ON THE NEUMANN-TO-DIRICHLET MAP

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The purpose of this paper is to explore
a matrix we shouldn't ignore.
While the properties of Lambda are known
as a response matrix it's not alone.
For this mapping there exists an inverse
whose properties are just as diverse.
Eta, we decided, is its name.
If you don't like it, we're to blame
— what a shame
— you're so lame
— end of game
. . . we're the same.

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1. Index of Terminology

Convention 1.1. The *Kirchhoff Matrix*, denoted $\mathbf{K}$, will be written here in the conventional block form as

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix},$$

where the submatirx $\mathbf{A}$ contains boundary-boundary information, $\mathbf{B}$ and $\mathbf{B}^T$ contain boundary-interior information, and $\mathbf{C}$ contains interior-interior information.

Definition 1.2. The *Dirichlet-to-Neumann Map*, denoted $\mathbf{\Lambda}$, is defined to be the unique linear operator taking boundary electric potentials to their generated boundary currents.

Definition 1.3. The *Neumann-to-Dirichlet Map*, denoted $\mathbf{H}$, is defined to be the unique linear operator taking boundary currents to their generated boundary electric potentials, with the stipulation that the sum of the electric potentials on the boundary of every connected component of the graph is equal to zero.

Definition 1.4. The column vector, ϵ_i, is defined by the rule

$$\epsilon_i(p) = \begin{cases} 1, & p \in \partial V_i \\ 0, & p \in \partial V \setminus \partial V_i \end{cases}$$

Furthermore, the matrix $\epsilon$is constucted so that the ith column of $\epsilon$is ϵ_i. For convenience, let $\mathbf{E} = \epsilon\epsilon^T$.

Definition 1.5. There exists a *generalized path*, $p \leftrightarrow q$, between two boundary vertices p and q if and only if there exists a sequence of neighboring vertices in G which begins with p and ends with q.

Definition 1.6. If $P = (p_1, p_2, ..., p_k)$ and $Q = (q_1, q_2, ..., q_k)$ are sets of boundary vertices, then there exists a *generalized k-connection* from P to Q if and only if there exists a permutation, σ, such that there exist generalized paths $\{p_i \leftrightarrow q_{\sigma(i)}\}$ that are vertex disjoint.

2. Derivations of the Response Matrices

2.1. **The Dirichlet-to-Neumann Map.** Consider the electrical network $\Gamma = (G, \gamma)$ with V being the set of vertices, ∂V the set of boundary vertices, and $\text{int}V = V \setminus \partial V$ the set of interior vertices. We will further require that G be a graph with boundary, each of whose m connected components contains a boundary vertex; for simplicity we will call this condition *boundedness*. Furthermore we will denote each connected component of G as $G_i = (V_i, E_i)$, $1 \le i \le m$. Let $\mathbf{K}$ be the Kirchhoff matrix of Γ, partitioned in the conventional manner with boundary vertices followed by interior vertices such that

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix}$$

Let ψ be a column vector corresponding to the currents on the boundary and $\mathbf{u}$ be a column vector corresponding to the electric potential at the vertices of G (partitioned appropriately, with $\mathbf{u} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$). Thus,

$$\mathbf{Ku} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \mathbf{u} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \psi \\ 0 \end{bmatrix} \tag{1}$$

Note that $\mathbf{e}^T\psi = 0$, where $\mathbf{e}^T = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}$. The vector $\mathbf{u}$ is not uniquely determined by ψ, since the addition of a constant vector to a solution will also yield a solution.

Lemma 2.1. Let $\Gamma = (G, \gamma)$ be a connected electrical network, where V is the set of vertices. Let $\mathbf{K}$ be the Kirchhoff matrix of $\mathbf{\Gamma}$, partitioned in the conventional manner. Then, the submatrix $\mathbf{C}$ of $\mathbf{K}$ is nonsingular if and only if $\partial V \neq \emptyset$.

Proof. By definition, the row and column sums of $\mathbf{K}$ are zero. That is, the constant vector $\mathbf{e}$ is in the nullspace of $\mathbf{K}$, which implies that $\det \mathbf{K} = 0$. Furthermore, the determinant of any principal proper submatrix of $\mathbf{K}$ is nonzero. If $\partial V = \emptyset$, then $\mathbf{C} = \mathbf{K}$, which implies that $\mathbf{C}$ is singular. On the other hand, if $\partial V \neq \emptyset$, then $\mathbf{C}$ is a principal proper submatrix of $\mathbf{K}$, which implies that $\mathbf{C}$ is nonsingular. □

Lemma 2.2. Let $\Gamma = (G, \gamma)$ be an electrical network and $\mathbf{K}$ be the Kirchhoff matrix of $\mathbf{\Gamma}$, partitioned as usual. Then, the submatrix $\mathbf{C}$ of $\mathbf{K}$ is nonsingular if and only if $\mathbf{G}$ is bounded.

Proof. Order the interior nodes of $\mathbf{G}$ such that

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & 0 & \dots & 0 \\ 0 & \mathbf{C}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{C}_m \end{bmatrix},$$

where $\mathbf{C}_k$ corresponds to interior-interior connections within the kth connected component of G. It follows that

$$\det \mathbf{C} = \prod_{i=1}^{m} \det \mathbf{C}_i.$$

That is, $\mathbf{C}$ is singular if and only if there exists i such that $\mathbf{C}_i$ is singular.

By the previous lemma, $\mathbf{C}$ is nonsingular if and only if for all $1 \leq i \leq m$, $\partial V_i \neq \emptyset$. Thus, the submatrix $\mathbf{C}$ of K is nonsingular if and only if G is bounded. □

Given that $\mathbf{C}$ is invertible we may use (1) to recover $\mathbf{y}$ from $\mathbf{x}$, as follows.

$$\mathbf{B}^T\mathbf{x} + \mathbf{C}\mathbf{y} = 0$$
$$\Rightarrow -\mathbf{C}^{-1}\mathbf{B}^T\mathbf{x} = \mathbf{y}$$

Now we may construct a linear mapping from boundary electric potentials to boundary currents, which we will refer to as the response matrix or the Dirichlet-to-Neumann Map.

$$\psi = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} = (\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T)\mathbf{x} \equiv \mathbf{\Lambda}\mathbf{x} \tag{2}$$

Lemma 2.3. The vector space $\langle \epsilon_i \rangle_{i=1}^{m}$ is a subspace of the nullspace of $\mathbf{\Lambda}$.

Proof. We will prove the equivalent claim that for all $1 \leq i \leq m$, $\mathbf{\Lambda}\epsilon_i = 0$. Since G is bounded, the Dirichlet problem on Γ is well-posed, given the input potential ϵ_i. Define $\mathbf{u}$ by the rule

$$\mathbf{u}(p) = \begin{cases} 1, & p \in V_i \\ 0, & p \in V \setminus V_i \end{cases}.$$

This potential satisfies both the given boundary data and Kirchhoff's law on the interior of V. Since this Dirichlet problem is well-posed, this solution is, in fact, the unique solution of this problem. Because $\mathbf{u}$ is locally constant, the current generated is identically zero on V. $\square$

Lemma 2.4. The kernel of the response matrix of a bounded, connected graph is $\langle \epsilon_i \rangle_{i=1}^m$.

Proof. In this case, $\langle \epsilon_i \rangle_{i=1}^m = \langle e \rangle$ is the space of constant vectors. By the results in [Morrow's book], null $\Lambda = 1$. By the previous lemma, $\langle e \rangle$ is a subspace of the nullspace of Λ. Both spaces have the same dimension, so they are equal. $\square$

Theorem 2.5. *The kernel of the response matrix of a bounded network is* $\langle \epsilon_i \rangle_{i=1}^m$.

Proof. Order the boundary vertices so that $\mathbf{\Lambda}$ is the following block form:

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_1 & 0 & \dots & 0 \\ 0 & \mathbf{\Lambda}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{\Lambda}_m \end{bmatrix},$$

where $\mathbf{\Lambda}_i$ denotes the Dirichlet-to-Neumann map, computed for the ith connected component of G. The dimension of the nullspace of $\mathbf{\Lambda}$ is equal to the sum over i of the dimension of the nullspace of $\mathbf{\Lambda}_i$. It follows that the dimension of the nullspace of $\mathbf{\Lambda}$ is exactly m. Since the vectors $\epsilon_1, \dots, \epsilon_m$ are linearly independent, their span is also of dimension m. By Lemma 2.3, the span of the vectors $\epsilon_1, \dots, \epsilon_m$ is contained in the nullspace of $\mathbf{\Lambda}$. These spaces are of equal dimension; the claim follows. $\square$

2.2. **The Neumann-to-Dirichlet Map.** Although the boundary currents ψ are uniquely determined by the given boundary electric potentials $\mathbf{x}$, the reverse is not true. In order to ensure the uniqueness of $\mathbf{x}$ given ψ, we require that the sum of the electric potential of the boundary vertices of each connected component of G be equal to zero. This condition can be represented symbolically as

$$\sum_{p \in \partial V_i} \mathbf{u}(p) = 0 \text{ for all } 1 \leq i \leq m$$

where ∂V_i refers to the boundary vertices of G_i. Define ϵ_i to be a column vector of boundary data equal to the characteristic function of ∂V_i, and define ϵ to be the matrix whose columns are ϵ_i. Then, ${\epsilon_i}^T \mathbf{x} = 0$ for all $1 \leq i \leq m$; that is,

$$\epsilon^T x = 0. \tag{3}$$

Expressing (2) and (3) in matrix form,

$$\begin{bmatrix} \mathbf{\Lambda} \\ \epsilon^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} \psi \\ 0 \end{bmatrix}.$$

Multiplying on the left by $\begin{bmatrix} \mathbf{\Lambda} & \epsilon \end{bmatrix}$, we have

$$\begin{aligned} \begin{bmatrix} \mathbf{\Lambda} & \epsilon \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda} \\ \epsilon^T \end{bmatrix} \mathbf{x} &= \mathbf{\Lambda}\psi \\ \Rightarrow \left(\mathbf{\Lambda}^2 + \mathbf{E}\right) x &= \mathbf{\Lambda}\psi. \end{aligned} \tag{4}$$

Lemma 2.6. The matrix $\mathbf{\Lambda}^2 + \mathbf{E}$ is invertible.

Proof. Note that

$$\begin{aligned}\left(\mathbf{\Lambda}^2+\mathbf{E}\right)\mathbf{x} &= \left[\begin{array}{cc}\mathbf{\Lambda} & \epsilon\end{array}\right]\left[\begin{array}{c}\mathbf{\Lambda}\\ \epsilon^T\end{array}\right]\mathbf{x}=0\\ \Rightarrow \mathbf{x}^T\left[\begin{array}{cc}\mathbf{\Lambda} & \epsilon\end{array}\right]\left[\begin{array}{c}\mathbf{\Lambda}\\ \epsilon^T\end{array}\right]\mathbf{x}&=0\\ \Rightarrow \|\mathbf{\Lambda}\mathbf{x}\|^2+\sum_{1\le i\le m}\|\epsilon_i^T\mathbf{x}\|^2&=0.\end{aligned}$$

The equality $\mathbf{\Lambda}\mathbf{x}=0$ holds if and only if $\mathbf{x}$ is constant on connected components, and $\epsilon_i^T\mathbf{x}=0$ if and only if the entries of $\mathbf{x}$ corresponding to vertices in ∂V_i sum to zero. The only vector that satisfies both of these conditions is the zero vector; the claim follows. □

We conclude $\mathbf{x}=\left(\mathbf{\Lambda}^2+\mathbf{E}\right)^{-1}\mathbf{\Lambda}\psi\equiv\mathbf{H}\psi$ where $\mathbf{H}$ is *a* Neumann-to-Dirichlet Map in that it acts on currents producing electric potentials which generate these respective currents.

Theorem 2.7. *The kernels of the response matrices* $\mathbf{\Lambda}$ *and* $\mathbf{H}$ *are identical.*

Proof. Since $\mathbf{\Lambda}^2+\mathbf{E}$ is invertible, $\mathbf{H}\xi=\left(\mathbf{\Lambda}^2+\mathbf{E}\right)^{-1}\mathbf{\Lambda}\xi=0$ if and only if $\mathbf{\Lambda}\xi=0$. □

It follows that the vectors ϵ_i form a basis for the kernel of $\mathbf{H}$.

Theorem 2.8. *The Neumann-to-Dirichlet Map is symmetric.*

Proof. Let $\mathbf{H}=\left(\mathbf{\Lambda}^2+\mathbf{E}\right)^{-1}\mathbf{\Lambda}$ be the Neumann-to-Dirichlet Map for some electrical network, $\Gamma=(G,\gamma)$.

Since rows and columns of $\mathbf{E}$ are in the kernel of $\mathbf{\Lambda}$,

$$\begin{aligned}\left(\mathbf{\Lambda}^2+\mathbf{E}\right)\mathbf{\Lambda}&=\mathbf{\Lambda}^3=\mathbf{\Lambda}\left(\mathbf{\Lambda}^2+\mathbf{E}\right)\\ \Rightarrow \mathbf{\Lambda}\left(\mathbf{\Lambda}^2+\mathbf{E}\right)^{-1}&=\left(\mathbf{\Lambda}^2+\mathbf{E}\right)^{-1}\mathbf{\Lambda}\end{aligned}\tag{5}$$

Since the inverse of a transpose is the transpose of an inverse, the square of a transpose is the transpose of a square, and $\mathbf{E}$ is symmetric,

$$\mathbf{H}=\left(\mathbf{\Lambda}^2+\mathbf{E}\right)^{-1}\mathbf{\Lambda}=\mathbf{\Lambda}\left(\mathbf{\Lambda}^2+\mathbf{E}\right)^{-1}=\mathbf{H}^T.$$ □

Notice that in (4), multiplying any columns of ϵ by nonzero scalars will have no effect on our imposed stipulation, (3), due to its homogeneity, and the above argument will still be sound. Let us consider the case on the connected graph (where $\epsilon=\mathbf{e}$) where an arbitrary scalar, α, is introduced. Then let $\mathbf{H}_\alpha=\left(\mathbf{\Lambda}^2+\alpha^2\mathbf{E}\right)^{-1}\mathbf{\Lambda}$, which is an equally valid Neumann-to-Dirichlet Map.

Theorem 2.9. *The equality* $\mathbf{H}_\alpha=\mathbf{H}$ *holds for all* $\alpha\neq 0$.

Proof. Let's take the derivative of $\mathbf{H}_\alpha$ with respect to α.

$$\begin{aligned}\frac{d}{d\alpha}\mathbf{H}_\alpha&=\frac{d}{d\alpha}\left(\mathbf{\Lambda}^2+\alpha^2\mathbf{E}\right)^{-1}\mathbf{\Lambda}\\ &=-\left(\mathbf{\Lambda}^2+\alpha^2\mathbf{E}\right)^{-1}(2\alpha\mathbf{E})\left(\mathbf{\Lambda}^2+\alpha^2\mathbf{E}\right)^{-1}\mathbf{\Lambda}\\ &=-\left(\mathbf{\Lambda}^2+\alpha^2\mathbf{E}\right)^{-1}(2\alpha\mathbf{E})\mathbf{\Lambda}\left(\mathbf{\Lambda}^2+\alpha^2\mathbf{E}\right)^{-1}\\ &=0,\end{aligned}$$

since $\mathbf{\Lambda}$ commutes with $\left(\mathbf{\Lambda}^2+\alpha^2\mathbf{E}\right)^{-1}$ by the same reasoning as in (5), and because the rows and columns of $\mathbf{E}$ are in the kernel of $\mathbf{\Lambda}$. Thus we may conclude that $\mathbf{H}_\alpha = \left(\mathbf{\Lambda}^2+\alpha^2\mathbf{E}\right)^{-1}\mathbf{\Lambda} = \left(\mathbf{\Lambda}^2+\mathbf{E}\right)^{-1}\mathbf{\Lambda} = \mathbf{H}$ for all $\alpha \neq 0$. □

Observe that this argument is generalizable to arbitrary bounded graphs.

Theorem 2.10. *The equality* $\mathbf{\Lambda} = \left(\mathbf{H}^2+\mathbf{E}\right)^{-1}\mathbf{H}$ *holds.*

Proof. For the sake of understanding, we will assume the equality $\mathbf{\Lambda} = \left(\mathbf{H}^2+\mathbf{E}\right)^{-1}\mathbf{H}$ deduce an indisputably true statement through a chain of if-and-only-if correspondences. Supposing that $\mathbf{H} = \left(\mathbf{\Lambda}^2+\mathbf{E}\right)^{-1}\mathbf{\Lambda}$, consider the following chain of equivalent statements.

$$\begin{aligned}
\mathbf{\Lambda} &= \left(\mathbf{H}^2+\mathbf{E}\right)^{-1}\mathbf{H} \\
\Leftrightarrow \mathbf{\Lambda} &= (((\mathbf{\Lambda}^2+\mathbf{E})^{-2}\mathbf{\Lambda}^2+\mathbf{E})^{-1}(\mathbf{\Lambda}^2+\mathbf{E})^{-1}\mathbf{\Lambda}) \\
\Leftrightarrow \mathbf{\Lambda} &= ((\mathbf{\Lambda}^2+\mathbf{E})^{-2}\mathbf{\Lambda}^2+\mathbf{E}^2)^{-1}\mathbf{\Lambda} \\
\Leftrightarrow ((\mathbf{\Lambda}^2+\mathbf{E})^{-1}\mathbf{\Lambda}^2+\mathbf{E}^2)\mathbf{\Lambda} &= \mathbf{\Lambda} \\
\Leftrightarrow (\mathbf{\Lambda}^2+\mathbf{E})^{-1}\mathbf{\Lambda}^3 &= \mathbf{\Lambda} \\
\Leftrightarrow \mathbf{\Lambda}^3 &= \mathbf{\Lambda}(\mathbf{\Lambda}^2+\mathbf{E}),
\end{aligned}$$

which is obviously true. The claim follows. □

Let $D = \{\xi : \epsilon^T\xi = 0\}$ and let n be the number of boundary vertices for a given electrical network. Since the boundary currents produced by $\mathbf{\Lambda}$ always sum to zero, we may consider $\mathbf{\Lambda}$ to be a map from $\mathbb{R}^n \to D$. That is, $\mathbf{\Lambda} : \mathbb{R}^n \to D$. If we restrict the domain of $\mathbf{\Lambda}$ to D, then $\mathbf{\Lambda}|_D : D \to D$ is an invertible map; denote its inverse by $\mathbf{H}$. Notice that $\mathbf{H}$ is not defined on all of $\mathbb{R}^n$, but we may extend its domain to include vectors that correspond to illegal currents (ones disobeying Kirchhoff's law) where $\mathbf{H}$ orthogonally projects such a vector to a legal current vector. This is done by requiring that $\mathbf{H}\epsilon = 0$ and extending $\mathbf{H}$ linearly.

3. Properties of Neumann-to-Dirichlet Map

Lemma 3.1. The Neumann-to-Dirichlet Map is positive semi-definite.

Proof. We know that $\mathbf{\Lambda}$, the response matrix for any given electrical network, $\Gamma = (G, \gamma)$, is positive semi-definite. Therefore,

$$\begin{aligned}
\psi^T\mathbf{H}\psi &= \mathbf{x}^T\psi \\
&= \mathbf{x}^T\mathbf{\Lambda}\mathbf{x} \geq 0.
\end{aligned}$$

□

Theorem 3.2. *The determinants of principal proper submatrices of the Neumann-to-Dirichlet Map for connected networks are positive.*

Proof. Since $\mathbf{H}$ is positive semi-definite, the determinant of any principal proper submatrix of $\mathbf{H}$ is non-negative. It remains to be shown that the determinant of every principal proper submatrix is non-zero. Let $P \subsetneq \partial V$ and $S = \partial V \setminus P$ Assume to the contrary that $\det \mathbf{H}(P;P) = 0$. Then there exists a non-zero $\mathbf{x}$ such that

$$\begin{bmatrix} \mathbf{H}_{PP} & \mathbf{H}_{PS} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix} = 0,$$

which implies that there exists a non-zero $\mathbf{y}$ such that

$$\begin{aligned}\begin{bmatrix}\mathbf{H}_{PP} & \mathbf{H}_{PS}\\ \mathbf{H}_{PS}^T & \mathbf{H}_{SS}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ 0\end{bmatrix} &= \begin{bmatrix}0\\ \mathbf{y}\end{bmatrix}\\ \Rightarrow \begin{bmatrix}\mathbf{\Lambda}_{PP} & \mathbf{\Lambda}_{PS}\\ \mathbf{\Lambda}_{PS}^T & \mathbf{\Lambda}_{SS}\end{bmatrix}\begin{bmatrix}0\\ \mathbf{y}\end{bmatrix} &= \begin{bmatrix}\mathbf{x}\\ 0\end{bmatrix}\\ \Rightarrow \mathbf{\Lambda}_{SS}\mathbf{y} = 0,\end{aligned}$$

which contradicts the invertibility of principal proper submatrices of $\mathbf{\Lambda}$. The claim follows. □

Lemma 3.3. The diagonal entries of the Neumann-to-Dirichlet Map are positive.

Alternate Proof. $\mathbf{H}_{ij}$ is the jth entry of $\mathbf{H}$ acting on $\mathbf{e}_i$, the ith unit basis vector for $\mathbb{R}^n$. We must orthogonally project $\mathbf{e}_i$ to produce a legal current vector (one obeying Kirchhoff's Current Law on each connected component). For the sake of unobtrusive notation, we will consider a connected graph, and the argument will generalize to bounded graphs.

$$\mathbf{H}\mathbf{e}_i = \mathbf{H}(\mathbf{e}_i - \alpha\mathbf{e})$$

In order to have a legal current, $1 - \alpha n = 0$, where n is equal to the number of boundary vertices. This implies that $\alpha = 1/n$. That is, the projection of $\mathbf{e}_i$ into a legal current spaces is

$$(\mathbf{e}_i - \mathbf{e}/n)(j) = \begin{cases}(n-1)/n & \text{if } j = i\\ -1/n & \text{if } j \neq i\end{cases}$$

For the sake of generalization, if G were bounded and not necessarily connected where $i \in \partial V_k$ and $n_k = |\partial V_k|$, the projection into a legal current space would be

$$\begin{cases}(n_k-1)/n_k & \text{if } j = i\\ -1/n_k & \text{if } j \neq i \text{ but } j \in \partial V_k\\ 0 & \text{if } j \notin \partial V_k\end{cases}$$

If the maximum electric potential were to occur at a vertex other than i, then the current from this vertex would be positive into the rest of the graph, a contradiction. Therefore, the maximum electric potential must occur at the ith vertex. Due to our requirement that the electric potentials on the boundary sum to zero, (3), the electric potential at the ith vertex must be positive. □

Corollary 3.4. *For $1 \leq i \leq n$, $\eta_{ii} > \eta_{ij}$ for all $j \neq i$.*

Lemma 3.5. The Neumann-to-Dirichlet Map is not necessarily a Kirchhoff matrix.

Proof. If the Neumann-to-Dirichlet Map, $\mathbf{H}$, were a Kirchhoff matrix, then

$$\begin{aligned}\mathbf{H}_{ij} > 0 \text{ for } i = j\\ \mathbf{H}_{ij} \leq 0 \text{ for } i \neq j\end{aligned}$$

Note that extending the previous argument to show that the off-diagonal entries of the Neumann-to-Dirichlet Map are negative is impossible. Consider the following counterexample.

In Figure 1 the currents, denoted in parentheses, are consistent with the previously prescribed method while the electric potentials do not observe the required

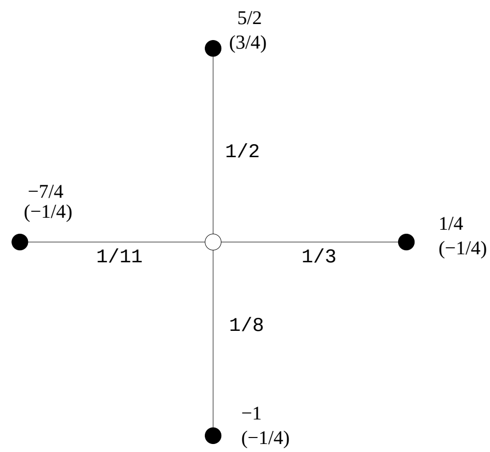

FIGURE 1. Electrical network of a $\bigstar_4$ graph with vertex labels to the left of/below the vertices (boundary vertices are solid black while the interior vertex is white), current in parentheses, electric potential to the right of current, and conductivities to the left of/below the edges connecting the vertices.

sign conventions of a Kirchhoff matrix i.e. the electric potential on vertex four is positive. Thus, in general, $\mathbf{H}$ is not a Kirchhoff matrix. □

4. Entry Relationships

4.1. Generalizations of Relationships.

Definition 4.1. There exists a *generalized path*, $p \leftrightarrow q$, between two boundary vertices p and q if and only if there exists a sequence of vertices in G whose edges join p to q.

Definition 4.2. If $P = (p_1, p_2, ..., p_k)$ and $Q = (q_1, q_2, ..., q_k)$ are sets of boundary vertices, then there exist a *generalized k-connection* from P to Q if and only if there exist vertex-disjoint generalized paths $\{p_i \leftrightarrow q_i\}$.

Note that most literature refer to paths as being sequences of interior vertices, and subsequently, k-connections are defined in terms of these paths. Note that generalized paths do not distinguish between interior and boundary vertices, and that generalized k-connections are defined in terms of generalized paths.

Theorem 4.3. *If the pq-th entry in the Neumann-to-Dirichlet Map, η_{pq}, is not equal to zero, then there exists a generalized path between p and q, $p \leftrightarrow q$.*

Proof by Contrapositive. Assume that there does not exist a generalized path between p and q. That is, the connected component containing p and that containing q are distinct. Observe that η_{pq} reflects the effect of the current at q on the voltage at p. Since p and q lie in distinct connected components, they are effectively independent of one another; it follows that $\eta_{pq} = 0$. □

Definition 4.4. Given a graph $G = (V, E)$ and two disjoint sets of vertices $P, Q \subset V$, a cutset of the pair $(P; Q)$ is a set $R \subset V$ such that for every $p \in P$ and $q \in Q$,

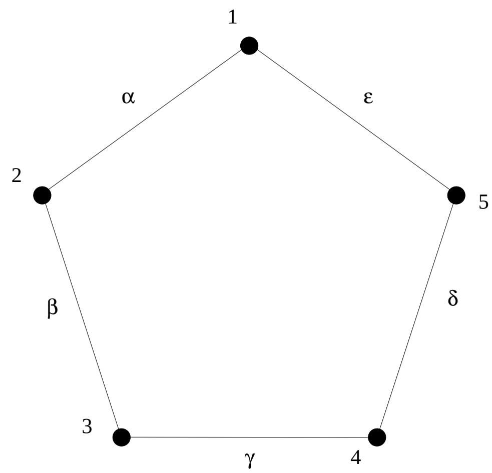

FIGURE 2. The pentagon graph demonstrates that we should consider generalized connection, and not just connections through the interior.

every path from p to q intersects R. A minimal cutset of the pair $(P;Q)$ is a cutset of P and Q of minimal cardinality among all such cutsets.

So, a cutset R of a pair $(P;Q)$ can be thought of as a set of vertices such that, when R is removed from G, along with any edges that had included any vertex $v \in R$, the resulting graph has the property that for every $p \in P$ and $q \in Q$, there is no generalized connection between p and q. Using this definition, we can prove an interesting theorem.

Theorem 4.5. *Suppose that $\Gamma = (G, \gamma)$ is a resistor network, and that there does not exist a generalized 2-connection between $P = (i,j) \subset V$ and $Q = (k,l) \subset V$ with $P \cap Q = \emptyset$. Then $\eta_{ik} + \eta_{jl} = \eta_{il} + \eta_{jk}$.*

Proof. Equivalently to saying that there exists no generalized 2-connection between P and Q, we can say that the minimal cutset R between P and Q in G is of cardinality 1. Assume that R can be chosen such that $R \cap P = R \cap Q = \emptyset$; if not, the claim follows trivially.

We will now construct two sets of vertices $V_1, V_2 \subset V$. Let V_1 contain all vertices v, with the condition that there exists a path p, either from k to R or from l to R, such that v is an element of this path. Let $V_2 = V \setminus V_1$. Consider the Neumann problem on $G_1 = (V_1, E_1)$, where $E_1 \subset E$ is formed by removing all elements of E that do not include any element of V_1, with boundary current vector $\psi_1 = [\delta_{n,k} - \delta_{n,l}]$. Note that since the sum of the elements of ψ_1 is zero, it is a valid current vector. Requiring the normalization on boundary voltages employed in this paper, there exists a unique voltage $\mathbf{u}_1$, defined on V_1, such that $I_{\mathbf{u}_1}|_{\partial V_1} = \psi_1$.

Consider next the Neumann problem on G, with boundary current vector $\psi = [\delta_{n,k} - \delta_{n,l}]$. That is, ψ is simply ψ_1 on ∂V_1 and identically zero on ∂V_2. Requiring the same normalization as before, this problem is well-posed. I claim that

$$\mathbf{u}(v) = \begin{cases} \mathbf{u}_1(v), & v \in V_1 \\ \mathbf{u}_1(R), & v \in V_2 \end{cases}$$

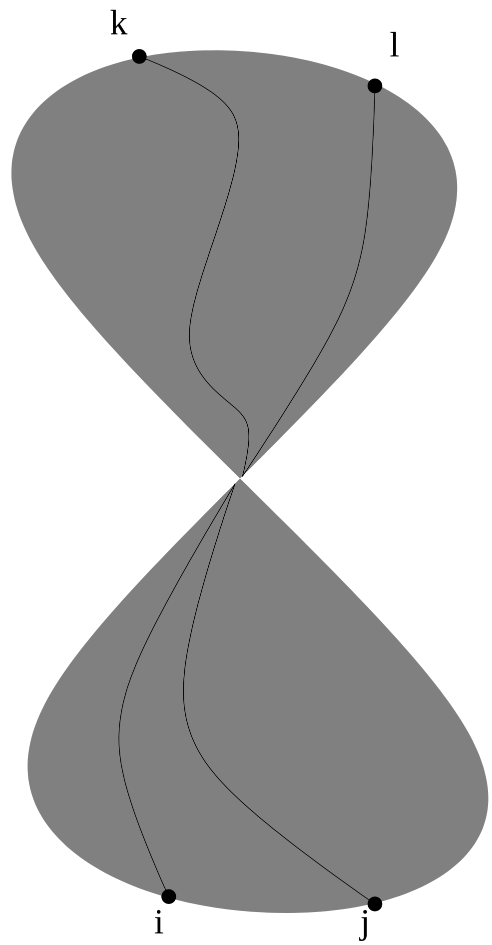

FIGURE 3. Say that there is a missing 2-connection between (i, j) and (k, l). We can then visualize our network in this manner.

solves this Neumann problem. This requires a bit of work. Say that this potential function is imposed on G. Consider the following lemma:

Lemma 4.6. The current flowing into R from the graph V_1 is zero.

Proof. If R is an interior node, then this follows easily from Kirchhoff's law, since $\mathbf{u}_1$ solves the Neumann problem on G_1. If R is a boundary node, then this follows from the requirement that $I_{\mathbf{u}_1}$ is identically zero on $V_1 \setminus \{k, l\}$. $\square$

It follows that no current flows into V_2, so Kirchhoff's law is be satisfied on the interior of V_2. Since $\mathbf{u}_1$ satisfies the Neumann problem on G_1, Kirchhoff's law is also satisfied on the interior of V_1. By definition, $I_{\mathbf{u}}$ assumes the correct boundary values. It follows, provided that $\mathbf{u}$ is appropriately normalized, that $\mathbf{u}$ solves the Neumann problem on G.

As previously mentioned, the Neumann problem on G is well-posed. Since $\mathbf{u}$ is a valid solution of this problem, $\mathbf{u}$ is the unique solution of this problem. It follows that for the imposed current vector ψ, $\mathbf{u}_i = \mathbf{u}_j$. In terms of the elements of the

Neumann-to-Dirichlet map, $\mathbf{u}_i = \eta_{ik} - \eta_{il}$, while $\mathbf{u}_j = \eta_{jk} - \eta_{jl}$. It follows that $eta_{ik} - \eta_{il} = \eta_{jk} - \eta_{jl}$, which implies that $\eta_{ik} + \eta_{jl} = \eta_{il} + \eta_{jk}$. □

This is a somewhat strange relation — it really doesn't feel as straightforward as the determinantal results for $\mathbf{\Lambda}$ felt. It turns out that there is a determinantal generalization of this result. Bizarre!

Theorem 4.7. *Suppose that $\Gamma = (G, \gamma)$ is a resistor network, and that there does not exist a generalized k-connection between $P = (p_1, \ldots, p_k) \subset V$ and $Q = (q_1, \ldots, q_k) \subset V$ with $P \cap Q = \emptyset$. Let $\mathbf{L} = [\ell_{ij}]$ be a matrix of linear relations, where $\ell_{ij} = \left(\eta_{p_i q_j} - \eta_{p_i q_k}\right) - \left(\eta_{p_k q_j} - \eta_{p_k q_k}\right)$. Then $\det \mathbf{L} = 0$.*

Proof. By the assumption that there exists no generalized k-connection between P and Q, the matrix $\mathbf{\Lambda}\left(\partial V \setminus Q; \partial V \setminus P\right)$ is singular. That is, there exists a nonzero vector $\mathbf{x}$ such that

$$\begin{bmatrix} \mathbf{\Lambda}_{PS} & \mathbf{\Lambda}_{PQ} \\ \mathbf{\Lambda}_{SS} & \mathbf{\Lambda}_{SQ} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $[\mathbf{x}_1\ \mathbf{x}_2]^T$ is the appropriate partitioning of $\mathbf{x}$ and $S = \partial V - (P \cup Q)$. It follows that

$$\begin{bmatrix} \mathbf{\Lambda}_{PP} & \mathbf{\Lambda}_{PS} & \mathbf{\Lambda}_{PQ} \\ {\mathbf{\Lambda}_{PS}}^T & \mathbf{\Lambda}_{SS} & \mathbf{\Lambda}_{SQ} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Finally, it follows that

$$\begin{bmatrix} \mathbf{\Lambda}_{PP} & \mathbf{\Lambda}_{PS} & \mathbf{\Lambda}_{PQ} \\ {\mathbf{\Lambda}_{PS}}^T & \mathbf{\Lambda}_{SS} & \mathbf{\Lambda}_{SQ} \\ {\mathbf{\Lambda}_{PQ}}^T & {\mathbf{\Lambda}_{SQ}}^T & \mathbf{\Lambda}_{QQ} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{y} \end{bmatrix},$$

where $\mathbf{y} = {\mathbf{\Lambda}_{SQ}}^T \mathbf{x}_1 + \mathbf{\Lambda}_{QQ} \mathbf{x}_2$. Rewriting this in terms of the Neumann-to-Dirichlet map, we have

$$\begin{bmatrix} \mathbf{H}_{PP} & \mathbf{H}_{PS} & \mathbf{H}_{PQ} \\ {\mathbf{H}_{PS}}^T & \mathbf{H}_{SS} & \mathbf{H}_{SQ} \\ {\mathbf{H}_{PQ}}^T & {\mathbf{H}_{SQ}}^T & \mathbf{H}_{QQ} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + b\mathbf{e}, \tag{6}$$

where b is a real constant and $\mathbf{e}$ is a vector of ones of appropriate length. (Note that this constant vector is necessary, since while $\mathbf{H}$ maps into the space of vectors with element sum zero, the element sum of $\mathbf{x} = [\mathbf{x}_1\ \mathbf{x}_2]^T$ is not necessarily zero.) By the first line of (6), $\mathbf{H}_{PQ}\mathbf{y} = b\mathbf{e}$, where $\mathbf{e}$ is again a vector of ones of appropriate length. Furthermore, not all of the elements of $\mathbf{y}$ are 0, for if $\mathbf{y}$ was composed entirely of zeros, then $\mathbf{x}$ would also be composed entirely of zeros, contradicting the fact that $\mathbf{x}$ is non-zero.

Since $\mathbf{y}$ is a current vector, its elements sum to zero. It follows that we can express y_n as $-\sum_{i=1}^{n-1} y_i$; thus, it follows from the previous equation that

$$\widetilde{\mathbf{H}_{PQ}}\widetilde{\mathbf{y}} = b\mathbf{e},$$

where $\widetilde{\mathbf{H}_{PQ}}$ is the $k \times (k-1)$ matrix with ij-th element equal to $\eta_{p_i q_j} - \eta_{p_i q_k}$, and $\mathbf{y}$ is the vector $(y_1\ \ldots\ y_{n-1})^T$. By subtracting the k-th row from all other row in this matrix equation, we are finally left with the equation

$$\mathbf{L}\widetilde{\mathbf{y}} = \mathbf{0},$$

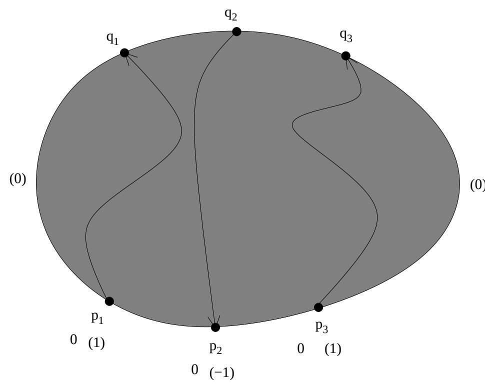

FIGURE 4. Say that there is a missing 2-connection between (i, j) and (k, l). We can then visualize our network in this manner.

where $\mathbf{L}$ is the $(k-1)\times(k-1)$ matrix with ij-th entry $\left(\eta_{p_i q_j} - \eta_{p_i q_k}\right) - \left(\eta_{p_k q_j} - \eta_{p_k q_k}\right)$. Recall that not all of the elements of $\mathbf{y}$ are zero. Further, it cannot be that the only nonzero entry of $\mathbf{y}$ is the n-th entry, since the sum over the elements of $\mathbf{y}$ is zero. It follows that $\widetilde{\mathbf{y}}$ is not composed entirely of zeros. That is, the kernel of $\mathbf{L}$ is nontrivial. The claim follows. $\square$

4.2. **Some Useful Inequalities.** Using current-pattern arguments similar to those of the previous section, we can produce inequalities strikingly similar to the inequalities described in [Morrow's paper] for the Dirichlet-to-Neumann map. Consider the following theorem.

Theorem 4.8. *Suppose that $\Gamma = (G, \gamma)$ is a resistor network, and that there exists a generalized k-connection between $P = (p_1, \ldots, p_k) \subset V$ and $Q = (q_1, \ldots, q_k) \subset V$ with $P \cap Q = \emptyset$. Then $\eta_{p_1 q_1} - \eta_{p_1 q_2} - (\eta_{p_2 q_1} - \eta_{p_2 q_2}) > 0$.*

Let's impose the following boundary conditions: the current on $\partial V - (P \cup Q)$ is identically zero; the current at p_1 is equal to 1; the current at p_2 is -1; and the voltage on P is identically zero. Recall a theorem from [Nate's other paper]:

Theorem 4.9. *Consider a network $\Gamma = (G, \gamma)$ with boundary. Partition the boundary ∂V of G into three disjoint subsets, N_B, N_C, and N_N, where $|N_B| = |N_N|$. Suppose that ϕ_B defines a voltage function on N_B, ϕ_V defines a voltage function on N_V, and ψ_B defines a current function on N_B. There exists a unique γ-harmonic potential u, defined on V, such that $u|_{N_B} = \phi_B$, $u|_{N_V} = \psi_B$, and $I|_{N_C} = \psi_C$ if and only if the submatrix $\mathbf{\Lambda}(N_B \cup N_C; N_C \cup N_N)$ of the response matrix is nonsingular.*

Since we are working on a circular planar network, and since there exists a generalized k-connection between P and Q, the matrix $\mathbf{\Lambda}(N_B \cup N_C; N_C \cup N_N) = \mathbf{\Lambda}(\partial V \setminus Q; \partial V \setminus P)$ is nonsingular. It follows that these boundary conditions induce a well-posed problem. By Lemma [write such a lemma], there exists $A > 1$ such that $q_1 = -A$ and $q_2 = A$.

Normalize the potential on V such that the boundary voltages sum to zero. This normalization preserves the equality $\mathbf{u}_{p_1} = \mathbf{u}_{p_2}$. In terms of the Neumann-to-Dirichlet map, $\mathbf{u}_{p_1} = \eta_{p_1 p_1} - \eta_{p_1 p_2} + A\left(\eta_{p_1 q_2} - \eta_{p_1 q_1}\right)$ and $\mathbf{u}_{p_2} = \eta_{p_2 p_1} - \eta_{p_2 p_2} +$

$A\left(\eta_{p_2q_2} - \eta_{p_2q_1}\right)$; equating these voltages, it follows that

$$(\eta_{p_1p_1} - \eta_{p_1p_2}) + (\eta_{p_2p_2} - \eta_{p_1p_2}) = A\left((\eta_{p_1q_1} - \eta_{p_1q_2}) - (\eta_{p_2q_1} - \eta_{p_2q_2})\right). \tag{7}$$

By Corollary 3.4, the left side of (7) is positive; since A is positive, the claim follows.