DETERMINANT OF A PRINCIPLE PROPER SUBMATRIX OF THE KIRCHHOFF MATRIX

MATTHEW J. LEWANDOWSKI

Abstract. In this paper, we develop a new method for analyzing the principle submatrix of the Kirchhoff matrix corresponding to the interior nodes of an electrical network. More specifically, we show that the determinant is always positive, and we give a means to calculate the determinant in terms of tree diagrams associated with the network.

Contents

1. Introduction

We start with an electrical network $\Gamma = (G, \gamma)$. G is a graph with a collection of vertices V (also referred to as nodes).We consider the partition of V into two sets, ∂V which denotes the set of boundary vertices, and $intV$ which denotes the set of interior vertices. We require that $\partial V \bigcap intV = \{\emptyset\}$, thus $V = \partial V \bigcup intV$ and $|V| = |\partial V| + |intV|$. We label the vertices with numbers $i \in [1, |V|]$ by first numbering the boundary nodes, and then numbering the interior nodes. That is, for $v_i \in \partial V$, we require $i \in [1, |\partial V|]$, and for $v_i \in intV$, we require that $i \in [|\partial V| + 1, |V|]$. For notational convenience, the symbol i will often be used to refer to vertex i, so it will make sense to write something like $i \in \partial V$. We also consider a relation $\sim$ on V which defines when two vertices are adjacent in the graph G. If two vertices are related by $\sim$ then we say $i \sim j$, and that (i, j) is the edge that joins vertex i to vertex j. We are not considering directed graphs, so we let $(i, j) = (j, i)$. Also, we do not allow an edge to be between a node and itself, thus $i \nsim i$. Let the set of edges

Date: August 29, 2008.

$E = \{(i,j) | i \sim j\}$. It will be helpful to label certain subsets of E. Let $E_{bb} = \{(i,j) | i \sim j \text{ and } i \in \partial V \text{ and } j \in \partial V\}$ be the set of boundary vertex to boundary vertex edges; let $E_{ii} = \{(i,j) | i \sim j \text{ and } i \in intV \text{ and } j \in intV\}$ be the set of interior vertex to interior vertex edges; and let $E_{ib} = \{(i,j) | i \sim j \text{ and } i \in \partial V \text{ and } j \in intV\}$ be the set of interior vertex to boundary vertex edges. Clearly, the sets are pairwise disjoint, and $E = E_{bb} \bigcup E_{ii} \bigcup E_{ib}$. We define $\gamma : E \to \mathbb{R}_+$ which assigns to each edge a conductance which is greater than zero. Since $(i,j) = (j,i)$, we have $\gamma(i,j) = \gamma(j,i) \equiv \gamma_{i,j}$. We define the Kirchhoff matrix, $K = (k_{i,j})$ for Γ as follows:

$$k_{i,j} = \begin{cases} -\gamma_{i,j} & \text{if } i \sim j \\ \displaystyle\sum_{k: i \sim k} \gamma_{i,k} & \text{if } i = j \\ 0 & \text{if } i \nsim j \text{ and } i \neq j \end{cases} \tag{1}$$

We will find it convenient to write K in block form, $K = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$, where the dimensions of A are $|\partial V| \times |\partial V|$, the dimensions of B are $|\partial V| \times |intV|$, and the dimensions of C are $|intV| \times |intV|$.

This paper will be concerned with analyzing the determinant of C. We know from arguments using the positive semidefiniteness of K that $\det C > 0$ [1]. Starting with Peter Mannisto's formulation of the determinant as a sum over loop partitions [2], we will show that $\det C > 0$ by considering only arguments related to paths and loop partitions. Note: in figures, interior nodes will be represented with open circles, while boundary nodes will be represented with filled circles.

2. A Review of Loop Partitions

This section is a review of Peter Mannisto's discussion in Section 2 [2]. The determinant of an $n \times n$ matrix M can be computed as

$$\det M = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)}.$$

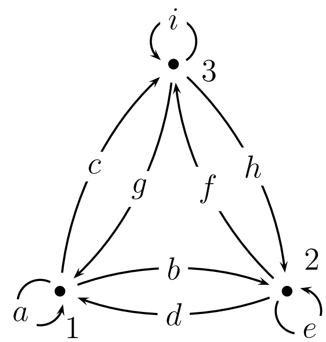
First, we give a few definitions.

Definition 2.1. The **associated graph**, G_M, of the $n \times n$ matrix M is a graph with n vertices, where for each nonzero $m_{i,j}$, we create a directed edge from vertex i to vertex j with weight $m_{i,j}$. Let V_M be the vertices of G_M and E_M be the edges.

Figure 1 shows an example of a matrix and its associated graph. Note that $m_{i,j} \neq m_{j,i}$ in general.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

(a) M



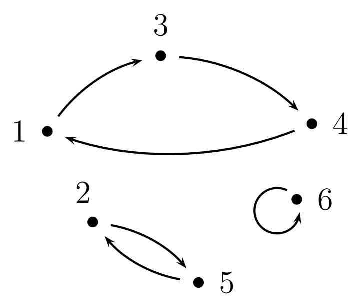
(b) G_M

FIGURE 1. a matrix and its associated graph

Definition 2.2. A **loop partition** of G_M is a set of directed edges such that every node has exactly one edge leading into it and exactly one edge leading away from it.

(134)(25)

(a) permutation



(b) loop partition

FIGURE 2. a permutation and its corresponding loop partition

Figure 2 shows a permutation on six elements and its corresponding loop partition. Notice that by the above definition for associated graph, our graph G is not the associated graph of K. For one thing, G does not have directed edges. It also does not contain any edges between a node and itself, which are allowed by the definition of associated graph.

In the associated graph, however, we allow **self loops**. A self loop is a single directed edge that leaves node i and returns to node i. Since it is a single edge, the loop only contains the node i. Since permutations can be decomposed uniquely into cycles, there is a natural bijection between permutations and loop partitions, namely sending the cycle $c = (i_1 i_2 ... i_k)$ to the loop with directed edges $l_c = (i_1, i_2)(i_2, i_3) \cdots (i_{k-1}, i_k)(i_k, i_1)$. An element of S_n, which is a product of cycles, would thus be sent to the loop partition of G_M which is the union of the loops associated with each cycle. Thus every permutation $\sigma \in S_n$ corresponds to a loop partition of a graph with n vertices, and we see that cycles in the permutation σ correspond to loops in the loop partition, and distinct cycles correspond to distinct loops.

Not all edges are used in each loop partition. Let $\omega(L)$ be the product of all weights corresponding to edges in a loop partition L, G_M be the associated graph to the matrix M, $|L|_c$ be the number of disjoint loops of L (including self loops), and $\mathscr{L}(G_M)$ be the set of all loop partitions on G_M. Then Peter Mannisto showed that

$$\det M = \sum_{L \in \mathscr{L}(G_M)} (-1)^{-n+|L|_c} \omega(L) \tag{2}$$

In words, we can calculate the determinant of M by summing over the contributions from all loop partitions of G_M. Each loop partition gives a contribution, with appropriate sign, that is equal to the product of all weights of edges included in the loop partition. In Figure 3 below, we show a matrix M and give the contribution to $\det M$ of the loop partition in Figure 2(b).

$$M = \begin{pmatrix} * & * & \alpha & * & * & * \\ * & * & * & * & \delta & * \\ * & * & * & \beta & * & * \\ \gamma & * & * & * & * & * \\ * & * & * & * & * & \zeta \end{pmatrix}$$

(a) M (* denotes arbitrary value)

$$-\alpha\beta\delta\gamma\zeta$$

(b) contribution of loop partition in Figure 2(b)

FIGURE 3. a loop partition and its contributions to the determinant

3. The Determinant of C

For the submatrix $C = (c_{i,j})$ of K, we have from equation 2

$$\det C = \sum_{L \in \mathscr{L}(G_C)} (-1)^{-n+|L|_c} \omega(L)$$

where $n = |intV|$. Some care must be taken when reexpressing this equation in terms of our network Γ, since G_C is a directed graph, and our network is not directed. By definition, we have $\omega(L) = \prod_{(i,j) \in L} c_{i,j}$. As a notational convenience, we define $\gamma_{i,i} \equiv k_{i,i} = \sum_{k: i \sim k} \gamma_{i,k}$. There is no edge (i,i) in our graph G because we excluded this possibility at the beginning, but an edge (i,i) can exist in the loop partition L. Thus, $\gamma_{i,i}$ does not correspond to an actual conductance on Γ, but $c_{i,i}$ is a valid edge weight for the loop partition L. Then, $\omega(L) = (-1)^{|L|_E} \prod_{(i,j)\in L} \gamma_{i,j}$, where $|L|_E$ is the number of edges not associated with self loops used in the loop partition L. Thus

$$\det C = \sum_{L\in\mathscr{L}(G_C)} (-1)^{-n+|L|_c+|L|_E} \prod_{(i,j)\in L} \gamma_{i,j}$$

Since each vertex has only one edge to it and one edge away from it, $|L|_E$ is also the number of nodes included in cycles that are not self loops. Thus, $n-|L|_E$ is the number of nodes included only in self loops, so $|L|_c-(n-|L|_E)$ is the number of cycles that are not self loops. We call this number $\ell(L)$. Therefore,

$$\det C = \sum_{L\in\mathscr{L}(G_C)} (-1)^{\ell(L)} \prod_{(i,j)\in L} \gamma_{i,j}. \tag{3}$$

4. Edge Diagrams and Tree Diagrams

If we expand the the determinant in (3), we obtain a sum of factors, each of which is a product of conductance. Keep in mind that each $\gamma_{i,i} = \sum_{k:i\sim k}\gamma_{i,k}$ is a sum of conductances. If $k\in\partial V$, then $\gamma_{i,k}$ corresponds to the conductance of an edge that connects an interior node to a boundary node. None of these edges are represented in the loop partitions since $\mathscr{L}(G_C)$ is a collection of edges associated with the matrix C which relates interior nodes of G only.

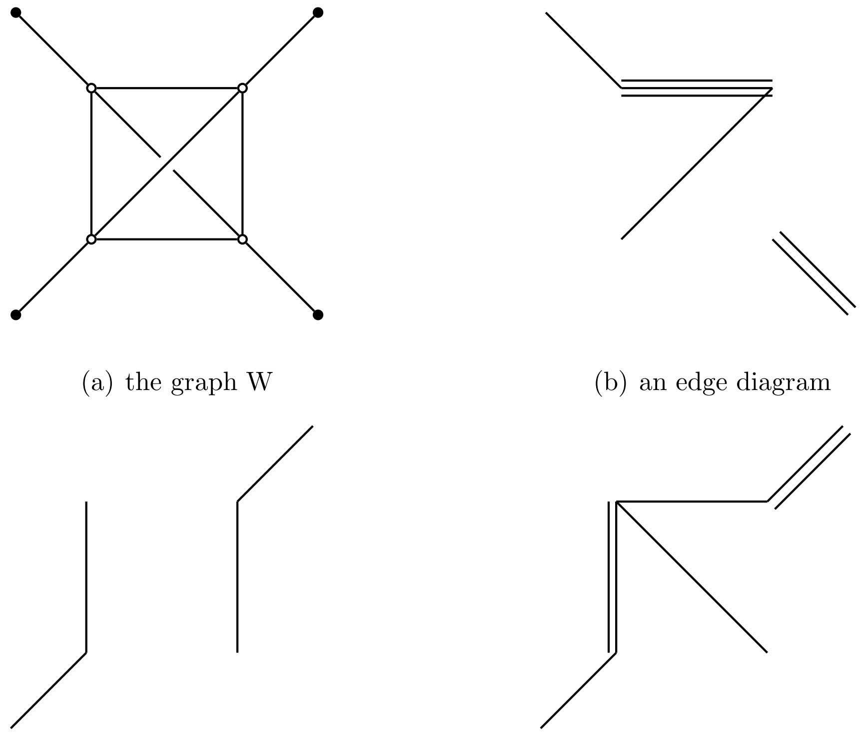
Definition 4.1. For any graph G, define an **edge diagram** of G as a multiset that contains $u(e)$ copies of each edge $e\in E$ where E is the edge set for the graph G, and $u(e)\in\mathbb{Z}$ is a nonnegative integer. An edge diagram is specified completely by this function $u: E\to\mathbb{Z}$, and is denoted $\mathcal{E}_u$. If $u(e)=0$ then we say that $e\notin\mathcal{E}_u$. We use $\mathbf{E}(G)$ to denote the set of edge diagrams associated with G.

It is called a diagram because it will be helpful to make a picture to depict the set. Figure 4 gives an example of this situation.

Definition 4.2. When the edge set E is part of an electrical network Γ, the **value** of an edge diagram is given by $\mathbf{value}(\mathcal{E}_u) \equiv \prod_{e\in E}\gamma(e)^{u(e)}$.

In this new language, $\det C$ is a sum, with appropriate signs, of values of edge diagrams. The result of this paper will be to show that $\det C$ is actually a sum of values of a specific subset of edge diagrams, which we refer to as tree diagrams.

Definition 4.3. A **tree diagram** is an edge diagram $\mathcal{E}_u\in\mathbf{E}(G)$ that has the following properties:



(a) the graph W (b) an edge diagram

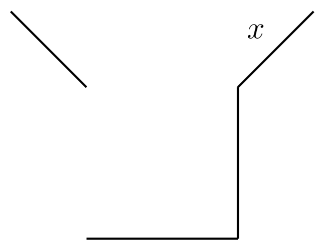
(c) another edge diagram (d) one more edge diagram

FIGURE 4. edge diagrams

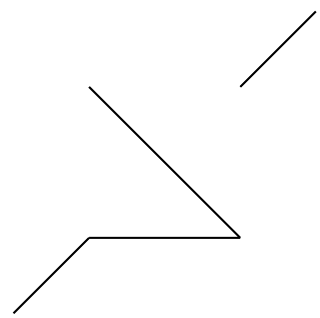
(1) $e \in \mathcal{E}_u \Rightarrow u(e) = 1$. Since there is at most one of each edge, this set is in bijective correspondence with a subgraph of G. Thus it makes sense to language about graphs when talking about this set.
(2) The connected components of $\mathcal{E}_u$ are "trees". I use quotes here because this is not the standard use of the term tree. These trees are not graphs, for they have no vertices. However, they are in bijective correspondence with subgraphs of G, which do have vertices.
(3) $\forall\ i \in intV\ \exists\ e \in \mathcal{E}_u$ such that $e = (i, j)$ for some $j \neq i$ (i.e. every interior vertex is an endpoint for some edge in some component tree of $\mathcal{E}_u$).
(4) For every component tree, there exists a unique $q \in \partial V$ such that $(q, p) \in \mathcal{E}_u$ for some $p \in intV$, and for any $r \in \partial V$, $(r, p) \in \mathcal{E}_u \Rightarrow r = q$. Edge (p, q) is called a **root**.
(5) $p \in \partial V$ and $q \in \partial V \Rightarrow (p, q) \notin \mathcal{E}_u$.

Such an edge diagram is denoted $\mathcal{T}_u$, and the set of all tree diagrams for a given graph G is denoted $\mathbf{T}(G)$. Figure 4(d) is an example of a tree diagram for W given in Figure 7(a);

below are two more examples for that W.



(a) x is a root



(b)

FIGURE 5. example of two tree diagrams for W in Figure 7(a)

5. A Tree Diagram Approach to the Determinant of C

Given an electrical network $\Gamma = (G, \gamma)$, $G = (E, V, \sim)$, the set $\mathbf{T}$ of tree diagrams of G, and the associated Kirchhoff matrix $K = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$, we have to following theorem.

Theorem 5.1. *The determinant of C is the sum of values of all tree diagrams of G. That is*

$$\det C = \sum_{\mathcal{T} \in \mathbf{T}(G)} \mathbf{value}(\mathcal{T}) \tag{4}$$

Before giving the proof, it will be helpful to introduce the following lemma. In general, there may be more than one interior to boundary edge at a given interior node i. This lemma will show that instead of considering each of these edges separately, we may consider one interior to boundary edge at i whose conductance is the sum of the conductances of the actual interior to boundary edges at i. Given an electrical network as described above, consider a new network $\Gamma' = (G', \gamma')$, $G' = (E', V', \sim')$ where we group all interior to boundary edges at interior node i into a single interior to boundary edge whose conductance is the sum of the original conductances. If there were no interior to boundary edges at interior node i, then we do not add a boundary edge.

More specifically, we let $intV'$ correspond exactly to $intV$, and we let E'_{ii} correspond exactly to E_{ii}; we will often give corresponding pieces (edges or vertices) the same name, but specify where they are. Then, for every $j \in intV$, compute $\varphi_j \equiv c_{j,j} - \sum_{\substack{k \in intV \\ k \neq j}} \gamma_{k,j}$ (this is the sum of conductances connected to node $j \in intV$ that are not associated with edges in E_{ii}). If $\varphi_j \neq 0$, then let there be a boundary vertex of V' labeled ∂_j connected by an edge (∂_j, j) to $j \in intV'$ such that $\gamma'(\partial_j, j) = \varphi_j$. If $\varphi_j = 0$, create no such vertex or edge. Thus $\partial V' = \{\partial_j | \varphi_j \neq 0\}$. Let $E'_{ib} = \{(\partial_j, j) | \varphi_j \neq 0\}$ and let $E'_{bb} = \{\emptyset\}$. Also, let $\gamma'|_{E'_{ii}} = \gamma|_{E_{ii}}$ and $\gamma'|_{E_{ib}}$ be defined as above. If we write out the Kirchhoff matrix for Γ' in the usual block decomposition $K' = \begin{pmatrix} A' & B' \\ B'^t & C' \end{pmatrix}$, we notice that $C' = C$ since we have kept all of the interior information from Γ in the new network Γ'. The matrices A' and B' can be defined using the above information, but are of little significance here. Let $\mathbf{T}'(G')$ denote the set of tree diagrams of G'. Call Γ' the **boundary reduced** version of Γ. Now we introduce the lemma.

Lemma 5.2.

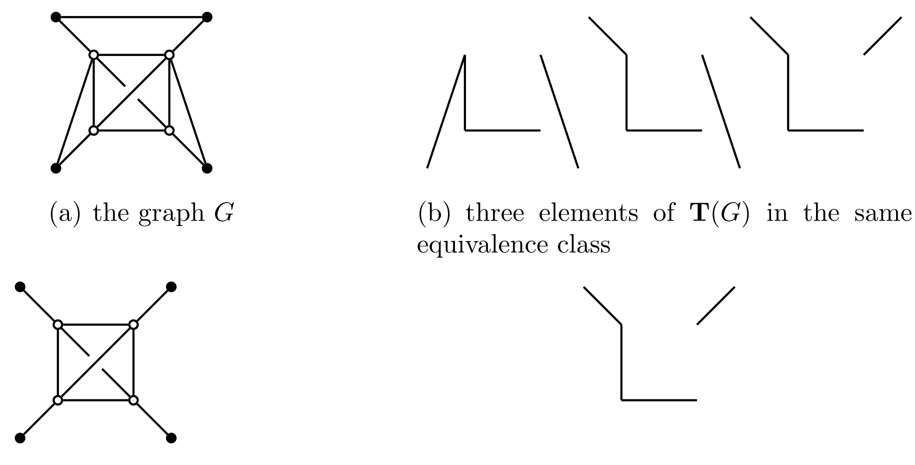
$$\det C = \sum_{\mathcal{T} \in \mathbf{T}(G)} \mathbf{value}(\mathcal{T}) \iff \det C' = \sum_{\mathcal{T} \in \mathbf{T}'(G')} \mathbf{value}(\mathcal{T})$$

In words, this says that if Theorem 5.1. holds for the electrical network Γ, then it holds for the boundary reduced electrical network Γ'.

Proof of Lemma 5.2.

We have $C = C' \Rightarrow \det C = \det C'$. Thus, it remains to show that $\sum_{\mathcal{T} \in \mathbf{T}} \mathbf{value}(\mathcal{T}) = \sum_{\mathcal{T} \in \mathbf{T}'} \mathbf{value}(\mathcal{T})$. First we define an equivalence relation on the set of tree diagrams $\mathbf{T}$. For any tree diagram $\mathcal{T}$, let $\epsilon(\mathcal{T}) \subset E_{ii}$ be the set of interior vertex to interior vertex edges in $\mathcal{T}$. For each component tree of $\mathcal{T}$, there is a root. Let the set of interior vertices that are connected to a root be denoted $r(\mathcal{T}) \subset V$. Define an equivalence class by setting $\mathcal{T}_1 \sim \mathcal{T}_2$ if $\epsilon(\mathcal{T}_1) = \epsilon(\mathcal{T}_2)$ and $r(\mathcal{T}_1) = r(\mathcal{T}_2)$. Then the equivalence class can be denoted by $\mathcal{C}_{\epsilon,r} = \{\mathcal{T} \in \mathbf{T} | \mathcal{T} \sim \mathcal{T}_1\}$ where $\epsilon = \epsilon(\mathcal{T}_1)$ and $r = r(\mathcal{T}_1)$. Given $\mathcal{T}' \in \mathbf{T}'$, it is the only tree diagram with the combination $\epsilon(\mathcal{T}')$ and $r(\mathcal{T}')$ since there is at most one boundary to interior edge at any interior vertex. It is natural to label a tree diagram of $\mathbf{T}'$ by its set of interior to interior edges ϵ and its set of interior nodes r included in a root, $\mathcal{T}'_{\epsilon,r}$. There is a natural bijective correspondence between equivalence classes $\mathcal{C}_{\epsilon,r} \subset \mathbf{T}$ and elements $\mathcal{T}'_{\epsilon,r} \in \mathbf{T}'$. Figure 6. depicts the grouping of equivalence classes.

The problem has been rephrased to showing that $\sum_{(\epsilon,r)} \sum_{\mathcal{T} \in \mathcal{C}_{\epsilon,r}} \mathbf{value}(\mathcal{T}) = \sum_{(\epsilon,r)} \mathbf{value}(\mathcal{T}'_{\epsilon,r})$, where the sums are over all valid (ϵ, r) that define tree diagrams. Since the outermost sums are the same on each side, it will suffice to show that $\sum_{\mathcal{T} \in \mathcal{C}_{\epsilon,r}} \mathbf{value}(\mathcal{T}) = \mathbf{value}(\mathcal{T}'_{\epsilon,r})$, for an arbitrary valid (ϵ, r). But $\mathbf{value}(\mathcal{T}'_{\epsilon,r}) = \big(\prod_{e \in \epsilon} \gamma'(e)\big)\big(\prod_{a \in r} b'_{\partial_a,a}\big) = \big(\prod_{e \in \epsilon} \gamma(e)\big)\big(\prod_{a \in r} \varphi_a\big)$. Here, the first product is over interior to interior edges, and the second is over interior to boundary edges. On the other hand, $\sum_{\mathcal{T} \in \mathcal{C}_{\epsilon,r}} \mathbf{value}(\mathcal{T}) = \sum_{\mathcal{T} \in \mathcal{C}_{\epsilon,r}} \Big[\prod_{a \in r} \gamma(a, \sigma_{\mathcal{T}}(a)) \prod_{e \in \epsilon(\mathcal{T})} \gamma(e)\Big]$ where $\sigma_{\mathcal{T}}(a) \in \partial V$ is the boundary node that is connected to interior node a in $\mathcal{T}$. But $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{C}_{\epsilon,r} \Rightarrow \epsilon(\mathcal{T}_1) = \epsilon(\mathcal{T}_2)$, so the last product is independent of $\mathcal{T}$ and can be pulled out of the summation. However, $\sum_{\mathcal{T} \in \mathcal{C}_{\epsilon,r}} \prod_{a \in r} \gamma(a, \sigma_{\mathcal{T}}(a))$ is just the expansion of $\prod_{a \in r} \varphi_a$ as a sum.



(a) the graph G

(b) three elements of $\mathbf{T}(G)$ in the same equivalence class

(c) the graph G'

(d) the corresponding element of $\mathbf{T}'(G')$

FIGURE 6. the equivalence classes

Thus, $\sum_{\mathcal{T}\in\mathcal{C}_{\epsilon,r}} \mathbf{value}(\mathcal{T}) = \left(\prod_{e\in\epsilon}\gamma(e)\right)\left(\prod_{a\in r}\varphi_a\right)$. We conclude that $\sum_{\mathcal{T}\in\mathcal{C}_{\epsilon,r}} \mathbf{value}(\mathcal{T}) = \mathbf{value}(\mathcal{T}'_{\epsilon,r})$, and finally that $\det C = \sum_{\mathcal{T}\in\mathbf{T}} \mathbf{value}(\mathcal{T}) \iff \det C' = \sum_{\mathcal{T}\in\mathbf{T}'} \mathbf{value}(\mathcal{T})$.

□

With this lemma in hand, given an electrical network Γ, we form the corresponding boundary reduced network Γ' as described above, and prove Theorem 5.1. for the boundary reduced network Γ' (although in the following proof, the boundary reduced network is written without the primes).

Proof of Theorem 5.1.
We will proceed by induction on the number of interior nodes, n. First, we prove the statement for $n = 1$. Let there be one interior node, p and one boundary node, q. Let $\gamma(p,q) = \gamma$ be the conductance between nodes p and q. The Kirchhoff matrix is
$K = \begin{pmatrix} \gamma & -\gamma \\ -\gamma & \gamma \end{pmatrix}$ and so $C = (\gamma)$. Thus, $\det C = \gamma$. On the other hand, there is only one tree diagram: $\mathbf{T} = \{\mathcal{T}_\gamma\}$, $\mathcal{T}_\gamma = \{(p,q)\}$, and $\mathbf{value}(\mathcal{T}_\gamma) = \gamma$. Thus, $\det C = \mathbf{value}(\mathcal{T}_\gamma) = \sum \mathbf{value}(\mathcal{T})$.

Now consider the case that $n = k+1$, i.e. there are $k+1$ interior nodes. We wish to show that assuming the theorem is true for all networks with k interior nodes implies that it is true for networks with $k+1$ interior nodes. Let $\Gamma = (G, \gamma)$ be the boundary reduced version of the network that we are interested in. Let $G = (E, V, \sim)$, $E = E_{ii} \bigcup E_{bb} \bigcup E_{ib}$, $V = \partial V \bigcup intV$, and $\partial V \bigcap intV = \{\emptyset\}$ as usual (remember that in the boundary reduced version, $E_{bb} = \{\emptyset\}$). We have $|intV| = k+1$. In particular, the dimension of C is $(k+1)\times(k+1)$. For convenience, and to minimize confusion, we relabel the interior nodes from one to $|intV|$. Thus, $\gamma_{i,j}$ for $i < |intV|$ and $j < |intV|$ refers to the conductance between interior nodes i and j. Using the earlier notation, $\varphi_j = c_{j,j} - \sum_{\substack{m \in intV \\ m \neq j}} \gamma_{m,j}$, and letting $\sum_{\substack{m \in intV \\ m \neq j}} \gamma_{m,j} \equiv \sigma_j$ for notational convenience, C has the following form:

$$C = \begin{pmatrix} \varphi_1 + \sigma_1 & -\gamma_{1,2} & -\gamma_{1,3} & \cdots & -\gamma_{1,k+1} \\ -\gamma_{1,2} & \varphi_2 + \sigma_2 & -\gamma_{2,3} & \cdots & -\gamma_{2,k+1} \\ -\gamma_{1,3} & -\gamma_{2,3} & \varphi_3 + \sigma_3 & \cdots & -\gamma_{3,k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma_{1,k+1} & -\gamma_{2,k+1} & -\gamma_{3,k+1} & \cdots & \varphi_{k+1} + \sigma_{k+1} \end{pmatrix}.$$

In order to calculate the determinant of C, we use the linearity of the determinant function with respect to the columns of the matrix. Thus, we can say (using $|\ |$ to denote the determinant),

$$|C| = \begin{vmatrix} \varphi_1 + \sigma_1 & -\gamma_{1,2} & -\gamma_{1,3} & \cdots & -\gamma_{1,k+1} \\ -\gamma_{1,2} & \varphi_2 + \sigma_2 & -\gamma_{2,3} & \cdots & -\gamma_{2,k+1} \\ -\gamma_{1,3} & -\gamma_{2,3} & \varphi_3 + \sigma_3 & \cdots & -\gamma_{3,k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma_{1,k+1} & -\gamma_{2,k+1} & -\gamma_{3,k+1} & \cdots & \varphi_{k+1} + \sigma_{k+1} \end{vmatrix}$$

$$= \begin{vmatrix} \varphi_1 & -\gamma_{1,2} & \cdots & -\gamma_{1,k+1} \\ 0 & \varphi_2 + \sigma_2 & \cdots & -\gamma_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -\gamma_{2,k+1} & \cdots & \varphi_{k+1} + \sigma_{k+1} \end{vmatrix} + \begin{vmatrix} \sigma_1 & -\gamma_{1,2} & \cdots & -\gamma_{1,k+1} \\ -\gamma_{1,2} & \varphi_2 + \sigma_2 & \cdots & -\gamma_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{1,k+1} & -\gamma_{2,k+1} & \cdots & \varphi_{k+1} + \sigma_{k+1} \end{vmatrix}$$

$$= \varphi_1 \begin{vmatrix} \varphi_2 + \sigma_2 & -\gamma_{2,3} & \cdots & -\gamma_{2,k+1} \\ -\gamma_{2,3} & \varphi_3 + \sigma_3 & \cdots & -\gamma_{3,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{2,k+1} & -\gamma_{3,k+1} & \cdots & \varphi_{k+1} + \sigma_{k+1} \end{vmatrix} + \begin{vmatrix} \sigma_1 & -\gamma_{1,2} & -\gamma_{1,3} & \cdots & -\gamma_{1,k+1} \\ -\gamma_{1,2} & \varphi_2 + \sigma_2 & -\gamma_{2,3} & \cdots & -\gamma_{2,k+1} \\ -\gamma_{1,3} & -\gamma_{2,3} & \varphi_3 + \sigma_3 & \cdots & -\gamma_{3,k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma_{1,k+1} & -\gamma_{2,k+1} & -\gamma_{3,k+1} & \cdots & \varphi_{k+1} + \sigma_{k+1} \end{vmatrix}$$

Next, we expand the second term in a similar fashion, but this time we split up the second column. We will continue this process until we have expanded each column. After expanding the $k+1$ columns, there is a term that is the determinant of a matrix whose row sums are zero; thus this determinant is zero, and we are left with all terms that look like the first term above, which is a constant multiplied by the determinant of a $k \times k$ matrix. We will use the symbol $C(i)$ to denote the general term like the matrix whose determinant is multiplied by

φ_1 above,

$$C(i) = \begin{array}{cc} \begin{array}{ccccccc} 1 & 2 & \cdots & i-1 & i+1 & \cdots & k+1 \end{array} & \\ \begin{pmatrix} \sigma_1 & -\gamma_{1,2} & \cdots & -\gamma_{1,i-1} & -\gamma_{1,i+1} & \cdots & -\gamma_{1,k+1} \\ -\gamma_{1,2} & \sigma_2 & \cdots & -\gamma_{2,i-1} & -\gamma_{2,i+1} & \cdots & -\gamma_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ -\gamma_{1,i-1} & -\gamma_{2,i-1} & \cdots & \sigma_{i-1} & -\gamma_{i-1,i+1} & \cdots & -\gamma_{i-1,k+1} \\ -\gamma_{1,i+1} & -\gamma_{2,i+1} & \cdots & \gamma_{i-1,i+1} & \varphi_{i+1} + \sigma_{i+1} & \cdots & -\gamma_{i+1,k+1} \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ -\gamma_{1,k+1} & -\gamma_{2,k+1} & \cdots & -\gamma_{i-1,k+1} & -\gamma_{i+1,k+1} & \cdots & \varphi_{k+1} + \sigma_{k+1} \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ \vdots \\ i-1 \\ i+1 \\ \vdots \\ k+1 \end{array} \end{array} \tag{5}$$

In words, to obtain $C(i)$, we start with the matrix C. We then delete the i^{th} row and column. Then, for all $j < i$ we subtract φ_j from the diagonal. Notice that C(1) agrees with the determinant term above. This gives

$$\det C = \sum_{i=1}^{k+1} \varphi_i \det C(i) \tag{6}$$

The labels on the side of equation (5) indicate how we will index the matrix $C(i)$. We are NOT labeling the rows and columns continuously from one to k, as would normally be done with a $k \times k$ matrix. Instead, we skip i. Thus, $(C(i))_{i,q}$ and $(C(i))_{p,i}$ are NOT entries of the matrix $C(i)$ for any p or q. (Notice that, conveniently, the rows and columns of $C(i)$ are indexed to match the indices of the conductances which are the entries of $C(i)$.) Thus,

$$(C(i))_{p,q} = \begin{cases} -\gamma_{p,q} & \text{if } p \neq q \\ \sigma_p & \text{if } p = q \text{ and } p < i \\ \varphi_p + \sigma_p & \text{if } p = q \text{ and } p > i \end{cases} \tag{7}$$

Each $C(i)$ allows a natural extension to an electrical network $\Gamma_i = (G_i, \gamma_i)$. Let V_i be the vertex set of G_i and let E_i be the edge set of G_i. $C(i)$ is $k \times k$. Thus, it represents a network with k interior nodes. This network can be gotten in the following way from $C(i)$. For every $j < i$, delete vertex $\partial_j \in \partial V$ and delete edge $(\partial_j, j) \in E$; let node j remain an interior node of G_i. Also, delete vertex $\partial_i \in \partial V$, and delete edge $(\partial_i, i) \in E$, but turn node $i \in intV$ into a boundary node $i \in \partial V_i$; this is why G_i has one less interior node than G. We can consider G_i to be a subgraph of G where we have changed one interior node of G into a boundary node. More formally, but equivalently, we have

$$\begin{aligned} V_i &= V \setminus \{\partial_j | j \leq i\} \\ \partial V_i &= \partial V \setminus \{\partial_j | j \leq i\} \cup \{i\} \\ intV_i &= intV \setminus \{i\} \\ E_i &= E \setminus \{(\partial_j, j) | j \leq i\} \end{aligned}$$

Note that edges $(k,i) \in E$ where $k \in intV$ were interior to interior edges in G, but now are interior to boundary edges in G_i. We use the conductances of Γ to define the conductances of Γ_i,

$$\gamma_i(p,q) = \gamma(p,q) \text{ for } p \in V_i \text{ and } q \in V_i \tag{8}$$

With conductances defined by equation (8), we now have a network $\Gamma_i = (G_i, \gamma_i)$. It has an associated Kirchhoff matrix $K_i = \begin{pmatrix} A_i & B_i \\ B_i^t & C_i \end{pmatrix}$. However, with the above definitions, we can see that $C(i) = C_i$. To show this notice that, from equation (8) when $p \neq q$, $(C_i)_{p,q} = -\gamma_i(p,q) = -\gamma_{p,q}$. Additionally, $(C_i)_{p,p} = \sum_{q \sim p} \gamma_i(p,q) = \sum_{q \sim p} \gamma(p,q)$. Since the only changes made when going from G to G_i was at nodes $q \leq i$ as described above, we have $\sum_{q \sim p} \gamma(p,q) = \begin{cases} \sigma_p & \text{if } p < i \\ \varphi_p + \sigma_p & \text{if } p > i \end{cases}$. Finally, this gives,

$$(C_i)_{p,q} = \begin{cases} -\gamma_{p,q} & \text{if } p \neq q \\ \sigma_p & \text{if } p = q \text{ and } p < i \\ \varphi_p + \sigma_p & \text{if } p = q \text{ and } p > i \end{cases} \tag{9}$$

which, by inspection, tells us that $C(i) = C_i$. This indeed allows us to assume that, using the induction hypothesis,

$$\det C(i) = \sum_{\mathcal{T} \in \mathbf{T}(G_i)} \mathbf{value}(\mathcal{T}) \tag{10}$$

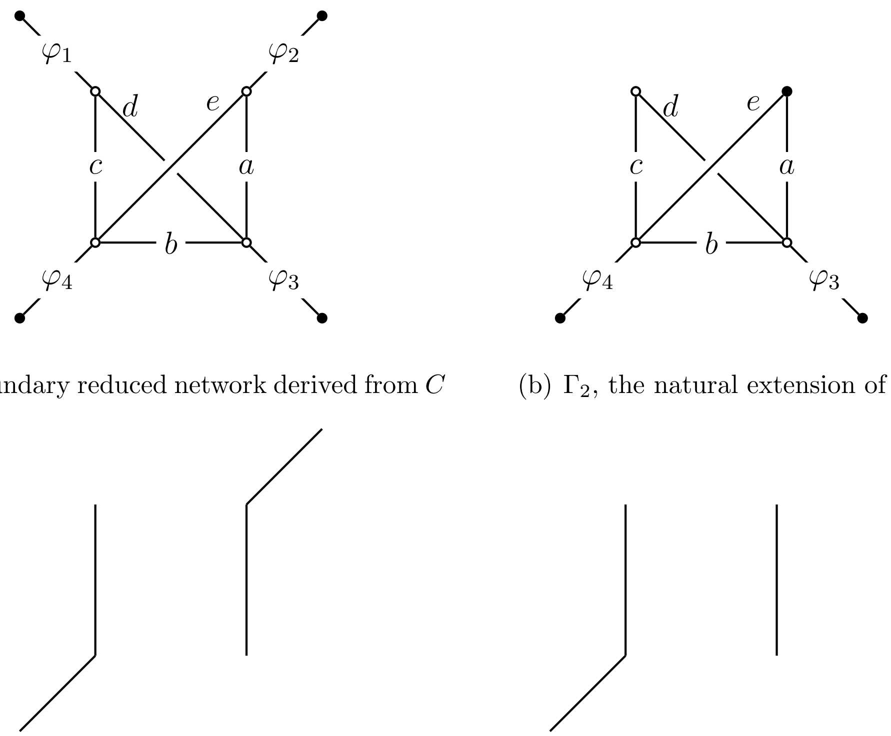
where $\mathbf{T}(G_i)$ is the set of tree diagrams for the graph G_i.

The diagrams help to see what is going on.

$$C = \begin{pmatrix} \varphi_1 + c + d & 0 & -d & -c \\ 0 & \varphi_2 + a + e & -a & -e \\ -d & -a & \varphi_3 + a + b + d & -b \\ -c & -e & -b & \varphi_4 + b + c + e \end{pmatrix}$$

$$C_2 = \begin{pmatrix} c + d & -d & -c \\ -d & \varphi_3 + a + b + d & -b \\ -c & -b & \varphi_4 + b + c + e \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\varphi_3 - a & 0 \\ 0 & 0 & -\varphi_4 - e \end{pmatrix}$$



(a) boundary reduced network derived from C

(b) Γ_2, the natural extension of C_2

(c) a tree diagram $\mathcal{T} \in \mathbf{T}(G)$

(d) corresponding tree diagram $\mathcal{I} \in \mathbf{T}(G')$

FIGURE 7. example: notice $\mathbf{value}(\mathcal{T}) = \varphi_2 \cdot \mathbf{value}(\mathcal{I})$

Using equation (6) and equation (10), we have that

$$\det C = \sum_{i=1}^{k+1} \varphi_i \sum_{\mathcal{J} \in \mathbf{T}(G_i)} \mathbf{value}(\mathcal{J}) \tag{11}$$

We now wish to show that 1) for every $\mathcal{T} \in \mathbf{T}(G)$ we have $\mathbf{value}(\mathcal{T}) = \varphi_i \cdot \mathbf{value}(\mathcal{J})$ for some $\mathcal{J} \in \mathbf{T}(G_i)$, and 2) for every $\mathcal{J} \in \mathbf{T}(G_i)$ we have $\varphi_i \cdot \mathbf{value}(\mathcal{J}) = \mathbf{value}(\mathcal{T})$ for some $\mathcal{T} \in \mathbf{T}(G)$. An example of this correspondence is given in Figure 7.

For 1), consider some $\mathcal{T} \in \mathbf{T}(G)$. Let $i = \min_k\{k|(\partial_k, k) \in \mathcal{T}\}$. Let $t_j \subset \mathcal{T}$ be a component tree of $\mathcal{T}$. Then $\mathcal{T} = \bigcup_J t_j$, where $j \in J$ for the appropriate indexing set J. By assumption, there is one component tree t_i such that $(\partial_i, i) \in t_i$. Every component tree t_k for $k \in J \setminus \{i\}$ (which is just a set of edges) can also be considered as a component tree t_k^* in G_i, and $\mathbf{value}(\bigcup_{k\in J\setminus\{i\}} t_k) = \mathbf{value}(\bigcup_{k\in J\setminus\{i\}} t_k^*)$ since the edges are exactly the same in both graphs. We know $t_i = \{(\partial_i, i)\} \bigcup_{k\in K} e_k$ for some set of edges $\{e_k\}$ and indexing set K. Consider $t_k^* = \bigcup_{k\in K} e_j$ as a subset of E_i. t_i^* is a tree in G_i because t_i was a tree in G, and because there is only one boundary node in t_i^* (namely i). Let $\mathcal{J} = \bigcup_J t_j^*$. Then $\mathcal{J}$ is a tree diagram in G_i because each connected component is a tree, it contains every interior node (because $\mathcal{T}$ did), because every component tree contains one boundary node, any edge is only included once, and there are no boundary to boundary edges. $\mathcal{T}$ has one more edge than $\mathcal{J}$, namely (∂_i, i); $\gamma(\partial_i, i) = \varphi_i$. It follows from above that $\mathbf{value}(\mathcal{T}) = \varphi_i \cdot \mathbf{value}(\mathcal{J})$ as desired.

For 2), consider some $\mathcal{J} \in \mathbf{T}(G_i)$. Let $t_j^* \subset \mathcal{J}$ be a component tree of $\mathcal{J}$. Then $\mathcal{J} = \bigcup_J t_j^*$, where $j \in J$ for the appropriate indexing set J. Assume $(i,p) \notin \mathcal{J}$ for any p. Then each t_j^* is also a set of edges in G, and is also a tree in G; call the set of edges t_j in G. $\{(\partial_i, i)\} \bigcup_J t_j = \mathcal{T}$ ise trivially a tree diagram of G. Since $\mathcal{T}$ has one more edge than $\mathcal{J}$, and because this edge has conductance φ_i, it follows that $\mathbf{value}(\mathcal{T}) = \varphi_i \cdot \mathbf{value}(\mathcal{J})$. However, if there exists $(i,p) \in \mathcal{J}$, then it occurs in only one component tree of G_i; call it t_i^*. As edges in G, the set t_i^* does not contain a boundary node, since i is not a boundary node in G. However, it is still a tree. Thus, if we let t_i be the set of edges t_i^*, then $(\partial_i, i) \bigcup t_i$ is a tree that satisfies Definition 4.3 (4) and (5). Since all other t_j^* are trees in G as well, we can consider them trees $t_j \subset E$. Let $\mathcal{T} = \{(\partial_i, i)\} \bigcup_J t_j$. $\mathcal{T}$ satisfies Definition 4.3 (1)-(5), and thus is a tree diagram in G. Since $\gamma(\partial_i, i) = \varphi_i$, it follows that $\mathbf{value}(\mathcal{T}) = \varphi_i \cdot \mathbf{value}(\mathcal{J})$ as desired.

The induction is now complete.

□

6. The Nature of Determinant of C

With Theorem 5.1 in hand, we have the following theorem about $\det C$. We already know from arguments about the positive semidefiniteness of K that $\det C > 0$. However, Theorem 5.1 allows us to prove this fact in a different way.

Theorem 6.1.
Let $\Gamma = (G, \gamma)$ be an electrical network, where G is a finite graph with edge set E and vertex

set V, such that every interior vertex has a path to the boundary, and γ is a non-negative function on E. Let $K = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$ be the associated Kirchhoff matrix in block form, then $\det C > 0$.

Proof of Theorem 6.1.
Equation (4) shows that if $\gamma : E \to \mathbb{R}_+$, then each **value** is a positive term. Thus, We immediately see that $\det C \geq 0$. It remains to show that there is at least one tree diagram associated with G. Because of Lemma 5.2, we can assume that Γ is in boundary reduced form. Thus, every boundary node $\partial_i \in \partial V$ is connected by an edge (∂_i, i) to vertex $i \in intV$. Let $X = \{(\partial_i, i) | \partial_i \in \partial V\}$. Choose an interior node a and connect it to the boundary by a path α; use A to denote the set of edges of α. Next, choose another interior node not included in an edge of X or A and connect it to the boundary with a path β. Find the smallest set of continuous edges of β starting at b such that the last edge connects to a vertex that is included in an edge of either X or A; call this set of edges B. Then choose an interior vertex not included in an edge of any of A, B, or X and repeat the process. Collect all of these edges in a set called $\mathcal{T}$. Then $\mathcal{T}$ is a spanning tree for G (includes all interior nodes and is a union of component trees), each component tree includes exactly one boundary node, has no boundary to boundary edges, and contains no double edges. Therefore $\mathcal{T}$ is a tree diagram of G. □

7. Spanning Trees

We begin this section with a couple definitions.

Definition 7.1. A **graphical tree** is a graph in which, given any two vertices, they are connected by exactly one path. This is the typical definition of "tree".

Definition 7.2. A **spanning tree** of a graph G is a subgraph of G that is a single connected component, is a graphical tree, and contains all the vertices of G.

Kirchhoff's Matrix Tree theorem gives a way to calculate the number of spanning trees of a connected graph. This result comes as a corollary of Theorem 5.1.

Corollary 7.3 (Kirchhoff's Matrix Tree Theorem)**.** *Given a graph G with vertex set V and edge set E, define an electrical network Γ on G by assigning a conductance of 1 to every edge in G. Promote one vertex in V to be a boundary vertex. Create the associated Kirchhoff matrix $K = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$ in the usual block form. Then the number of spanning trees of G is equal to* $\det C$.

Proof of Corollary 7.3. After promoting a vertex of V to a boundary node, we can divide V into two sets: ∂V and $intV$ such that $V = \partial V \bigcup intV$ and $\partial V \bigcap intV = \{\emptyset\}$. Call the new graph G', which is a graph with boundary. First of all, since there is a conductance of 1 on every edge, and since $\det C$ has one term from each tree diagram of G', $\det C$ is equal to the number of tree diagrams of G'. Every tree diagram of G' contains every vertex of V and has no cycles. Thus, every tree diagram of G' is a spanning tree of G. The set of edges of every spanning tree of G clearly satisfies conditions (1)-(5) of Definition 4.3 for G', and thus is a tree diagram on G'. Thus, spanning trees of G are in one to one correspondence with tree diagrams of G', so the number of tree diagrams of G' is equal to the number of spanning trees of G. Since $\det C$ is equal to the number of tree diagrams of G', it is also equal to the number of spanning trees of G. □

Using a similar technique, we can compute the number of spanning trees of a graph that contains a given edge, a given two edges, or a given three edges. (This should be generalizable, but I didn't have time to finish it.) First, we will generalize the notation.

- For the remainder of this section let Γ, G, G', K, A, B, C, V, and E be defined as in Corollary 7.3 and its proof.

- For the electrical network which is the same as Γ except has a conductance of 2 at edges $F = \{e_1, e_2, ..., e_k\}$, let $C(F)$ be the block matrix of the associated Kirchhoff matrix corresponding to the interior to interior connections. Call the new network Γ_F.
- Let $\mathbf{value}_F$ denote the value function associated with the electrical network Γ_F, and let **value** denote the value function associated with the electrical network Γ.
- Let $\mathcal{N}(F)$ denote the number of spanning trees of G containing the edges in the subset $F \subset E$.
- Let $\mathbf{T}(G')$ be the set of tree diagrams of G'.

Proposition 7.4. *For a given edge $e \in E$, let $F = \{e\}$. Then $\mathcal{N}(F) = \det C(F) - \det C$.*

Proof of Proposition 7.4.
$\det C(e) - \det C = \sum_{\mathcal{T} \in \mathbf{T}(G')} \big(\mathbf{value}_F(\mathcal{T}) - \mathbf{value}(\mathcal{T})\big)$. $\mathbf{value}_F(\mathcal{T}) = \begin{cases} 2 & \text{if } e \in \mathcal{T} \\ 1 & \text{if } e \notin \mathcal{T} \end{cases}$ and $\mathbf{value}(\mathcal{T}) = 1$. Then $\mathbf{value}_F(\mathcal{T}) - \mathbf{value}(\mathcal{T})$ is 0 if $e \notin \mathcal{T}$ and is 1 if $e \in \mathcal{T}$. Thus $\det C(e) - \det C = \sum_{\mathcal{T}: e \in \mathcal{T}} 1 = \mathcal{N}(F)$. □

Proposition 7.5. *For two distinct edges $e_1 \in E$ and $e_2 \in E$, let $F = \{e_1, e_2\}$, $F_1 = \{e_1\}$, and $F_2 = \{e_2\}$. Then $\mathcal{N}(F) = \det C(F) - \det C(F_1) - \det C(F_2) + \det C$.*

Proof of Proposition 7.5.
Let $\mathcal{T}_0$ denote a tree diagram of G' such that $e_1 \notin \mathcal{T}_0$, and $e_2 \notin \mathcal{T}_0$. Let $\mathcal{T}_1$ denote a tree diagram of G' such that only one of the edges e_1, e_2, is contained in $\mathcal{T}_1$. Let $\mathcal{T}_2$ denote a tree diagram of G' such that both of the edges e_1 and e_2 are contained in $\mathcal{T}_1$. We have $\det C(F) -$

$\det C(F_1) - \det C(F_2) - \det C = \sum_{\mathcal{T}\in\mathbf{T}(G')} \big(\mathbf{value}_F(\mathcal{T}) - \mathbf{value}_{F_1}(\mathcal{T}) - \mathbf{value}_{F_2}(\mathcal{T}) + \mathbf{value}(\mathcal{T})\big)$
Let $\mathbf{v}(\mathcal{T}) \equiv \mathbf{value}_F(\mathcal{T}) - \mathbf{value}_{F_1}(\mathcal{T}) - \mathbf{value}_{F_2}(\mathcal{T}) + \mathbf{value}(\mathcal{T})$. We know that

$$\mathbf{value}_F(\mathcal{T}_i) = \begin{cases} 1 & \text{if } i = 0 \\ 2 & \text{if } i = 1 \\ 4 & \text{if } i = 2 \end{cases}$$

$$\mathbf{value}_{F_1}(\mathcal{T}_i) + \mathbf{value}_{F_2}(\mathcal{T}_i) = \begin{cases} 2 & \text{if } i = 0 \\ 3 & \text{if } i = 1 \\ 4 & \text{if } i = 2 \end{cases}$$

$$\mathbf{value}(\mathcal{T}_i) = 1 \text{ for } i = 0, 1, 2$$

It then follows that $\mathbf{v}(\mathcal{T}_0) = 0$, $\mathbf{v}(\mathcal{T}_1) = 0$, and $\mathbf{v}(\mathcal{T}_2) = 1$. Then $\sum_{\mathcal{T}\in\mathbf{T}(G')} \mathbf{v}(\mathcal{T}) = \sum_{\mathcal{T}_0} \mathbf{v}(\mathcal{T}_0) + \sum_{\mathcal{T}_1} \mathbf{v}(\mathcal{T}_1) + \sum_{\mathcal{T}_2} \mathbf{v}(\mathcal{T}_2)$ where $\sum_{\mathcal{T}_i}$ denotes summing over all tree diagrams that contain i of the given edges. The first two sums are zero and the third adds a one for every tree diagram containing e_1 and e_2. Thus, $\sum_{\mathcal{T}_2} \mathbf{v}(\mathcal{T}_2) = \mathcal{N}(F)$. It follows that $\mathcal{N}(F) = \det C(F) - \det C(F_1) - \det C(F_2) + \det C$. $\square$

Proposition 7.6. *For three distinct edges $e_1 \in E$, $e_2 \in E$, and $e_3 \in E$, let $F = \{e_1, e_2, e_3\}$, $F_i = \{e_i\}$, $F_{i,j} = \{e_i, e_j\}$ for $i \neq j$. Then $\mathcal{N}(F) = \det C(F_1) + \det C(F_2) + \det C(F_3) - \big(\det C(F_{1,2}) + \det C(F_{1,3}) + \det C(F_{2,3})\big) + \det C(F) - \det C$.*

Proof of Proposition 7.6.
The proof is similar to the preceding one. Let $\mathcal{T}_i$ denote a tree diagram of G' that contains i of the edges in F. We have

$$\begin{aligned} &\det C(F_1) + \det C(F_2) + \det C(F_3) + \det C(F) - \det C \\ &\qquad - \big(\det C(F_{1,2}) + \det C(F_{1,3}) + \det C(F_{2,3})\big) \\ &= \sum_{\mathcal{T}\in\mathbf{T}(G')} \big(\mathbf{value}_{F_1}(\mathcal{T}) + \mathbf{value}_{F_2}(\mathcal{T}) + \mathbf{value}_{F_3}(\mathcal{T}) - \mathbf{value}_{F_{1,2}}(\mathcal{T}) - \\ &\qquad \mathbf{value}_{F_{1,3}}(\mathcal{T}) - \mathbf{value}_{F_{2,3}}(\mathcal{T}) + \mathbf{value}_F(\mathcal{T}) - \mathbf{value}(\mathcal{T})\big). \end{aligned}$$

Let

$$\begin{aligned} \mathbf{v}(\mathcal{T}) \equiv\ & \mathbf{value}_{F_1}(\mathcal{T}) + \mathbf{value}_{F_2}(\mathcal{T}) + \mathbf{value}_{F_3}(\mathcal{T}) - \mathbf{value}_{F_{1,2}}(\mathcal{T}) \\ & - \mathbf{value}_{F_{1,3}}(\mathcal{T}) - \mathbf{value}_{F_{2,3}}(\mathcal{T}) + \mathbf{value}_F(\mathcal{T}) - \mathbf{value}(\mathcal{T}). \end{aligned}$$

We have

$$\mathbf{value}_{F_1}(\mathcal{T}_i) + \mathbf{value}_{F_2}(\mathcal{T}_i) + \mathbf{value}_{F_3}(\mathcal{T}_i) = 3 + i \text{ for } i = 0, 1, 2, 3$$

$$\mathbf{value}_{F_{1,2}}(\mathcal{T}_i) + \mathbf{value}_{F_{1,3}}(\mathcal{T}_i) + \mathbf{value}_{F_{2,3}}(\mathcal{T}_i) = \begin{cases} 3 & \text{if } i = 0 \\ 5 & \text{if } i = 1 \\ 8 & \text{if } i = 2 \\ 12 & \text{if } i = 3 \end{cases}$$

$$\mathbf{value}_F(\mathcal{T}_i) = 2^i \text{ for } i = 0, 1, 2, 3$$

$$\mathbf{value}(\mathcal{T}_i) = 1 \text{ for } i = 0, 1, 2, 3$$

Then it follows that $\mathbf{v}(\mathcal{T}_0) = 0$, $\mathbf{v}(\mathcal{T}_1) = 0$, $\mathbf{v}(\mathcal{T}_2) = 0$, and $\mathbf{v}(\mathcal{T}_3) = 1$. Then $\sum_{\mathcal{T}\in\mathbf{T}(G')} \mathbf{v}(\mathcal{T}) = \sum_{\mathcal{T}_0} \mathbf{v}(\mathcal{T}_0) + \sum_{\mathcal{T}_1} \mathbf{v}(\mathcal{T}_1) + \sum_{\mathcal{T}_2} \mathbf{v}(\mathcal{T}_2) + \sum_{\mathcal{T}_3} \mathbf{v}(\mathcal{T}_3)$. The first three sums are zero and the third adds a one for every tree diagram containing e_1, e_2, and e_3. Thus $\sum_{\mathcal{T}_3} \mathbf{v}(\mathcal{T}_3) = \mathcal{N}(\mathrm{F})$. It follows that $\mathcal{N}(F) = \det C(F_1) + \det C(F_2) + \det C(F_3) - \big(\det C(F_{1,2}) + \det C(F_{1,3}) + \det C(F_{2,3})\big) + \det C(F) - \det C$. $\square$

As stated before, this formula should generalize. I think that it will look something like the following. Let $F = \{e_1, ..., e_n\}$, $F_i = \{e_i\}$, $F_{i,j} = \{e_i, e_j\}$ for $1 \le i < j \le n$, $F_{i,j,k} = \{e_i, e_j, e_k\}$ for $1 \le i < j < k \le n$, and $F_{i,j,...,m} = \{e_i, e_j, ..., e_m\}$ for $1 \le i < j < ... < m \le n$. I postulate that

$$\begin{aligned}\mathcal{N}(F) &= \pm \det C(F) \pm \det C \\ &\pm \sum_i \det C(F_i) \pm \sum_{i<j} \det C(F_{i,j}) \pm \sum_{i<j<k} \det C(F_{i,j,k}) \pm \cdots \pm \sum_{i<j<...<m} \det C(F_{i,j,..,m})\end{aligned}$$

for some appropriate choice of signs. This may also be multiplied by an overall constant. Then, if we write the above formula in terms of values of tree diagrams, we will have

$$\mathcal{N}(F) = \sum_{\mathcal{T}\in\mathbf{T}(G')} \mathbf{v}(\mathcal{T})$$

where $\mathbf{v}(\mathcal{T})$ is defined similar to how it is in the proof of Proposition 7.6. This will then break up according to how many of the edges are contained in the tree diagram, ie,

$$\mathcal{N}(F) = \sum_{i=0}^{n} \sum_{\mathcal{T}_i} \mathbf{v}(\mathcal{T}_i)$$

The hope is that $\mathbf{v}(\mathcal{T}_i) = \begin{cases} 0 & \text{if } 1 \le i < n \\ 1 & \text{if } i = n \end{cases}$.

References

[1] Curtis, B., and James A. Morrow. "Inverse Problems for Electrical Networks." Series on applied mathematics – Vol. 13. World Scientific, ©2000.

[2] P. Mannisto, *Another Look at Connections and Determinants*, (2005).