ON THE RECOVERABILITY OF AMALGAMATED AND SEPARATED GRAPHS USING MEDIAL GRAPHS

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Abstract. This paper will deal with and describe a process to amalgamate graphs. The recoverability of the resulting graphs will be addressed as well as a reverse procedure to separate the graphs into recoverable components. Throughout the paper, circular planar graphs will be used and their medial graphs will be instrumental in the proofs.

Contents

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1. Introduction

This section will provide information about resistor networks, which is covered in detail in [1]. We will define the necessary terms to describe the inverse problem and what it means for a network to be recoverable.

We define a *graph with boundary* in the usual way. Let $G = (\text{int}V, \partial V, E)$ be a graph such that $V = \text{int}V \cup \partial V$ denotes the vertices of G and $\text{int}V \cap \partial V = \emptyset$. The set ∂V is the set of *boundary nodes* and $\text{int}V$ is the set of *interior nodes.* Also, $E \subseteq V \times V$ denotes the set of edges of G. Define a relation, $\sim$, on V such that for $p, q \in V$ we have $p \sim q$ if $pq \in E$.

A *conductivity* on a graph G is a function $\gamma : E \to \mathbb{R}^+$. Thus γ assigns a positive, real conductance to every edge in G. A *resistor network*, $\Gamma = (G, \gamma)$, is a pair consisting of a graph and a conductivity on that graph. For a *potential function* $u : V \to \mathbb{R}$ we can define a *current* function $I : E \to \mathbb{R}$ by Ohm's Law. That is, $I(pq) = \gamma(pq)(u(p) - u(q))$ for $pq \in E$.

Let $\Gamma = (G, \gamma)$ be a resistor network. A potential function, u, is called γ*-harmonic* if for each vertex in $\text{int}V$ we have

$$\sum_{q \sim p} \gamma(pq)(u(p) - u(q)) = 0.$$

This property is called *Kirchhoff's Law.* The Dirichlet Problem is then, given a boundary potential function $\phi : \partial V \to \mathbb{R}$ does there exist a unique γ- harmonic function $u : V \to \mathbb{R}$ such that $u(p) = \phi(p)$ for all $p \in \partial V$? The answer to this question is yes, as is shown in [1].

Given a boundary function on Γ we can determine uniquely the currents at each boundary node in ∂V. Supposing that $|\partial V| = n$, we define the *Dirichlet to Neumann* map $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ which takes boundary potentials to boundary currents. This map is linear and its matrix is called the *response matrix* for Γ. For the inverse problem on electrical networks, we want to recover γ from Λ. If this is possible we say that Γ is a *recoverable* network.

The response matrix, $\Lambda = [\lambda_{i,j}]$, for a network Γ with n boundary nodes is an $n \times n$ matrix where each boundary node corresponds to a row and column. If a boundary function is put on Γ with a value of 1 at boundary node j and 0 elsewhere, then $\lambda_{i,j}$ is the current flowing out of node i.

A *circular planar* graph is a planar graph with boundary which is embedded in a disk, D. The boundary nodes lie on circle which bounds D, while the interior nodes are completely contained in the interior of

D. The boundary nodes can be labeled in circular order around the boundary of D.

2. Medial Graphs

Consider a circular planar graph, G, embedded on a disk, D. The *medial graph*, $\mathcal{M}$, of G is a graph which has a vertex at every edge of G and an edge connecting any two vertices which lie on edges of G that share a vertex and a face. In addition, $\mathcal{M}$ has one vertex on each side of each boundary node of G on the boundary of D. In this way, each edge of G will have four edges of $\mathcal{M}$ corresponding to it. Thus we can define the *geodesics* of $\mathcal{M}$ which start and end on the boundary of D and transverse each other at each crossing, that is, at each edge of G.

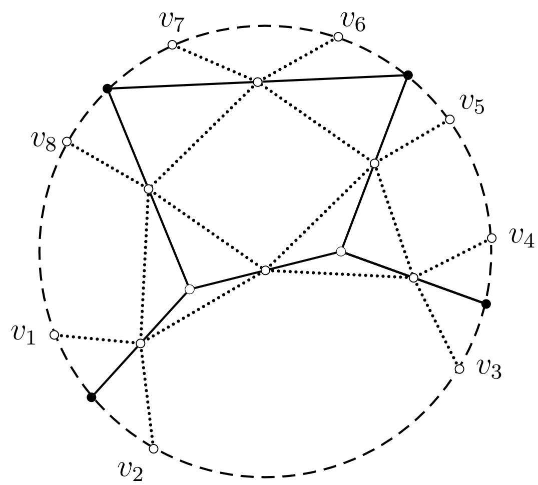
Each boundary node of G can be identified with two geodesics. These geodesics may or may not cross each other. If two geodesics cross more than once, or if one geodesic crosses itself, a *lens* is formed. See Figure 2. It was shown in [1] that a circular planar network is recoverable if and only if its medial graph does not contain a lens. Since lenses are dependent on geodesics crossing each other, it's important to realize which properties of a graph will produce crossing geodesics.

A boundary spike is defined to be a boundary node with degree one and an edge attached to an interior node. From the definition of a medial graph, we know the geodesics terminating on either side of a boundary spike will necessarily cross at the edge formed by the boundary spike.

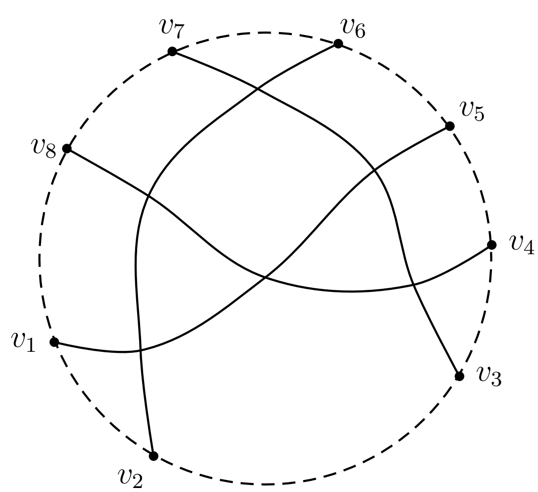
As is discussed in [1], if three geodesics form an empty triangle, i.e. a triangle with no geodesics passing through it, then a Y-Δ transformation in G will put one geodesic on the other side of the crossing made by the other two. In this way, if the two geodesics of a boundary node cross each other, we can do Y-Δ transformations to move the crossing so that no other geodesics are between it and the boundary node.

In Figure 3 consider the geodesics terminating at v_5 and v_6. The geodesic terminating at v_8 forms an empty triangle with these and by doing a Y-Δ transformation we've moved it to the other side of the crossing. Performing this again we have now moved the crossing of these two geodesics to the boundary node between their terminating points. This idea is summarized in the following lemma.

Lemma 2.1. *Let G be a circular planar graph and let $\mathcal{M}$ be its medial graph. Let g_1 and g_2 be two geodesics in $\mathcal{M}$ which terminate on either side of a boundary node, a. Then g_1 and g_2 will cross each other if and only if a is Y-Δ equivalent to a boundary spike.*



(a) A graph with its medial graph (dotted lines)



(b) The medial graph of the above graph showing the geodesics

FIGURE 1. Graph and medial graph

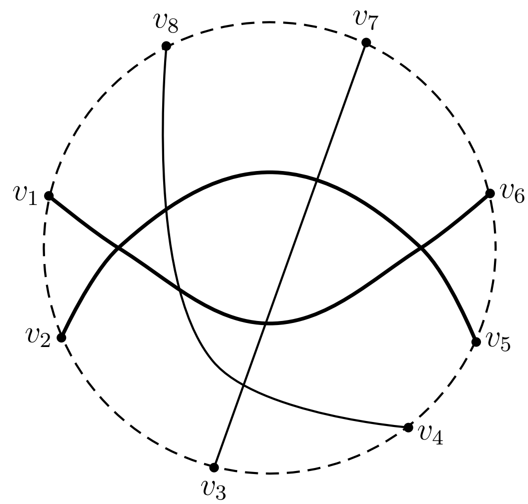


FIGURE 2. An example of a lens

Proof. Suppose g_1 and g_2 in $\mathcal{M}$ cross each other. If there are no other geodesics between the crossing and a, then their crossing must correspond to an edge which is directly attached to a. Thus, a is a boundary spike. If there are geodesics between a and the crossing then consider the one closest to the crossing. This will form an empty triangle and thus a Y-Δ transformation in G will bring the crossing closer to a. Continue in this process until no geodesics are between a and the crossing.

Now suppose that there exists G' which is Y-Δ equivalent to G such that in G', a is a boundary spike. Consider $\mathcal{M}'$, the medial graph of G'. The g_1 and g_2 in $\mathcal{M}'$, which terminate on either side of a, will necessarily cross each other because a is a boundary spike. Now perform Y-Δ transformations on G' until we arrive at G again. Since each transformation will not uncross any geodesics, it will only move the crossings around, g_1 and g_2 are still crossed. $\square$

3. Amalgamating Graphs

This section will discuss amalgamating graphs in a general sense. Much of this is discussed in [2] where most of the research was started. The notation in [2] will also be adopted. We will briefly discuss the matrix manipulations needed to amalgamate two graphs together.

Let G_1 and G_2 be two graphs with known response matrices Λ_1 and Λ_2. Order the boundary nodes in G_1 and G_2 so that the nodes at

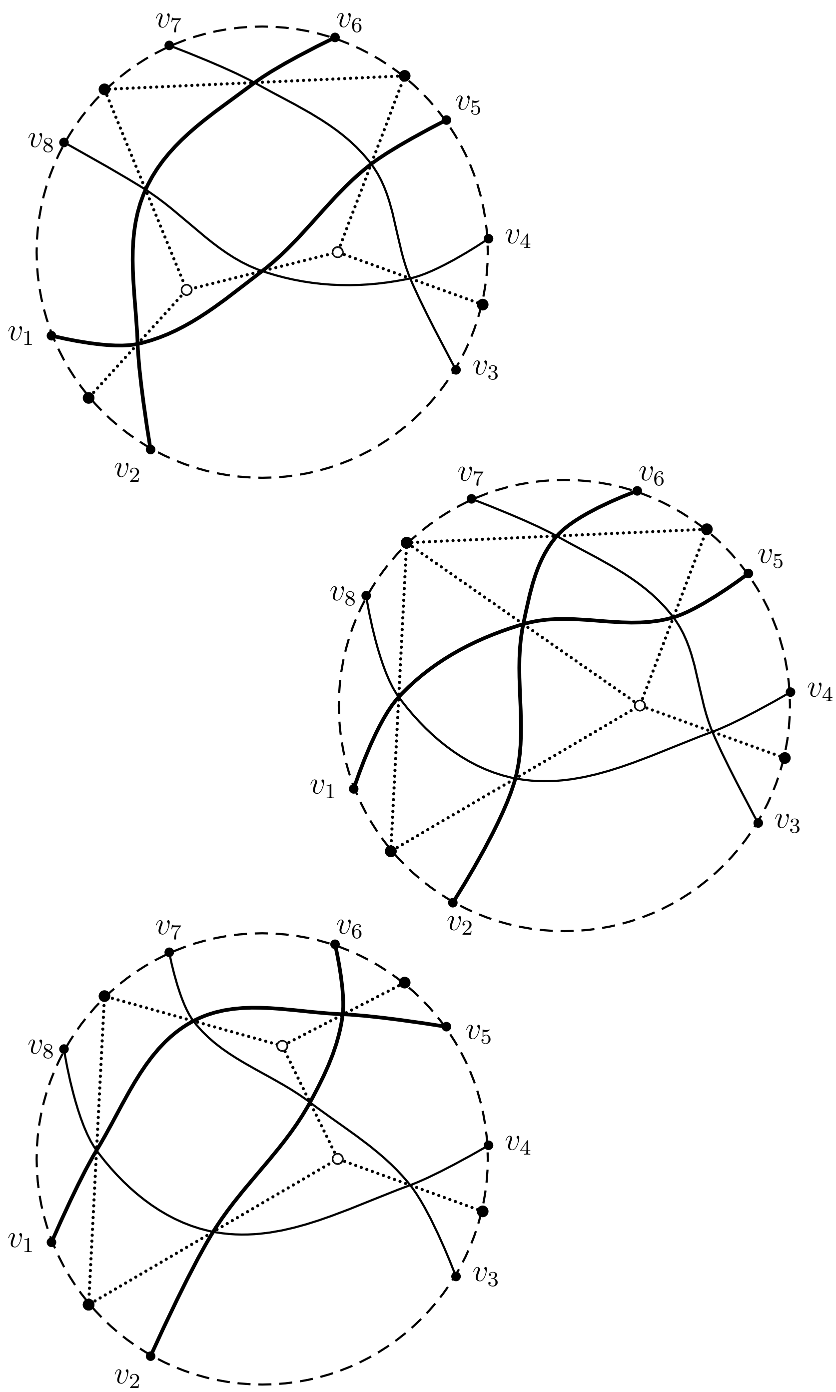


FIGURE 3. An example of Y-Δ transformations to move a crossing towards a boundary node

which we want to amalgamate are listed last. Call the set of boundary nodes to be amalgamated in G_1, L_1 and those in G_2, L_2. We must have $|L_1| = |L_2|$. Then the response matrices for G_1 and G_2 can be written in block form as

$$\Lambda_1 = \begin{pmatrix} A_1 & B_1 \\ B_1^T & C_1 \end{pmatrix} \qquad\qquad \Lambda_2 = \begin{pmatrix} A_2 & B_2 \\ B_2^T & C_2 \end{pmatrix}$$

where C_1 and C_2 correspond to L_1 and L_2. Next we identify the nodes in L_1 and L_2 to create a new graph which we'll call G^*. The response matrix is

$$\Lambda^* = \begin{pmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ B_1^T & B_2^T & C_1 + C_2 \end{pmatrix}.$$

There are no connections through the interior between boundary nodes which were not in L_1 or L_2. This explains the blocks of zeros in Λ^*. To make the nodes interior nodes, we take the *Schur complement* of Λ^* with respect to $C_1 + C_2$. Then

$$\Lambda = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} - \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} (C_1 + C_2)^{-}1 \begin{pmatrix} B_1^T & B_2^T \end{pmatrix}$$

is the response matrix for G with internalized boundary nodes.

Before we continue we want to make a small modification to the amalgamation process in order to preserve recoverability later on. In the case of amalgamating a boundary spike to another boundary node we will contract the boundary spike and identify its interior node with the other boundary node. Then we can interiorize that node. Figure 4 shows this process.

4. Recoverability of Amalgamated Graphs

Let G_1 and G_2 be two circular planar graphs which we will amalgamate to form G. We will consider various cases of amalgamation at one or more boundary nodes and the recoverability of the resulting graph.

Suppose that G_1 and G_2 each have at least one boundary spike and suppose we amalgamate G_1 and G_2 at these boundary spikes. Then in the medial graph of G the only geodesics which have been modified are the ones terminating on either side of the boundary spikes. We simply connect these geodesics when we amalgamate the graphs. Figure 5 shows an example of this process. Notice that no new crossings were created when we amalgamated. This leads to the following theorem.

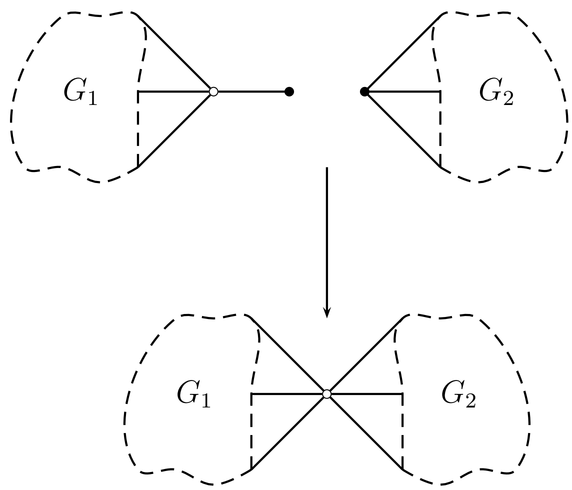


FIGURE 4. Amalgamation of a boundary spike

Theorem 4.1. *Let G_1 and G_2 be two recoverable, circular planar graphs such that G_1 and G_2 each have at least one boundary spike. Suppose that G_1 and G_2 are amalgamated to form G at this boundary spike. Then G is also recoverable.*

Proof. Note that G will still be circular planar since amalgamating will simply remove boundary nodes. Consider the medial graphs of G_1 and G_2. Call the geodesics which terminate on either side of the boundary spikes A_1, B_1, A_2 and B_2 accordingly. Since G_1 and G_2 are recoverable, there are no lenses in their medial graphs. Thus, A_1 and B_1 will cross only at the boundary spike in G_1 and nowhere else. A similar statement can be made about A_2 and B_2. When we amalgamate, A_1 and A_2 will become A in the medial graph of G and B_1 and B_2 will become B. But A and B will only cross at the point where the graphs were amalgamated. Since G is circular planar and there are no lenses in the medial graph of G, it must be recoverable. $\square$

The statement made in Theorem 4.1 can be made more general since we have a particular way to amalgamate at boundary spikes.

Corollary 4.1. *Let G_1 and G_2 be two recoverable, circular planar graphs such that G_1 has at least one boundary spike. Suppose that G_1 and G_2 are amalgamated to form G at this boundary spike in G_1 and at any boundary node in G_2. Then G is also recoverable.*

Proof. This follows from a similar argument as Theorem 4.1. As shown in Figure 4, when we amalgamate at a boundary spike, the boundary

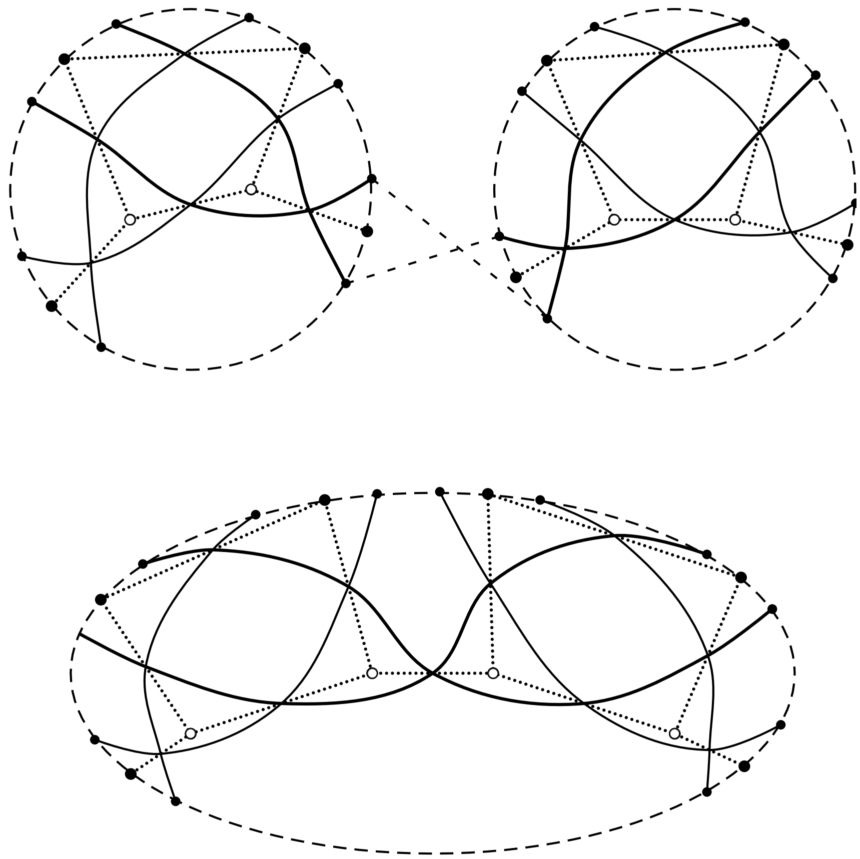


FIGURE 5. Connecting geodesics while amalgamating two graphs

spike is contracted. This means that the crossing that was made by the geodesics terminating there is no longer there in the amalgamated graph. Since both G_1 and G_2 are recoverable, there are no lenses. Thus the corresponding geodesics in G_2 to be connected can only cross once. Therefore the resulting geodesics in G can only cross at most once. □

We can also make a statement about the non-recoverability of amalgamated graphs since we know exactly when the geodesics in the medial graph will cross.

Corollary 4.2. *Let G_1 and G_2 be two circular planar graphs such that G_1 and G_2 each have at least one boundary node which is not a boundary spike but is Y-Δ equivalent to a boundary spike. Suppose that G_1 and*

G_2 are amalgamated to form G at this boundary node. Then G is not recoverable.

Proof. The geodesics terminating on either side of the boundary nodes to be amalgamated must cross at least once by Lemma 2.1. Since each boundary node is not a boundary spike the crossings will still be present in the medial graph of G. Since we have two geodesics which cross each other twice, the medial graph of G has a lens. Thus G is not recoverable. □

It was shown in [2] that two circular planar graphs when amalgamated in circular planar order will preserve circular planarity in the the resulting graph. Since we know which operations and boundary nodes will produce crossing geodesics in an amalgamated graph, we also know about the recoverability of graphs amalgamated at more than one boundary node. That is, given two recoverable, circular planar graphs, we can amalgamate them to get a recoverable graph at as many boundary nodes as we like, provided that they are in circular planar oder and for each pair being amalgamated, one of the two is a boundary spike.

5. Separation of Graphs

Since we have a process of amalgamating graphs using medial graphs it seems natural to try and separate a circular planar graph into recoverable pieces.

To separate a circular planar graph, first draw its medial graph, then look for crossings in the medial graph to separate at. It's necessary to be able to draw a continuous line from the boundary to the boundary which only crosses the medial graph at its vertices. Also, the separation line must cross the edges of the original graph transversely. In Figure 6 the x's mark where the separation line crosses the original graph (dotted lines). In this way, edges of the graph are broken into boundary spikes in the resulting graphs.

Because separation breaks a graph at the crossing of geodesics, creating a boundary spike in the resulting medial graph will produce no extra crossings. Thus, if a lens is broken during separation, the only time it will be present in the resulting graphs is if the separation line crosses one or both endpoints of a lens and crosses the lens nowhere else. In this way we can separate nearly any graph into recoverable graphs.

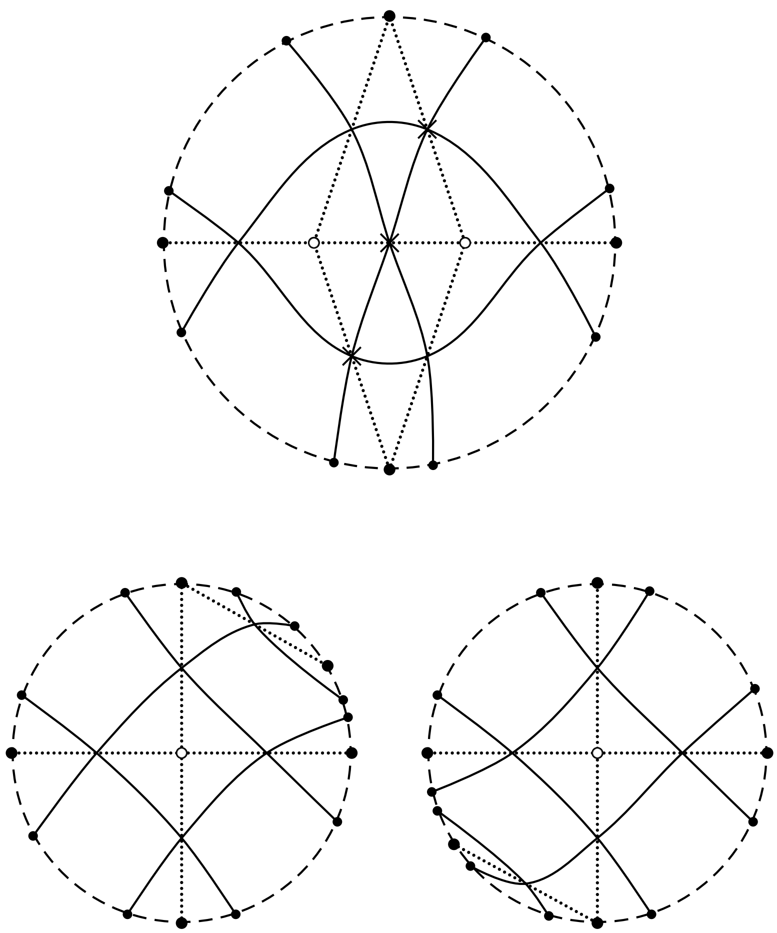


FIGURE 6. Separation of a graph using its medial graph

6. Conclusion

This paper presents a process for amalgamating circular planar graphs using their medial graphs as well as discusses the recoverability of the resulting graphs. Furthermore, a separation procedure, which completely reverses amalgamation, is given and can be used to separate nonrecoverable circular planar graphs into recoverable graphs.

Certainly the biggest limitation in this paper is the fact that all graphs are circular planar. It would be good to be able to extend recoverability results to general graphs which are not necessarily circular planar. It would also be interesting if this could be done in some way with medial graphs, albeit they may be degenerate in some form. The fact that circular planar graphs can be amalgamated to create non-circular planar graphs suggests that there may be a way to use the original medial graphs to obtain a statement about the resulting graph.

More work should be done on separating graphs and which sorts of graphs can be separated. A result along the lines of "Any circular planar graph can be separated into recoverable graphs" would be useful, but at the moment a simple series connection provides a counterexample. Nevertheless, a slightly weaker claim can certainly be made.

References

[1] Edward B. Curtis and James A. Morrow, *Inverse Problems for Electrical Networks*. Series on applied mathematics - Vol. 13. World Scientific, ©2000.

[2] Ryan K. Card and Brandon I. Muranaka, *Using Network Amalgamation and Separation to Solve the Inverse Problem*. 2003.