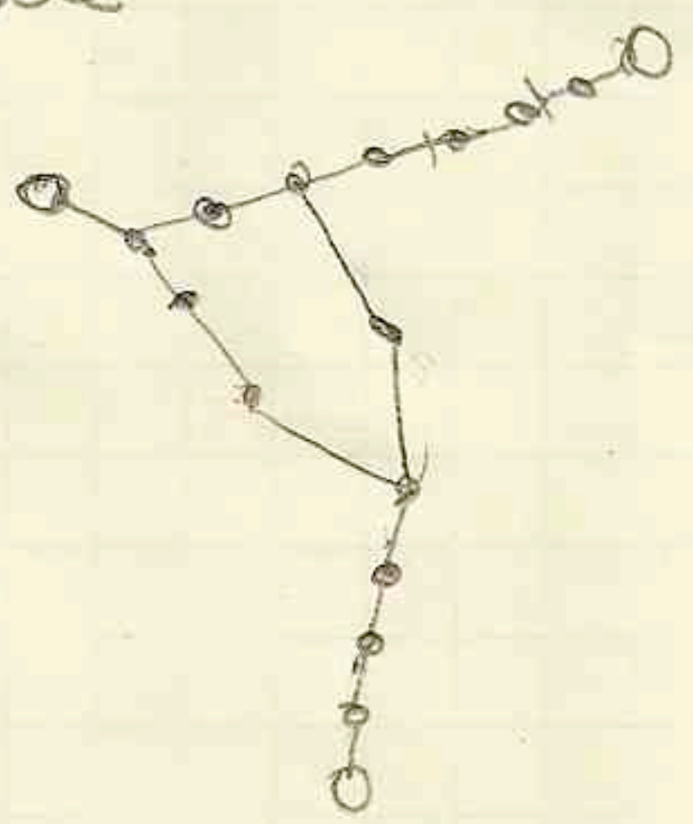
The model:

Given a graph with vertices and undirected edges $(V, E)$ where the vertices are partitioned into interior vertices $I$ and boundary vertices $\partial$ which are connected to an single interior vertex as a boundary spike

o— boundary node

•— interior node



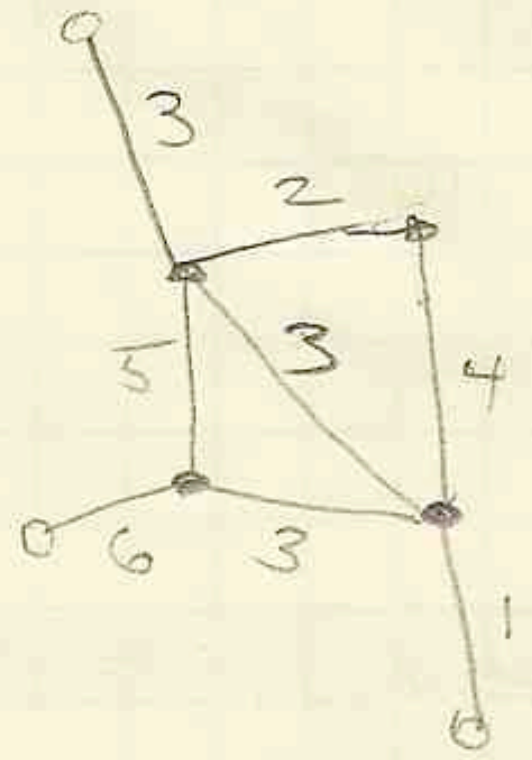
G

A particle is fed into the graph at one of the boundary vertices and made to step to the interior vertex and then allowed to random walk starting there one vertex at a time assuming equal probability to each avaible direction at any given time. The random walk ends when the particle reaches a boundary vertex. In this particular problem, we keep ~~track~~ the number of steps the random walk takes and the boundary vertex it exits at.

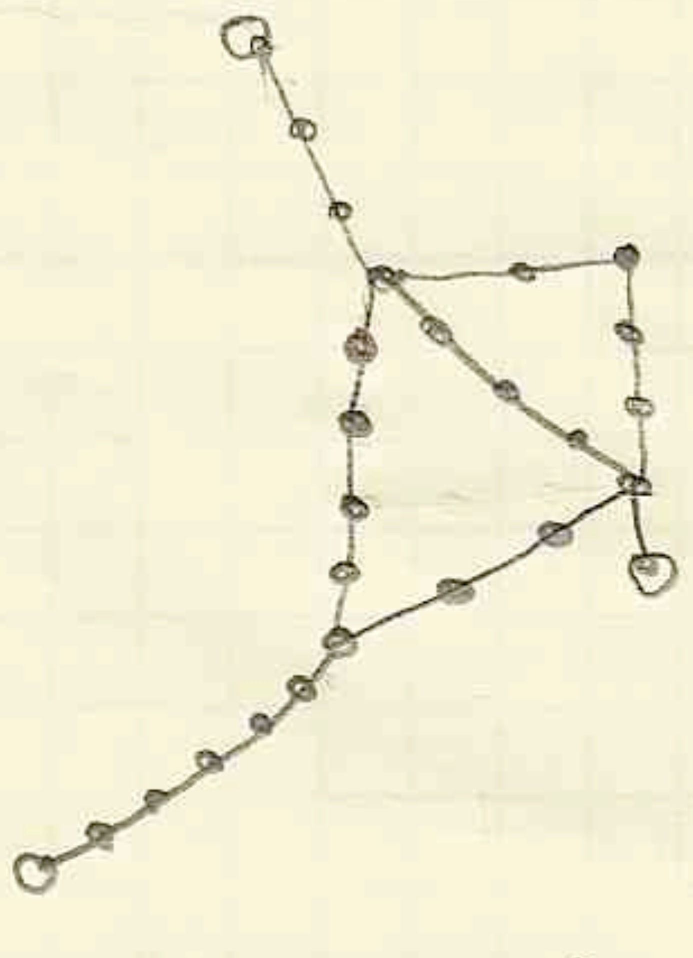
This is intended to model the behaviour of particle introduced at the boundary node of a resistance network and allowed to diffuse, as electron actually does, with the potential in the netwokle off.

My model imagines the network as being built of unit resistors attached in series to model any resistor connecting to nodes.

For example, the electric network with such labeled resistances as these:




might be modeled with the graph:

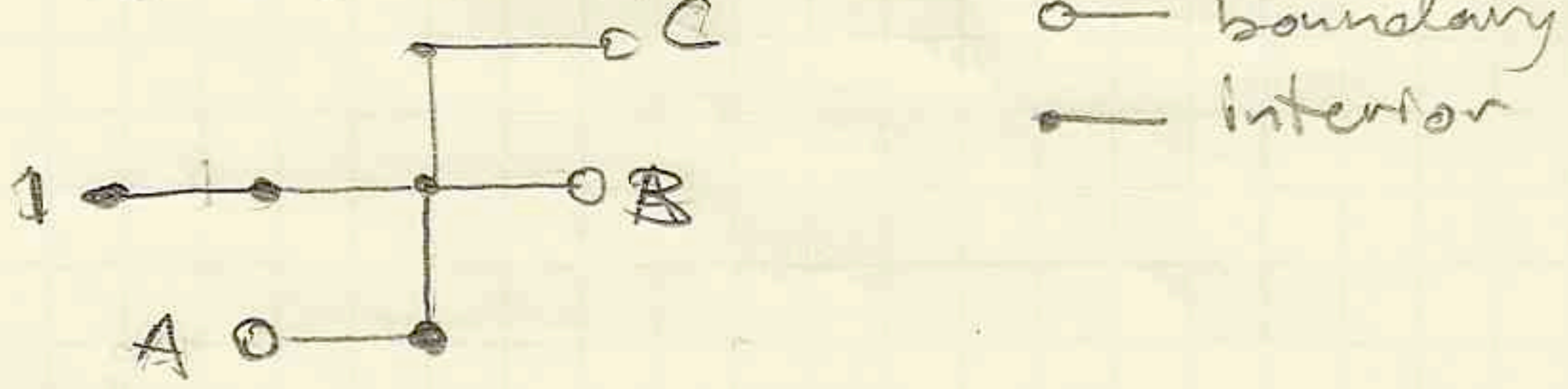


The motivation for doing this is to create a metric for measuring the "distance" between boundary node that, for, the purposes of inverse problems might improve upon typical might of shortest distance

In Megan McCormick's paper of 2005, one of the problems she contended with trying to solve the inverse problem she designed:

[If given the lengths of the shortest distances between boundary nodes, what can one deduce about the graph.]

was that the shortest path metric only provided information about the shortest path.

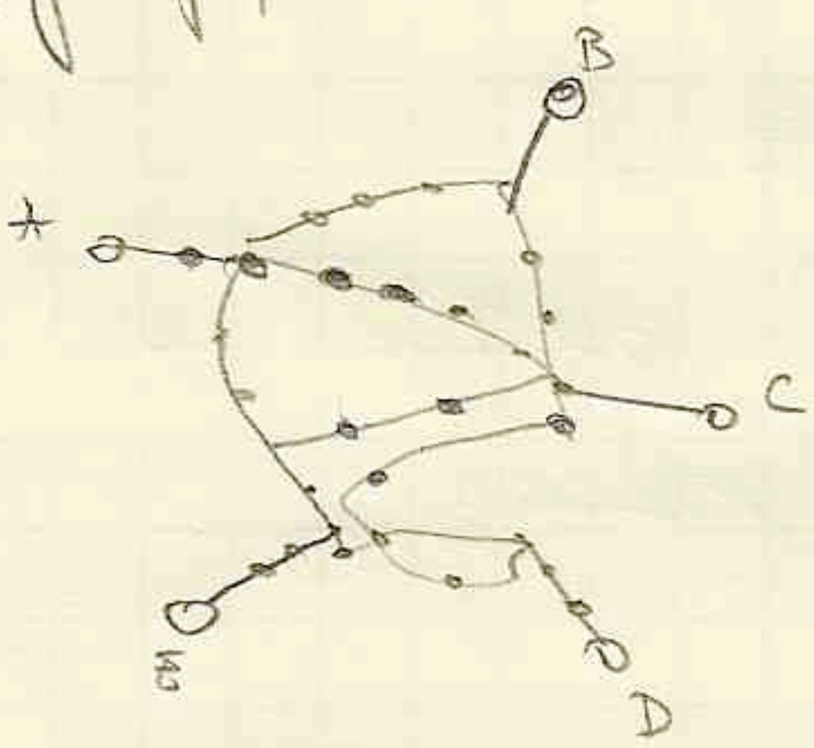



In this graph, shortest distance information given between boundary nodes A, B, C would provide no clue about the presence of the ~~the~~ edges leading to vertex 1.

It is hoped that using my proposed metric instead in which we measure the expected number of steps a random walk would require to get from each boundary vertex to each other boundary vertex, the presence of the edges leading to vertex I, could be detected as they would increase the expected number of step to get from one boundary vertex to another.

Let us be more precise

For a graph G




With boundary vertices, $\{A, B \ldots\}$

Let the space of outcomes be all possible paths from boundary node A to any boundary node (including A)

The probability of a given path that leaves the series of vertices $\{A, i_1, i_2 \ldots\ldots i_k\}$ is

$\frac{1}{d_A}\left(\frac{1}{d_{i_1}}\right)\left(\frac{1}{d_{i_2}}\right)\cdots\left(\frac{1}{d_{i_k}}\right)$ where $d_i$ is the degree of vertex $i$ representing the number different option for the particle at that point.

Let $X_A$(path) be the a random variable that represents the number of steps in a path leaving A and exiting any where

Let $X_{AB}$ (path) be the number of step in a path leaving A and exiting B and $X_{AB}$ (path) is zero for paths leaving A and exiting somewhere other than B.

$$E[X_A] = \sum_{\substack{\text{all paths} \\ p}} X_A(p)\,\text{prob}(p)$$

is the number of steps in a random walk starting at A ending at any boundary vertex

$E[X_{AB}]$ is, then, the portion of the above sum that comes from paths that begin at A and end at B.

My proposed metric $d(A,B)$ is a conditional expectation. Assuming a random walk gets from A to B, $d(A,B)$ is the expected number of steps in such a walk. If $A \neq B$

$$d(A,B) = \frac{E[X_{AB}]}{P_{AB}}$$

where $P_{AB}$ is the probability a walk beginning at A will end at B.

and

$d(A,A) \equiv 0$ (This additional part of the definition is needed for if we let a particle diffuse into the network at A it may take a long time to diffuse back out at A. This issue will come back to haunt us later)

Claim:

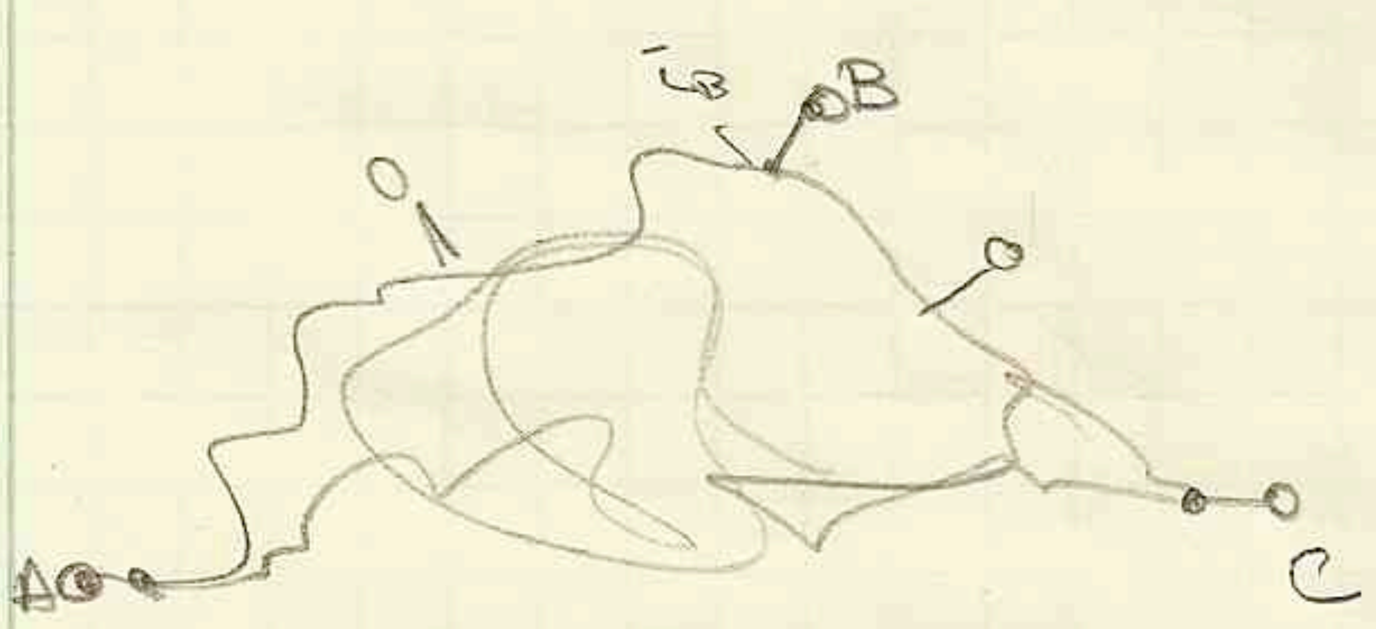
$d(A,B) = d(B,A)$

Any path from A to B leaves vertices $\{A, i_1 \cdots i_k\}$ has probability $\frac{1}{d_A} \cdot \frac{1}{d_{i_1}} \cdots \frac{1}{d_{i_k}} = \frac{1}{d_{i_1}} \cdots \frac{1}{d_{i_k}}$ and length $k+1$ steps. Reversing the path give a path from B to A leaving vertices $\{B, i_k \ldots i_1\}$ and has probability $\frac{1}{d_B} \cdot \frac{1}{d_{i_k}} \cdots \frac{1}{d_{i_1}}$ $= \frac{1}{d_{i_1}} \cdots \frac{1}{d_{i_k}}$ and length $k+1$ so $P_{AB} = P_{BA}$ and $E[X_{AB}] = E[X_{BA}]$

so $d(A,B) = d(B,A)$

Claim: $d(A,C) \leq d(A,B) + d(B,C)$

All the paths that go from A to B and the from B to C are almost paths from A to C




They just have an added trip from $i_B$ to B and back and so have probability reduced by $\frac{1}{d_{i_B}}$

so $P_{AB} \cdot P_{BC} < P_{AC}$

and path from $A \to i_B \to C$ are included in paths from $A \to C$ so

$$E[X_{AC}] \geq \left[ E[X_{AB}] + E[X_{BC}] \right] d_{i_B}$$

$$d(A,B) + d(B,C) = \frac{E[X_{AB}]}{P_{AB}} + \frac{E[X_{BC}]}{P_{BC}}$$

$$= \frac{P_{BC} E[X_{AB}] + P_{AB} E[X_{BC}]}{P_{AB} P_{BC}}$$

since $P_{BC}, P_{AC} < 1$

$$P_{BC} E[X_{AB}] + P_{AB} E[X_{BC}] \leq \left[ E[X_{AB}] + E[X_{BC}] \right] d_{i_B} \leq E[X_{AC}]$$

and since $P_{AB} P_{BC} < P_{AC}$

so

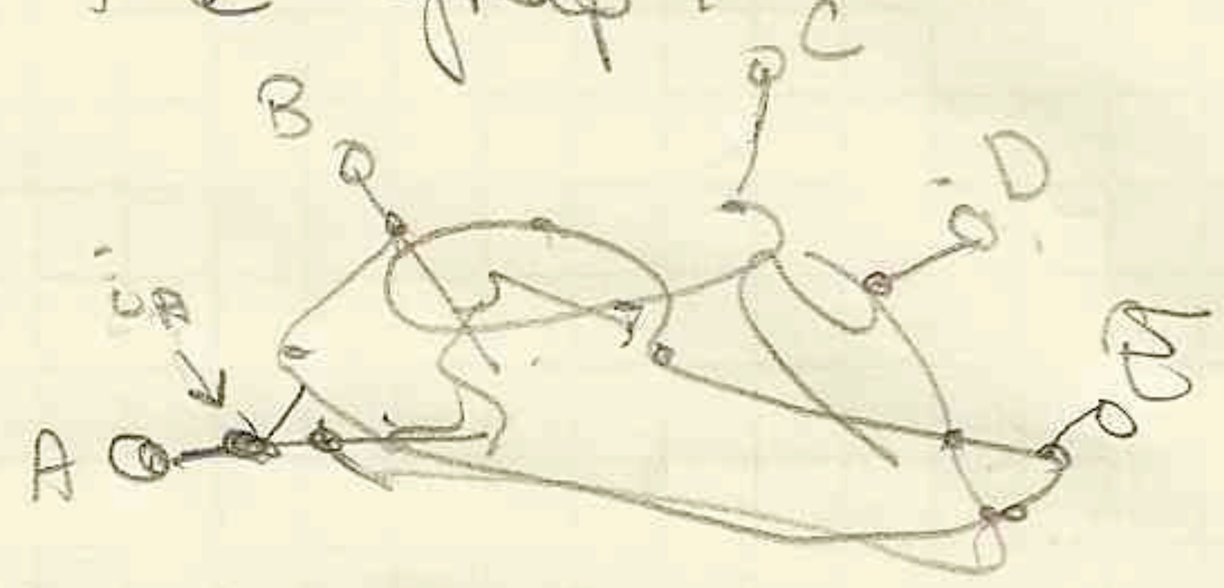
$$\frac{P_{BC} E[X_{AB}] + P_{AB} E[X_{BC}]}{P_{AB} P_{BC}} < \frac{E[X_{AC}]}{P_{AC}}$$

$$d(A,B) + d(B,C) \leq d(B,C)$$

Thus $d(A,B)$ is a metric

So is there an efficient way to calculate $d(A,B)$?
Yes, it is a bit unintuitive.
We need to calculate $P_{AB}$ and $E[X_{AB}]$

$P_{AB}$ is calculated by solving the Dirichlet problem on the graph




It is well know and is well discussed in
Random walks on Electric Networks by Doyle + Snell

that if we set boundary conditions one at B and zero at all the other boundary nodes
The probability of a particle starting a random walk at $i_A$ exiting at B is the value of the solution to the Direchlet problem at $i_B$, $P_{i_B}$
Since our particle diffuses with probability one from A to $i_A$ this is $P_{AB}$
[denote the solution to this boundary value problem at vertex $i$ as $P_{iB}$]

$E[X_{AB}]$ is commonly calculated using Markov chains, I find this approach unintuitive and maybe not a good place to start an inverse problem so I approach finding $E[X_{AB}]$ differently, but Markov chains work well for computers so I will discuss the approach but I wont prove anything. Again, Doyle and Snell is a good reference.

Let $P_{ij}$ be the transition matrix where

$P_{ij}$ is the probability a particle at $i$ move to $j$ after one step. In our case $P_{ij}$ is the equal to $1/d_i$ provide $i$ and $j$ are connected by an edge. Except at $P_{ki}$ where $k$ is a boundary node, there $P_{ki} = 1$ for $i = k$ and $P_{ki} = 0$ for $i \neq k$

In this way, in the particle starts at the $k$th vertex

$$(0, \ldots, \overset{k}{1}, \ldots \overset{m}{0}) P^n = (x_1 \ldots\ldots x_n)$$

$x_i$ is the probability the particle is at the $i$th vertex after $n$ steps

If
$$P = \begin{array}{cc} & \begin{array}{cc} \partial & I \end{array} \\ \begin{array}{c} \partial \\ I \end{array} & \left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & O \\ \hline R & Q \end{array}\right] \end{array}$$

and $N = (I - Q)^{-1}$

Then

$$B = NR$$

and the probability of a particle at the $i^{th}$ vertex exiting eventually at the $j^{th}$ vertex on the boundary is $B_{ij}$

Thus $P_{AB} = P_{i_A B} = B_{i_A B}$

and

$$V = N^2 R$$

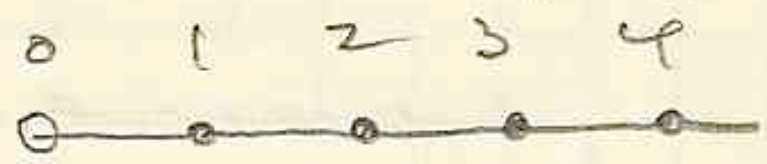
the expected number: $E[X_{AB}] = V_{i_A B}$

Since the transition matrix $P$ keeps particles on boundary vertices where they are we we would need to start a random walk from A to B and $i_A$ for the purposes of Markov Chains. Otherwise this approach works.

## A Different Way

I began by thinking about a problem that turned up again and again

On the graph




Assuming you don't exit at 0, how many steps in a random walk do it take to go from 1 to 2, 2 to 3, 3 to 4 etc.

This is accomplished by summing geometric series

For Example in 2 to 3: To find $P_{23}$

$$\text{prob}(2\to 3) + \text{prob}(2\to 1\to 2\to 3) + \text{pro}(2\to 1\to 2\to 1\to 2\to 3) + \cdots$$

$$= \frac{1}{2} + \frac{1}{2}\frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2} + \cdots$$

$$= \frac{1}{2}\left[1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots\right] = \frac{1}{2}\left[\frac{1}{1-\frac{1}{4}}\right] = \frac{1}{2}\,\frac{4}{3} = \frac{2}{3}$$

In this way we can find

$$\text{prob}(0\to 1) = 1,\ \text{prob}(1\to 2) = \frac{1}{2},\ P(2\to 3) = \frac{2}{3},\ P(3\to 4) = \frac{3}{4}$$

$$P(n-1\to n) = \frac{n-1}{n}$$

To find $E[X_{23}]$:

$$E[X_{23}] = \overset{2\to 3}{\frac{1}{2}(1\text{ step})} + \overset{2\to 1\to 2\to 3}{\frac{1}{2}\left(\frac{1}{4}\right)[3\text{ steps}]} + \overset{2\to 1\to 2\to 1\to 3}{\frac{1}{2}\left(\frac{1}{4}\right)^2[5\text{ steps}]} + \cdots$$

$$= \frac{1}{2}1 + \frac{1}{2}\left(\frac{1}{4}\right)[2\cdot 1+1] + \frac{1}{2}\left(\frac{1}{4}\right)^2[2\cdot 2+1] + \frac{1}{2}\left(\frac{1}{4}\right)^3[2\cdot 3+1]$$

$$= \frac{1}{2}\left[1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots\right] + \frac{1}{2}\left(\frac{1}{4}\right)2\left[1 + 2\cdot\left(\frac{1}{4}\right) + 3\cdot\left(\frac{1}{4}\right)^2 + \cdots\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{1}{4}}\right] + \frac{1}{4}\left[\frac{1}{1-\frac{1}{4}}\right]^2 = \frac{1}{2}\,\frac{4}{3} + \frac{1}{4}\left(\frac{4}{3}\right)^2 = \frac{2}{3} + \frac{4}{9} = \frac{10}{9}$$

$$d(2,3) = \frac{10/9}{(2/3)} = \frac{5}{3}$$

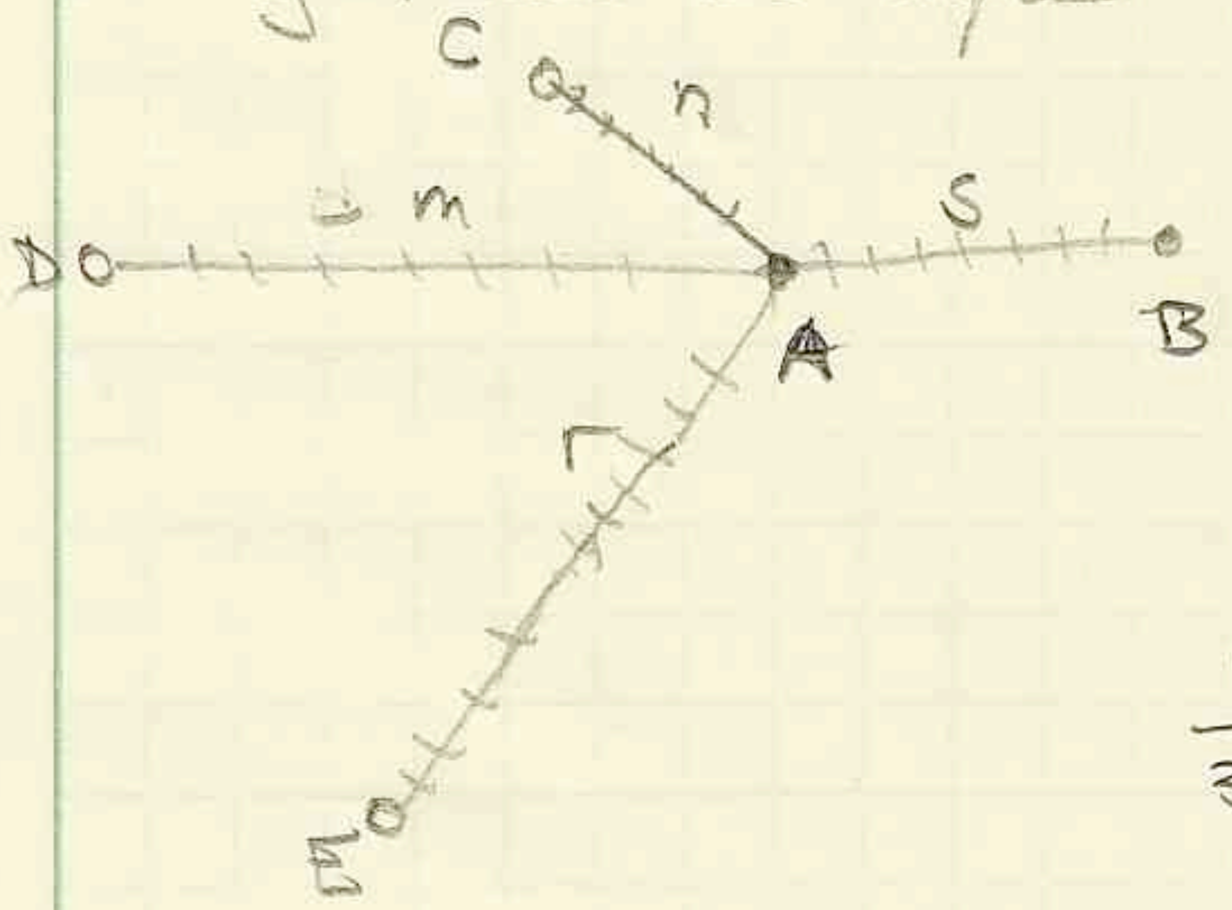
Continuing in this way we get

P: $1 \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \quad \cdots \quad 1-\frac{1}{n}$

d: $1 \quad \frac{3}{3} \quad \frac{5}{3} \quad \frac{7}{3} \quad \frac{9}{3} \quad \cdots \quad \frac{2n-1}{3}$


$d(0, A) = \frac{1}{3}\left[n^2 + 2\right]$ [sum of odds up to $2n-1$ is $n^2$]

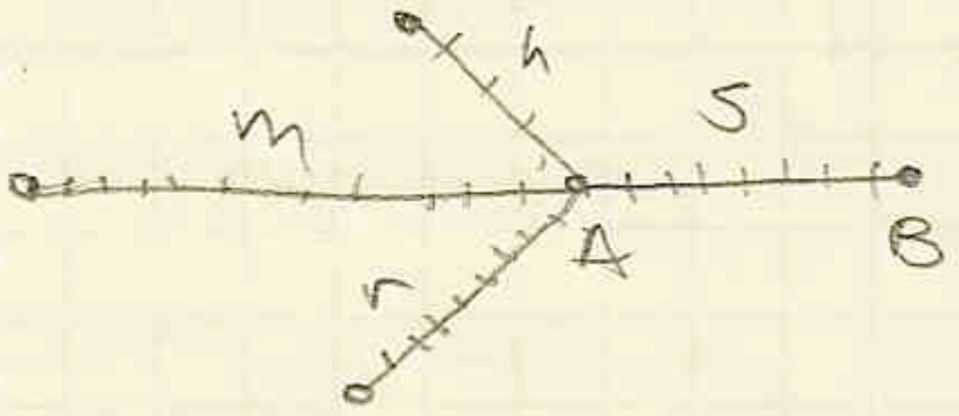
Using these techniques we find




Where C, D, E are boundary vertices
The Expected number of steps to go from A to B without going out at C, D, E

$$\frac{1}{3}\left[s^2 + 2\,\frac{m+n+s+r}{\frac{1}{m}+\frac{1}{n}+\frac{1}{s}+\frac{1}{r}}\right]$$

Also if C, D, E are <u>not</u> boundary vertices




expected number of step from A to B is

$$s^2 + 2(m+n+r)s$$

or is $s=1$

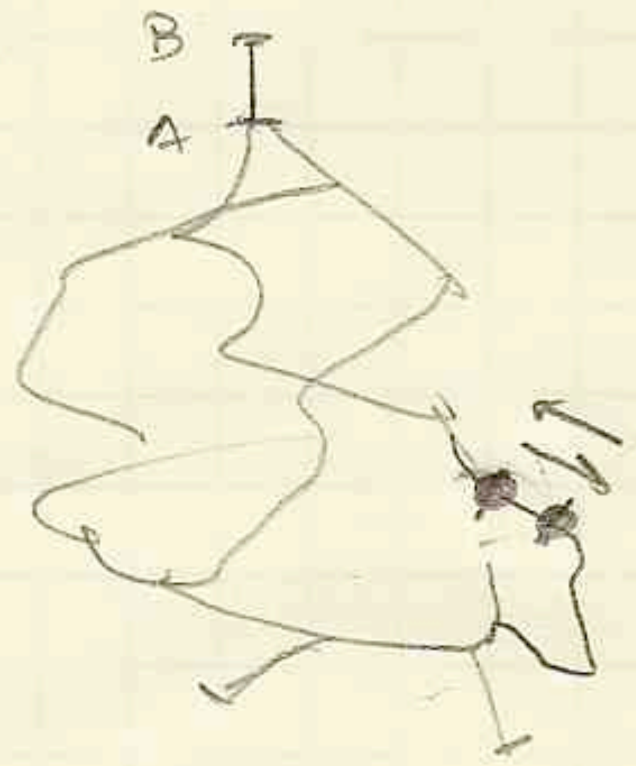
$= 1 + 2$ (# of edges to the left of A)

Turns out to be true for any Graph G with no boundary vertices


The expected number of steps from A to B is

$1 + 2$ (# of edges in back of A)

Proof:




Consider an arbitrary edge

Claim 1: The expected number of crossings in a random walk from A to B of that edge in the $\rightarrow$ direction is equal to the number of crossings in the $\leftarrow$ direction

The expect number crossings in $\rightarrow$ direction

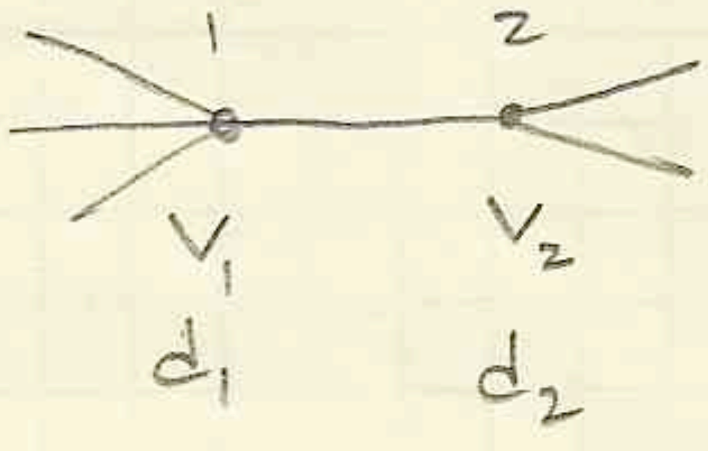
is $\sum_{\substack{\text{path } P \\ \text{from } A \to B}} \text{prob}(P)\,[\#\text{crossings} \rightarrow]$

for $\leftarrow$ direction: $\sum_{\substack{\text{path } p \\ A \to B}} \text{prob}(P)\,[\text{crossing} \leftarrow]$

All the paths that crosses the edge the same number of times in both directions contribute to both sums equally.

A path that crosses the edge m times in $\rightarrow$ direction and n times in the $\leftarrow$ direction can be reversed and still be a path from $A \to B$ [by changing the starting point] For that path we get m time in the $\leftarrow$ direct and n times in the $\rightarrow$ direction for m+n each way so the expect number both ways is equal.

Again consider an arbitrary edge connecting
vertices 1 and 2




let $v_1$ and $v_2$ the expected number of visits to vertex 1 or 2 respectively in a random walk from A to B
$d_1, d_2$ are the degrees of 1 and 2

Since $v_1$ is the expected number of visits to 1, the expected number of crossing from 1 to 2 should be $\frac{v_1}{d_1}$ similarly the expected number of crossings from 2 to 1 is $\frac{v_2}{d_2}$
By the last part these are equal

so $\frac{v_1}{d_1} = \frac{v_2}{d_2}$ this hold for any connect pair of vertices so the v's are proportional to the d's throughout the graph

Now let us calculate $v_A$ directly:

The probability of going $A \to B$ directly is $1/d_A$

the probability of instead going from A into the rest of the graph is $\frac{d_A - 1}{d_A}$ since there are no boundary vertices in the graph the particle will return to A so the expect number of visit to A is

$$v_A = \frac{1}{d_A}\cdot 1 + \frac{d_A - 1}{d_A}\frac{1}{d_A}\cdot 2 + \left(\frac{d_A - 1}{d_A}\right)^2 \frac{1}{d_A}\cdot 3 + \cdots$$

$$= \frac{1}{d_A}\left[1 + \left(\frac{d_A - 1}{d_A}\right)\cdot 2 + \left(\frac{d_A - 1}{d_A}\right)^2 \cdot 3 + \cdots\right]$$

$$= \frac{1}{d_A}\left[\frac{1}{1 - \frac{d_A - 1}{d_A}}\right]^2 = \frac{1}{d_A}\left[\frac{d_A}{1}\right]^2 = d_A$$

So $V_A = d_A$ so the factor of proportionality of the $V$'s and the $d$'s is one

so $V_i = d_i$

Thus for any arbitrary edge the expected crossings in each direction

is $\frac{V_1}{d_1} = \frac{d_1}{d_1} = 1$ and $\frac{V_2}{d_2} = \frac{d_2}{d_2} = 1$

so the expected number of crossing of each edge in a random walk from A to B is 2. Adding the step to go from A to B

$E[X_{AB}] = 1 + 2(\#\text{ of edges behind } A)$

$P_{AB} = 1$ since there are no other exits

$d(A,B) = 1 + 2(\#\text{ of edges behind } A)$

We can write in terms of vertices as

$$d(A,B) = \sum_{\substack{\text{vert.} \\ i \neq B}} V_i = \sum_{\substack{\text{vertices} \\ i \neq B}} d_i$$

We can try to extend this argument to the more complicated case where there are other boundary vertices aside from B

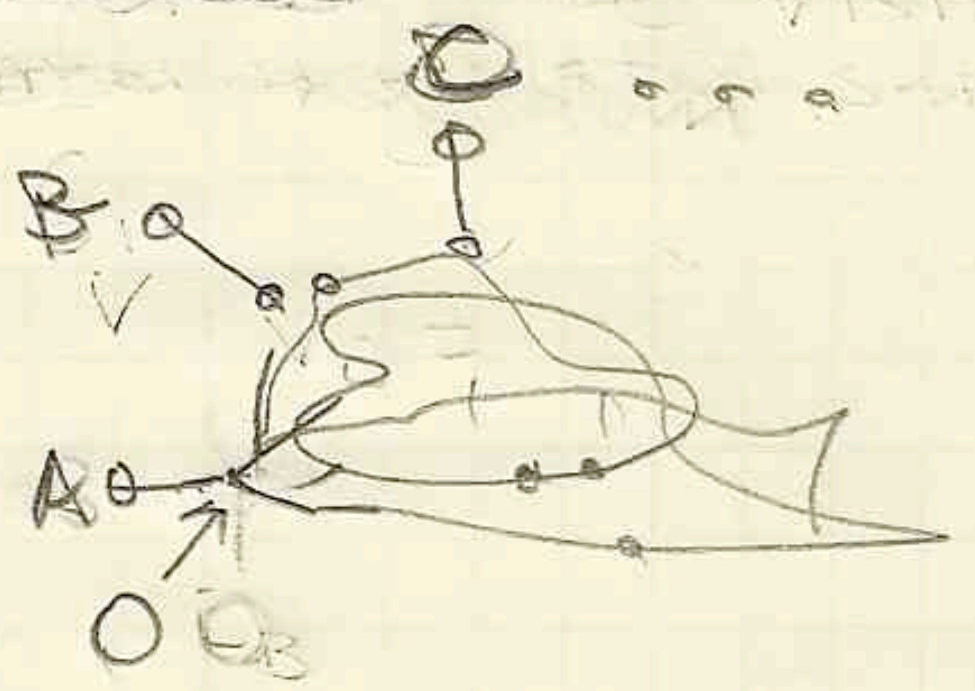
and get that

$$E[X_{AB}] = \sum_{\substack{\text{vertices} \\ i \neq B}} V_i = \sum d_i P_{iB}^2$$

Where $P_{iB}$ is the probability of a particle at vertex $i$ exiting at B, the solution to the Dirichelet problem with boundary value one at B and zero at all other boundary nodes

Unhappily the argument is faleacous, but the result is True

So we have to prove it by other means.
And let's shift our labeling a bit in the process




Let G be a graph with more than one boundary vertex consider a random walk starting at O exiting at any boundary vertex

Let $v_i$ be expected visits to vertex $i$ in the random walk beginning at O exiting at any boundary vertex $\{A, B, C, \ldots\}$
The $v_i$'s are related to their neighboring interior vertices that is

$$v_i = \sum_{\substack{i \sim j \\ j \text{ interior vertex}}} \frac{1}{d_j} v_j$$

this true for all interior vertices except for O because as the particle was put there from the outside it has one visit that did not come from other interior vertices

so $$v_0 = \left[\sum_{\substack{i \sim j \\ j \text{ interior}}} \frac{1}{d_j} v_j\right] + 1$$

Thus we have a matrix relating interior vertices

$$\begin{bmatrix} & & \\ & Q & \\ & & \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} = \begin{bmatrix} v_0 - 1 \\ v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}$$

the column of Q total one or less
less when the column is for a vertex that is next to a boundary vertex

so

$$[I - Q]\begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$I-Q$ is diagonal dominant so is invertable so there is a unique vector $\begin{bmatrix} v_0 \\ \vdots \\ v_k \end{bmatrix}$ that

solve the relations

claims

It is $v_i = d_i P_{iA}$ [where $P_{iA}$ is the probability of exiting at A if starting at $i$]

$P_{iA}$ is the solution to a Dirichelet probleem and so is harmonic at all interior points where boundary conditions are all zero except for A which is one

so ~~for~~ $i \neq 0$ $\quad P_{iA} = \sum_{\substack{i \sim j \\ j \text{ interior}}} \frac{P_{jA}}{d_i}$

so $\quad d_i P_{iA} = \sum_{\substack{i \sim j \\ j \text{ interior}}} P_{jA}\left(\frac{d_j}{d_j}\right) = \Sigma$

so $\quad v_i = \sum_{\substack{i \sim j \\ j \text{ interior}}} \frac{v_j}{d_j}$

for $i=0$

$$P_{iA} = \sum_{\substack{i \sim j \\ j \text{ interior}}} \frac{P_{jA}}{d_j} + P_{AA} \qquad P_{AA} = 1$$

so $\quad v_i = \sum_{\substack{i \sim j \\ j \text{ interior}}} \frac{v_j}{d_j} + 1$

so the expected number of step of a random walk begining at O exiting any where is the sum of all the visits to all the interior vertices [note the extra visit at O is the extra step needed to exit at a boundary vertex]

so $E[X_O] = \sum_{\substack{i \text{ interior} \\ \text{vertices}}} d_i P_{iA}$

If we let $V_A = P_{AA} d_A = 1$

and $V_k = P_{kA} d_A = 0$ for $k \neq A$ boundary vertex

so $E[X_A] = \sum_{\substack{i \text{ interior} \\ \text{vertex}}} d_i P_{iA} + 1 = \sum_{\substack{\text{all vertices} \\ i}} d_i P_{iA}$

Now suppose the particle starts at A what is the expected number of step to exit at B $d(A,B) = \dfrac{E[X_{AB}]}{P_{AB}}$

$P_{AB} = P_{OB}$ which we can find by solving the Dirichelet problem with one at B and zero at other boundary vertices

We will calculate $E[X_{AB}]$ from the visit function

Let $P^*_{Ai}$ be the probabolity of going from A to $i$ directly (visiting $i$ once)

Let $P^*_{i\partial}$ be the probabolity of going from $i$ to any boundary vertex directly (not returning to $i$)

Let $r_i$ be the probability of starting at $i$ and returning once

$$V_i = P_{Ai}^{*} P_{i\partial}^{*} \cdot 1$$
$$+ P_{Ai}^{*} r_i P_{i\partial}^{*} \cdot 2$$
$$+ P_{Ai}^{*} r_i^2 P_{i\partial}^{*} \cdot 3$$
$$+ \cdots$$

If we want $V_i'$ to be the portion of the above sum that exit at B

then if $P_{iB}^{*}$ be the probability of going directly to B from $i$ then

$$V_i' = P_{Ai}^{*} P_{iB}^{*} \cdot 1$$
$$+ P_{Ai}^{*} r_i P_{iB} \cdot 2$$
$$+ P_{Ai}^{*} r_i^2 P_{iB} \cdot 3$$
$$+ \cdots$$

so if we can convert $V_i$ to $V_i'$ by multiplying by $\frac{P_{iB}^{*}}{P_{i\partial}^{*}}$. We can exit from $i$ directly or return that is the entire range of options so $P_{i\partial}^{*} + r = 1$

or $P_{i\partial}^{*} = 1 - r$

also, $P_{iB} = P_{iB}^{*} + P_{iB}^{*} r_i + P_{iB}^{*} r_i^2 + P_{iB}^{*} r_i^3 + \cdots$

$$P_{iB} = P_{iB}^{*} \left[\frac{1}{1-r}\right]$$

$$P_{iB}^{*} = P_{iB}(1-r) \quad \text{so} \quad \frac{P_{iB}^{*}}{P_{i\partial}^{*}} = P_{iB}$$

so $V_i' = V_i \cdot P_{iB} = d_i P_{iA} P_{iB}$

so $E[X_{AB}] = \sum_{\substack{\text{all vertices} \\ i}} d_i P_{iA} P_{iB}$

So we have a way of calculating the expected number steps getting from boundary vertex A to boundary vertex B in terms of two solutions to the Dirichelêt problem one setting A to one other boundary vertices to zero the other setting B to one and the other boundary vertices to zero.

$$d(A,B) = \frac{\sum_{\substack{\text{all vert} \\ i}} d_i P_{iA} P_{iB}}{P_{i_A B}}$$

[ where $i_A$ is the interior vertex by A

Remember we defined $d(A,A) \equiv 0$

but

$$\frac{\sum_{\substack{\text{all vert} \\ i}} d_i P_{iA}^2}{P_{i_A A}}$$

is the expected number of steps for a particle leaving A to return and it is definitely non-zero

Perhaps we don't have so much as metric as something else.

We have an inner product over the vector space of solutions to the Dirichelet boundary value problem which is a perfectly good inner product over the vector space of all functions (vectors) on the graph.

One, of course, can design any sort of inner product one likes, but I'm going to argue that this might be special.

First, a random walk is sensitive to the geometric properties of space it walks through. In particular, dimension, in a flat d-dimesional space the return probability $P(\#\text{step}) \propto \frac{1}{(\#\text{steps})^{d/2}}$

So perhaps an inner product that falls out of a random walk product would make appropriate volumemetric adjustment to an integral (in this case a sum) This inner product might be imagined as the standard $L^2$ inner product adjusted for the inhomogeneities of the graph - by multiplying by the degree $d_i$ at each vertex.

Perhaps it would be worthwhile to look at selected sets of orthogonal function under this inner product.

Second, this inner product is closely related to the Dirichelet norm.

Recollect that the Dirichelet Norm for an electrical network is

$$W_\gamma(u) = \frac{1}{2} \sum_{\substack{i \sim j \\ i,j \\ \text{vertices}}} \gamma_{ij} (u_i - u_j)^2 \quad \left(= \frac{\Delta V^2}{R} = \text{Power}\right)$$

where $y$ and $u$ are equal on boundary nodes and $u$ is $\gamma$-harmonic

$$W_\gamma(u) \le W_\gamma(y)$$

in other words solution to the Dirichlet problem minimize the Dirichelet norm (minimum Power loss)

$W_\gamma(u)$ is a norm (or a semi-norm) over edges we can convert it into a norm of vertices

First, assume as I do $\gamma_{ij} = 1$

$$W_\gamma(u) = \frac{1}{2} \sum_{i \sim j} (u_i^2 - 2u_i u_j - u_j^2)$$

$$= \frac{1}{2} \sum_i \sum_{j \sim i} (u_i^2 - 2u_i u_j - u_j^2)$$

$$= \frac{1}{2} \left[ \sum_i d_i u_i^2 - 2u_i \sum_{i \sim j} u_j + \sum_{i \sim j} u_j^2 \right]$$

give the last term to thier respective vertices and you get

$$= \frac{1}{2} \sum_i \left[ 2d_i u_i^2 - 2u_i \frac{\sum_{i \sim j} u_j}{d_i} \right]$$

$$= \sum_i d_i u_i \left[ u_i - \frac{\sum u_j}{d_i} \right]$$

We recognize this as our inner product of $u$ with a function measuring how different $u$ is from harmonic

One may wonder if we redesign the random walk to run directly on networks where $\gamma$ varies, will the random walk inner product and Dirichlet norm still be closely related

Lastly lets convert the inner product to an inner product over edges just to see what it looks like.

Restricting our space to solutions to the Dirichlet problem on a graph with boundary vertices $\{A, B, C, D \ldots\}$

the functions on the vertices $P_{\hat{i}A}, P_{\hat{i}B}, P_{\hat{i}C} \cdots$

form a basis for that space.

$$\langle A, B \rangle = \sum_{\hat{i} \text{ vertex}} d_{\hat{i}} P_{\hat{i}A} P_{\hat{i}B} = \sum_{\hat{i} \text{ vert}} v_i' = \sum_{i} \sum_{\hat{i} \sim j} e(j \to i)$$

$e_i$

$e(j \to i)$ = expected number of times edge from $j$ to $i$ is crossed in that direction counting only paths the start at A and end at B

$$\left[ V_i' = \sum_{i \sim j} e(j \to i) \right]$$

$$\langle A, B \rangle = \sum_{i} d_i P_{iA} \left( \frac{\sum_{i \sim j} P_{jB}}{d_i} \right) = \sum_{i} P_{iA} \sum_{i \sim j} P_{jB}$$

$$= \sum_{i} \sum_{i \sim j} P_{iA} P_{jB}$$

$$V_i' = \sum_{i \sim j} e(j \to i) = \sum_{i \sim j} P_{iA} P_{jB}$$

$A \leftrightarrow B$ same argument shows $V_j' = \sum_{j \sim i} e(j \to i) = \sum_{i \sim j} P_{iB} P_{jA}$