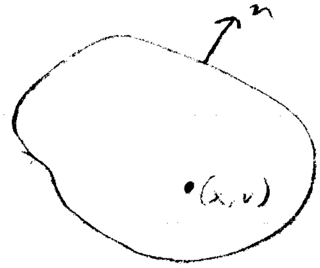
Continuous Transport Equation



$X \subset \mathbb{R}^n$ bounded domain with C^1 boundary

$x \in X$, $v \in S^{n-1}$

$f(x,v)$: particle density

$\Gamma = \partial X \times S^{n-1}$, $\Gamma_\pm = \{(x,v) \in \Gamma : \pm n \cdot v > 0\}$

$$Tf = -v \cdot \nabla_x f(x,v) - \sigma_a(x,v) f(x,v) + \int_{S^{n-1}} k(x,v,v') f(x,v') dv'$$

$\sigma \geq 0$
$k \geq 0$

σ_a : absorption (particle loss)
k : scattering (changing direction)

Forward problem: $\begin{cases} Tf = 0 \\ f|_{\Gamma_-} = f_- \end{cases}$

Specify f on Γ_- (incoming flux)
- then $\exists$ unique f satisfying
$Tf = 0$ on $X \times S^{n-1}$

Unique solution gives us access to "albedo" operator:

$$\mathcal{A} : f_- \mapsto f|_{\Gamma_+}$$

Inverse problem: Can we recover σ, k from $\mathcal{A}$ / does $\mathcal{A}$ determine σ, k?

Make the following assumptions: (define $\sigma_p(x,v') = \int_{S^{n-1}} k(x,v,v') dv$)

1. $\sigma_a(x,v) = \sigma_a(x)$
2. $0 \leq \sigma_a \in L^\infty(\mathbb{R}^n \times S^{n-1})$
3. $0 \leq k(x,v',\cdot) \in L^1(S^{n-1})$ a.e. and $\sigma_p \in L^\infty(\mathbb{R}^n \times S^{n-1})$

$\Rightarrow (\sigma_a, k)$ is an <u>admissible pair</u>

$$\tau_\pm(x,v) = \min\{t \in \mathbb{R} : (x \pm t v, v) \in \Gamma_\pm\}$$ ②

$$\tau(x,v) = \tau_-(x,v) + \tau_+(x,v)$$

And one of the following two conditions:

(a) $\|\tau\sigma_a\|_{L^\infty} < \infty$ $\quad \|\tau\sigma_p\| < 1$

or (b) $0 < \nu \leq \sigma_a(x,v) - \sigma_p(x,v)$

"absorption rate greater than production rate"

(a) or (b) imply "subcritical dynamics"

THM (Choulli, Stefanov): Let (σ_a, k), $(\hat{\sigma}_a, \hat{k})$ be two admissible pairs that satisfy (a) or (b). Assume the corresponding albedo operators coincide, $A = \hat{A}$. Then

- if $n \geq 3$, $\sigma_a = \hat{\sigma}_a$, $k = \hat{k}$
- if $n = 2$, $\sigma_a = \hat{\sigma}_a$

If $n=2$, no uniqueness for k
- need more assumptions

THM (Stefanov, Uhlmann) For transport equation in $\mathbb{R}^2$,
Define the class

$$\mathcal{U}_{\Sigma,\varepsilon} = \left\{ (\sigma_a(x), k(x, v', v)) : \|\sigma_a\|_{L^\infty} < \Sigma, \; \|k\|_{L^\infty} \leq \varepsilon \right\}$$

$\forall \Sigma > 0 \; \exists \; \varepsilon > 0$ s.t. a pair $(\sigma_a, k) \in \mathcal{U}_{\Sigma,\varepsilon}$ is uniquely determined in $\mathcal{U}_{\Sigma,\varepsilon}$ by A, its albedo operator

Applications - Medical Imaging, ...

Discrete Version

$G = (V, V_0, E)$ graph with boundary

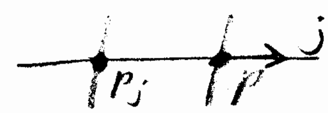
- Assume each interior node has even degree, and each boundary vertex has degree one

Let $M = \max_{p \in G} \deg(p)$

At each interior node p, number the edges incident to p by $1, \dots, k \le M$

$f(p,j)$ = particle density at node p traveling in direction j

Define p_j by:

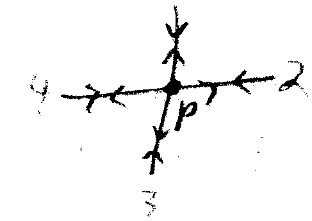


$$\begin{cases} Tf = -[f(p,j) - f(p_j,j)] - \sigma(p,j) f(p,j) + \sum_{i \neq j} k(p,i,j) f(p,i) = 0 \\ f(q, j_{in}) = f_-, \quad q \in \partial V \end{cases}$$

If a solution exists, we can take an albedo matrix A

$A f_- = f_+ \qquad f_+ = f(p, j_{out}) \quad p \sim q, \ q \in \partial$

Does A determine σ, k?

Example: 

Construct T

$$0 = -[f(p,N) - f(3)] - \sigma(p,N) f(p,N) + k(p,S,N) f(1) + k(p,W,N) f(2) + k(p,E,N) f(4)$$

$$0 = -[f(p,E) - f(4)] - \sigma(p,E) f(p,E) + k(p,S,E) f(1) + k(p,W,E) f(2) + k(p,N,E) f(3)$$

...

$$\begin{bmatrix} -1-\sigma_N & & & \\ & -1-\sigma_E & & \\ & & -1-\sigma_S & \\ & & & -1-\sigma_W \end{bmatrix} \begin{bmatrix} f_{pN} \\ f_{pE} \\ f_{pS} \\ f_{pW} \end{bmatrix} =$$

$$= -\begin{bmatrix} k_{SN} & k_{WN} & 1 & k_{EN} \\ k_{SE} & k_{WE} & k_{NE} & 1 \\ 1 & k_{WS} & k_{NS} & k_{ES} \\ k_{SW} & 1 & k_{NW} & k_{EW} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$$T f_+ = M f_-$$

$$A = T^{-1} M$$

- can read off all values of σ, k from the entries of $A = \{a_{ij}\}$

e.g. $a_{13} = \frac{1}{1+\sigma_N} \Rightarrow \sigma_N = \frac{1}{a_{13}} - 1$

$a_{14} = \frac{k_{EN}}{1+\sigma_N} \Rightarrow k_{EN} = a_{14}(1+\sigma_N) = \frac{a_{14}}{a_{13}}$

If we know A is albedo matrix for this G, ...

More generally, let G be a larger lattice graph

For an interior node p, $1 \le j \le 4$,

$$-[f(p,j) - f(p_i,j)] - \sigma(p,j) f(p,j) + k(p,i_1,j) f(p_{i_1}$$

$$+ k(p,i_2,j) f(p_{i_2}, i_2) + k(p,i_3,j) f(p_{i_3}, i_3)$$

$$-(1+\sigma(p,j)) f(p,j) + \text{(terms depending on int } V)$$

$$= (\text{lin. comb. of } f_-)$$

$$\left[\begin{array}{cc|c} -1-\sigma_{p,j} & T_1 & T_2 \\ & & -1-\sigma_{p,j} \\ \hline & 0 & T_4 \end{array}\right] \left[\begin{array}{c} f_+ \\ \hline f \end{array}\right] = \left[\begin{array}{c} M_1 \\ \hline M_2 \end{array}\right] \left[\begin{array}{c} f_- \\ \hline f_- \end{array}\right]$$

In each column of T, the sum of magnitudes of off diagonal entries is at most $1 + \sum_{i \ne j} k(p,i,j)$

$\Rightarrow$ If we impose $\sum_{i \ne j} k(p,j,i) < \sigma(p,j)$,

T is diagonally dominant & is invertible

We have proved:
PROP. If $(*)$ holds everywhere in interior of (G, σ, k), the forward problem has a unique solution.

$$\left[\begin{array}{c|c} T_1 & T_2 \\ \hline 0 & T_4 \end{array}\right]\left[\begin{array}{c} f_+ \\ \hline f \end{array}\right] = \left[\begin{array}{c} M_1 \\ \hline M_2 \end{array}\right]\left[\begin{array}{c} f_- \\ \hline f_- \end{array}\right]$$

Albedo (response) matrix

$$T_1 f_+ + T_2 f = M_1 f_-$$
$$T_4 f = M_2 f_-$$
$$\Rightarrow f = T_4^{-1} M_2 f_-$$
$$\Rightarrow T_1 f_+ + T_2 T_4^{-1} M_2 f_- = M_1 f_-$$
$$\Rightarrow f_+ = \underline{T_1^{-1}(M_1 - T_2 T_4^{-1} M_2)} f_-$$
$$= A f_-$$
$$A = T_1^{-1}(M_1 - T_2 T_4^{-1} M_2)$$

Want to recover σ, K from A

In general, cannot be done

1. $\sigma = \sigma(p)$, $k = k(p, i, j)$
2. $\sigma = \sigma(p)$, $K = K(p, i-j)$ e.g. $k(p, L), k(p, B)$ (left, back)
3. $\sigma = \sigma(p)$, $K = k(p)$ "isotropic scattering"

For each situation, which graphs are recoverable?

Note: Forward problem solved for general (G, σ, k), but for inverse problem, look at lattice graphs for now.