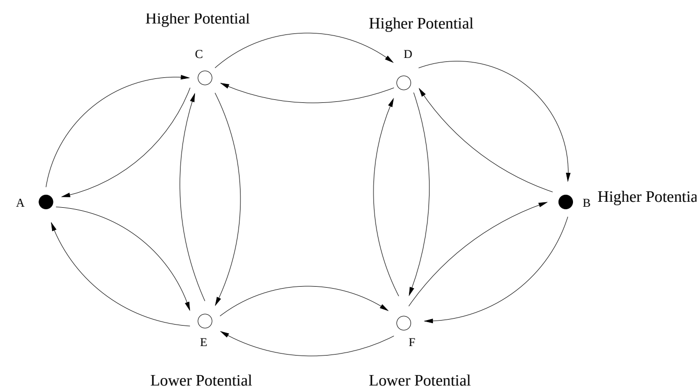

FIGURE 8.

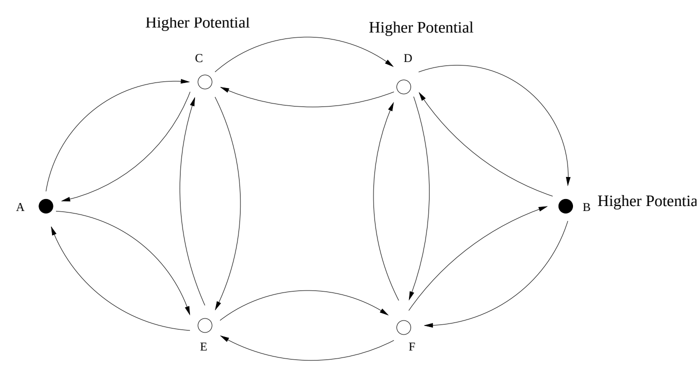

FIGURE 9.

increase, or the new extension would not be harmonic. Suppose the potential at one or more interior nodes does not increase when the potential at p is raised. At least one of these interior nodes is a neighbor of the node whose potential incresed, because the graph is connected. Call this node b. So there is more current flowing into b than there was before the potential at p was raised, because one of b's neighbors has a higher potential relative to b than before. But this current is more than the current that flows out of b, because none of the potentials in the interior decrease. So the extension is not harmonic. By this contradiction, the potential at every interior node increases when the potential at p is raised.

Consequently, the current will decrease at every boundary node other than p which is connected to the interior when the potential at p is raised. Thus the current at p is an increasing function of the voltage at p.

All three items hold for a graph with $n + 1$ interior nodes, and the theorem is proved by induction. $\square$

Given a graph consisting of connected components, Theorem 3.14 holds for each connected component. So a stronger result follows:

Corollary 3.15. *The conclusion of Theorem 3.14 holds for symmetric networks consisting of connected components that each have at least one boundary node.*

4. Future Research

(1) While having a symmetric network is a sufficient condition for having a unique harmonic extension for every set of boundary potentials, it is not a necessary condition. A more thorough examination of directed graphs may lead to a necessary and sufficient condition, or at least a weaker sufficient condition.

    Also, in our research we have found three types of graphs: those that have a unique harmonic extension for every set of boundary potentials, those that have a unique solution for some boundary potentials and not for others, and those that never have unique solutions. If sufficient and necessary conditions can be found for the first type of graphs, it seems likely that a classification system can be established for directed networks using these three categories.

(2) Our research has been on directed electrical networks with current moving continuously along the arcs. Ernie Esser suggested that someone look at computer networks where discrete packets of information move along arcs, with an upper bound at each node on the number or size of packets allowed.

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