LAYERED NETWORKS, THE DISCRETE LAPLACIAN, AND A CONTINUED FRACTION IDENTITY

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ABSTRACT. We find explicit formulas for the conductivities of the layers in a layered electrical network so that the square of its response map is the negative of the discrete Laplacian.

CONTENTS

1. INTRODUCTION

In [3] it was proved that for any N there are layered electrical networks with N boundary nodes with such that the square of the response map Λ (Dirichlet-to-Neumann map) satisfies the equation

$$\Lambda^2 = -\Delta,$$

where Δ is the (discrete) Laplacian on the discrete circle with N nodes. However the problem of explicitly giving the conductivities on the layers was left open. This problem was solved in the University of Washington NSF REU program in the summer of 2008. In this paper we prove a continued fraction identity which yields formulas for the conductivities of the layers. For example if the network has $N = 4p+1$ degree one boundary nodes (*spikes*) and $2p$ layers, the conductivities, of the alternating segments of radii and arcs of circles, starting from the outer edges are

$$\tan\left(\frac{2p\pi}{4p+1}\right), \cot\left(\frac{(2p-1)\pi}{4p+1}\right), \tan\left(\frac{(2p-2)\pi}{4p+1}\right), \ldots, \cot\left(\frac{\pi}{4p+1}\right).$$

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For example in Figure 1 the outer spikes have conductivity $\tan\left(\frac{4\pi}{9}\right)$, the outer arcs have conductivity $\cot\left(\frac{3\pi}{9}\right)$, the inner ray segments have conductivity $\tan\left(\frac{2\pi}{9}\right)$, and the inner arcs have conductivity $\cot\left(\frac{\pi}{9}\right)$. It is sometimes convenient to represent the Laplacian on the circle as $\frac{\partial}{\partial\theta^2}$ and then our main theorem is an explicit formula for Λ so that

$$\Lambda^2 = -\frac{\partial}{\partial\theta^2}. \tag{1.1}$$

We will refer to [1] and [3] for notation, definitions, and basic results.

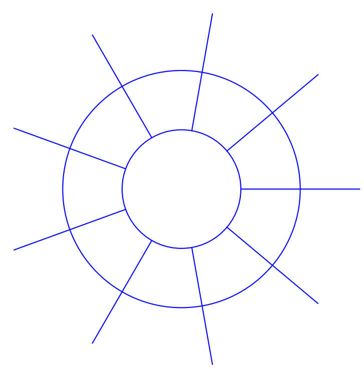

FIGURE 1. Four Layers, Nine Rays, No Central Vertex

2. FORMULATION OF THE PROBLEM

Let (V, E) be the vertices and edges of a graph (usually assumed connected) with a designated partition of vertices into interior vertices ($int(V)$) and boundary vertices ($\partial(V)$). The boundary is always assumed to be non-empty and we will assume no loops or multi-edges. Each edge $e \in E$ will have an assigned positive conductivity. It is convenient to associate a matrix, the Kirchhoff matrix K, to this network. We will index the vertices of G so that the boundary vertices are listed first. This indexing induces a block decomposition of K,

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}.$$

All of the information about the network is stored in K. The matrices K, A, C are symmetric and C is positive definite. Kirchhoff's current law for this network is expressed as $(Kx)_i = 0, i \in intV$ where x is a function defined on V. In [1] it is proved that for an arbitrary function ϕ defined on $\partial(V)$ there is a unique extension to a function u defined on all of V so that $(Ku)_i = 0, i \in intV$. The current flow ψ into the network at the boundary can be represented as a Schur complement,

$$\psi = (A - BC^{-1}B^T)\phi = \Lambda\phi.$$

The matrix Λ is the *response* or *Dirichlet-to-Neumann* map of the network.

The graph of a *layered* network is a graph conveniently represented in the unit disk in the plane by edges which are segments of radii and arcs of concentric circles (Figure 1). The boundary vertices are the vertices on the unit circle. It is convenient to coordinatize the vertices so that the arguments of the rays are located at angles $\theta = \frac{2\pi k}{N}, k = 0, \dots, N-1$. A *layer* is the set of segments of rays between successive concentric circles or the set of arcs on a circle. In a layered network conductivities are assumed constant on layers (conductivities depend only on the "radius"). We will express boundary values as a finite Fourier series. The function $e^{ik\theta}$ will symbolize the discrete function that assumes the values $e^{\frac{2\pi ijk}{N}}$ at the boundary vertices indexed by $j = 0 \dots N-1$.

For the rest of the paper we will assume that the number of boundary vertices $N = 2n+1$ is odd.

Let $D^2 = \Delta$ be the second difference operator (discrete Laplacian) on the discrete circle and let $L = -D^2$. The size of L will be implied by the context.

$$L = -D^2 = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}. \tag{2.1}$$

The eigenvectors of L are $e^{\pm ik\theta}$ and the eigenvalues are

$$(\omega_n^{(k)})^2 = 4\sin^2\left(\frac{k\pi}{2n+1}\right),\ k = 0, \dots n,$$

where we define $\omega_n^{(k)} = |e^{\frac{2\pi ik}{2n+1}} - 1| = 2\sin\left(\frac{k\pi}{2n+1}\right)$.

In [3] it is proved that the functions $e^{\pm ik\theta}, k = 0, \dots, n$ are an also eigenbasis for the response matrix Λ of a layered network with $2n+1$ boundary vertices. The non-zero eigenvalues are double eigenvalues. In [3] it is proved that the non-zero eigenvalues $\lambda_n^{(k)}$ of Λ are the values of the function $\mu\beta(\mu)$ at the numbers $\omega_n^{(k)}$, which are the positive square roots of the eigenvalues L and where $\beta(\mu)$ is the continued fraction

$$\beta(\mu) = \cfrac{1}{\frac{\mu}{c_n} + \cfrac{1}{c_{n-1}\mu + \ddots \cfrac{1}{\frac{\mu}{c_2} + \frac{1}{c_1\mu}}}},$$

and the c_i are the conductivities on the layers, with c_1 the conductivity on the innermost layer.

Since Λ and L have the same eigenvectors, to prove that $\Lambda^2 = L$ it will suffice to find conductivities c_j so that Λ^2 and L have the same eigenvalues. Hence we want to find c_j so that

$$\omega_n^{(k)} = \omega_n^{(k)}\beta(\omega_n^{(k)}).$$

This is equivalent to finding c_j so that $\beta(\omega_n^{(k)}) = 1$. Taking reciprocals, we have an interpolation problem: Find positive numbers c_j so that

$$1 = \frac{\mu}{c_n} + \cfrac{1}{c_{n-1}\mu + \cfrac{1}{\ddots \cfrac{1}{\frac{\mu}{c_2} + \frac{1}{c_1\mu}}}},$$

for $\mu = \omega_n^{(k)}, k = 1, \ldots, n$.

This is the correct formula for an even number of layers, with boundary spikes. It must be modified appropriately if the outer edges are arcs of circles and/or the number of layers is odd.

From now on we will use the notation $[a_j]_{j=1}^n$ for the continued fraction

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_2 + \frac{1}{a_1}}}}.$$

In this way, we can write:

$$[a_j]_{j=1}^{n+1} = a_{n+1} + \frac{1}{[a_j]_{j=1}^n}. \tag{2.2}$$

The identity we will prove is:

$$\left[\cot\left(\frac{2\pi j}{2n+1}\right)\mu\right]_{j=1}^n = 1, \tag{2.3}$$

for $\mu = \omega_n^{(k)}, k = 1, \ldots, n$. We introduce the following notation:

$$w = e^{-\frac{\pi i}{2n+1}},$$

and notice that

$$\begin{aligned}\cot\left(\frac{2\pi j}{2n+1}\right)\omega_n^{(k)} &= \frac{w^{-j}+w^j}{w^{-j}-w^j}(w^{-k}-w^k)\\ &= \frac{w^j+w^{-j}}{w^j-w^{-j}}(w^k-w^{-k}),\end{aligned}$$

since

$$w^{-k} - w^k = 2i\sin\left(\frac{k\pi}{2n+1}\right) = i\omega_n^{(k)},$$

and

$$\frac{w^{-j}+w^j}{w^{-j}-w^j} = \left(\frac{1}{i}\right)\cot\left(\frac{2\pi j}{2n+1}\right).$$

This is our theorem:

Theorem 2.1. *Let $n \geq 1$ and $w = e^{\frac{-\pi i}{2n+1}}$. For all $k = 1\ldots, n$,*

$$\left[\frac{w^j+w^{-j}}{w^j-w^{-j}}(w^k-w^{-k})\right]_{j=1}^n = 1. \tag{2.4}$$

3. OUR GOAL AND ITS REFORMULATION

Our goal is to show the following identity:

Main Theorem. *For integers k and n, with $n \geq k \geq 1$ we have:*

$$\left[\frac{w^j + w^{-j}}{w^j - w^{-j}}(w^k - w^{-k})\right]_{j=1}^{n} = 1, \tag{3.1}$$

where $w = e^{-\pi i/2n+1}$.

We can write $w = e^{\pi i}e^{-\pi i(2n/2n+1)} = -\zeta$, where $\zeta = e^{-\pi i(2n/2n+1)}$ is a primitive $(2n+1)$st root of unity. Letting x be formally indeterminate, we would like to find a general formula for the above continued fraction, with w replaced by $-x$, so that the entire expression may eventually be evaluated at $x = \zeta$. First, note that

$$\begin{aligned}\left[\frac{(-x)^j + (-x)^{-j}}{(-x)^j - (-x)^{-j}}((-x)^k - (-x)^{-k})\right]_{j=1}^{n} &= \left[\frac{x^j + x^{-j}}{x^j - x^{-j}}(-1)^k(x^k - x^{-k})\right]_{j=1}^{n} \\ &= (-1)^k \left[\frac{x^j + x^{-j}}{x^j - x^{-j}}(x^k - x^{-k})\right]_{j=1}^{n},\end{aligned}$$

so our task becomes finding a general way to compute $\left[\frac{x^j + x^{-j}}{x^j - x^{-j}}(x^k - x^{-k})\right]_{j=1}^{n}$, then evaluating at $x = \zeta$ to prove the following version of (3.1):

$$\left[\frac{\zeta^j + \zeta^{-j}}{\zeta^j - \zeta^{-j}}(\zeta^k - \zeta^{-k})\right]_{j=1}^{n} = (-1)^k. \tag{3.2}$$

4. A FEW SMALL CASES

The above is not difficult to accomplish for a few small values of k, as we demonstrate below.

Lemma 4.1. *For all integers $n \geq 1$, we have*

$$\left[\frac{x^j + x^{-j}}{x^j - x^{-j}}(x^1 - x^{-1})\right]_{j=1}^{n} = \frac{x^{n+1} - x^{-n-1}}{x^n - x^{-n}}.$$

Proof. We proceed by induction on n. For the base case $n = 1$, we have

$$\begin{aligned}\left[\frac{x^j + x^{-j}}{x^j - x^{-j}}(x - x^{-1})\right]_{j=1}^{1} &= \frac{x + x^{-1}}{x - x^{-1}}(x - x^{-1}) \\ &= \frac{x^2 - x^{-2}}{x - x^{-1}}.\end{aligned}$$

For the inductive step, we assume the formula is true for n and derive that it holds for $n+1$. Then we have

$$\begin{aligned}\left[\frac{x^j+x^{-j}}{x^j-x^{-j}}(x^1-x^{-1})\right]_{j=1}^{n+1} &= \frac{x^{n+1}+x^{-(n+1)}}{x^{n+1}-x^{-(n+1)}}(x^1-x^{-1})+\frac{1}{\left[\frac{x^j+x^{-j}}{x^j-x^{-j}}(x^1-x^{-1})\right]_{j=1}^{n}}\\ &= \frac{(x^{n+1}+x^{-(n+1)})(x-x^{-1})}{x^{n+1}-x^{-(n+1)}}+\frac{x^n-x^{-n}}{x^{n+1}-x^{-(n+1)}}\\ &= \frac{x^{n+2}-x^n+x^{-n}-x^{-(n+2)}+x^n-x^{-n}}{x^{n+1}-x^{-(n+1)}}\\ &= \frac{x^{n+2}-x^{-(n+2)}}{x^{n+1}-x^{-(n+1)}},\end{aligned}$$

as desired, completing the induction step. □

A similar re-expression is possible for $k=2$:

Lemma 4.2. *For all integers $n \geq 1$, we have*

$$\left[\frac{x^j+x^{-j}}{x^j-x^{-j}}(x^2-x^{-2})\right]_{j=1}^{n} = \frac{x^{2(n+1)}-2+x^{-2(n+1)}}{x^{2n}-2+x^{-2n}}.$$

Proof. Again we use a proof by induction. For the base case $n=1$, we have

$$\begin{aligned}\frac{x^1+x^{-1}}{x^1-x^{-1}}(x^2-x^{-2}) &= \frac{x^3+x-x^{-1}-x^{-3}}{x-x^{-1}}\\ &= \frac{(x^3+x-x^{-1}-x^{-3})(x-x^{-1})}{(x-x^{-1})(x-x^{-1})}\\ &= \frac{x^4+x^2-1-x^{-2}-x^2-1+x^{-2}+x^{-4}}{x^2-2+x^{-2}}\\ &= \frac{x^4-2+x^{-4}}{x^2-2+x^{-2}},\end{aligned}$$

as desired.

For the inductive step, we have

$$\begin{aligned}\left[\frac{x^j+x^{-j}}{x^j-x^{-j}}(x^2-x^{-2})\right]_{j=1}^{n+1} &= \frac{x^{n+1}+x^{-(n+1)}}{x^{n+1}-x^{-(n+1)}}(x^2-x^{-2})+\frac{1}{\left[\frac{x^j+x^{-j}}{x^j-x^{-j}}(x^2-x^{-2})\right]_{j=1}^{n}}\\ &= \frac{x^{n+1}+x^{-(n+1)}}{x^{n+1}-x^{-(n+1)}}(x^2-x^{-2})+\frac{x^{2n}-2+x^{-2n}}{x^{2(n+1)}-2+x^{-2(n+1)}}\end{aligned}$$

$$\begin{aligned}&= \frac{(x^{n+1}+x^{-(n+1)})(x^{n+1}-x^{-(n+1)})(x^2-x^{-2})}{(x^{n+1}-x^{-(n+1)})(x^{n+1}-x^{-(n+1)})}+\frac{x^{2n}-2+x^{-2n}}{x^{2(n+1)}-2+x^{-2(n+1)}}\\ &= \frac{x^{2n+4}-x^{2n}-x^{-2n}+x^{-(2n+4)}+x^{2n}-2+x^{-2n}}{x^{2(n+1)}-2+x^{-2(n+1)}}\\ &= \frac{x^{2(n+2)}-2+x^{-2(n+2)}}{x^{2(n+1)}-2+x^{-2(n+1)}}.\end{aligned}$$

□

We can use these two results to demonstrate that the main theorem holds for $k=1,2$.

Proposition 4.3. *The main theorem* (3.2) *holds for $k = 1$. In other words, for all $n \geq 1$, we have*

$$\left[\frac{\zeta^j + \zeta^{-j}}{\zeta^j - \zeta^{-j}}(\zeta - \zeta^{-1})\right]_{j=1}^{n} = -1,$$

where ζ is a primitive $2n+1$st root of unity.

Proof. By Lemma 4.1, it suffices to show that

$$-1 = \frac{\zeta^{n+1} - \zeta^{-(n+1)}}{\zeta^n - \zeta^{-n}}. \tag{4.1}$$

We can remove a common factor of $\zeta - \zeta^{-1}$ from both the numerator and denominator, finding

$$\begin{aligned}\frac{\zeta^{n+1} - \zeta^{-(n+1)}}{\zeta^n - \zeta^{-n}} &= \frac{\zeta^n + \zeta^{n-2} + \ldots + \zeta^{-n+2} + \zeta^{-n}}{\zeta^{n-1} + \zeta^{n-3} + \ldots + \zeta^{-n+3} + \zeta^{-n+1}} \\ &= \frac{(-\zeta^{n-1} - \zeta^{n-2} - \cdots - \zeta^{-n}) + \zeta^{n-2} + \ldots + \zeta^{-n+2} + \zeta^{-n}}{\zeta^{n-1} + \zeta^{n-3} + \ldots + \zeta^{-n+3} + \zeta^{-n+1}},\end{aligned}$$

where we have used the fact that $\zeta \neq 1$ is a root of $(x^{2n+1} - 1) = (x-1)(x^{2n} + x^{2n-1} + \ldots + 1)$, so that $\zeta^{2n} + \zeta^{2n-1} + \ldots + 1 = 0$ and $\zeta^n = -\zeta^{n-1} - \zeta^{n-2} - \ldots - \zeta^{-n}$.

$$\begin{aligned}&= \frac{-\zeta^{n-1} - \zeta^{n-3} - \ldots - \zeta^{-n+1}}{\zeta^{n-1} + \zeta^{n-3} + \ldots + \zeta^{-n+1}} \\ &= -1.\end{aligned}$$

□

Similarly, we have the following result corresponding to $k = 2$:

Proposition 4.4. *The main theorem* (3.2) *holds for $k = 2$; i.e. for all $n \geq 2$, we have*

$$\left[\frac{\zeta^j + \zeta^{-j}}{\zeta^j - \zeta^{-j}}(\zeta^2 - \zeta^{-2})\right]_{j=1}^{n} = 1,$$

where ζ is a primitive $2n+1$st root of unity.

Proof. By Lemma 4.2 and Equation (4.1), we can immediately write:

$$\begin{aligned}\left[\frac{\zeta^j + \zeta^{-j}}{\zeta^j - \zeta^{-j}}(\zeta^2 - \zeta^{-2})\right]_{j=1}^{n} &= \frac{\zeta^{2(n+1)} - 2 + \zeta^{-2(n+1)}}{\zeta^{2n} - 2 + \zeta^{-2n}} \\ &= \left(\frac{\zeta^{n+1} - \zeta^{-(n+1)}}{\zeta^n - \zeta^{-n}}\right)^2 \\ &= \left(\left[\frac{\zeta^j + \zeta^{-j}}{\zeta^j - \zeta^{-j}}(\zeta - \zeta^{-1})\right]_{j=1}^{n}\right)^2 \\ &= (-1)^2 = 1.\end{aligned}$$

□

At this point, it may seem obvious to attempt to prove some general identity of the form

$$\left[\frac{x^j + x^{-j}}{x^j - x^{-j}}(x^k - x^{-k})\right]_{j=1}^{n} = \left(\frac{x^{n+1} - x^{-(n+1)}}{x^n - x^{-n}}\right)^k.$$

Unfortunately, while this expression would quickly imply the main theorem, it is not valid for $k > 2$. Hence we must find a more accurate way to generalize Lemmas 4.1 and 4.2.

5. A General Computation

Consider rewriting the rational functions from Lemmas 4.1 and 4.2 in the following way:

$$\left[\frac{x^j+x^{-j}}{x^j-x^{-j}}(x^1-x^{-1})\right]_{j=1}^{n} = \frac{x^{n+1}-x^{-n-1}}{x^n-x^{-n}} = \frac{x(x^n)-x^{-1}(x^n)^{-1}}{x(x^{n-1})-x^{-1}(x^{n-1})^{-1}}$$
$$\left[\frac{x^j+x^{-j}}{x^j-x^{-j}}(x^2-x^{-2})\right]_{j=1}^{n} = \frac{x^{2(n+1)}-2+x^{-2(n+1)}}{x^{2n}-2+x^{-2n}} = \frac{x^2(x^n)^2-2+x^{-2}(x^n)^{-2}}{x^2(x^{n-1})^2-2+x^{-2}(x^{n-1})^{-2}}.$$

So it is conceivable that for arbitrary k, the continued fraction of level n should be expressible as a ratio of identical (Laurent) polynomials, with coefficients that are rational functions of x, and evaluated in the numerator at x^n and in the denominator at x^{n-1}. More precisely, for each n, let $\chi_n : \mathbb{Q}(x)[y, y^{-1}] \to \mathbb{Q}(x)$ be the homomorphism fixing $\mathbb{Q}(x)$ and sending y to x^n. For each k we would like there to be a $P_k \in \mathbb{Q}(x)[y, y^{-1}]$ so that for all n,

$$\frac{\chi_n(P_k)}{\chi_{n-1}(P_k)} = \left[\frac{x^j+x^{-j}}{x^j-x^{-j}}(x^k-x^{-k})\right]_{j=1}^{n}. \tag{5.1}$$

We can instead use Equation (2.2) to give a "recursive" requirement on P_k using:

$$\left[\frac{x^j+x^{-j}}{x^j-x^{-j}}(x^k-x^{-k})\right]_{j=1}^{n} = \frac{x^n+x^{-n}}{x^n-x^{-n}}(x^k-x^{-k}) + \frac{1}{\left[\frac{x^j+x^{-j}}{x^j-x^{-j}}(x^k-x^{-k})\right]_{j=1}^{n-1}}.$$

We will require,

$$\frac{\chi_n(P_k)}{\chi_{n-1}(P_k)} = \frac{x^n+x^{-n}}{x^n-x^{-n}}(x^k-x^{-k}) + \frac{\chi_{n-2}(P_k)}{\chi_{n-1}(P_k)} \text{ or, alternatively,} \tag{5.2}$$

$$\chi_n(P_k) = \left(\frac{x^n+x^{-n}}{x^n-x^{-n}}(x^k-x^{-k})\right)\chi_{n-1}(P_k) + \chi_{n-2}(P_k), \tag{5.3}$$

together with a "base case":

$$\frac{\chi_1(P_k)}{\chi_0(P_k)} = \left[\frac{x^j+x^{-j}}{x^j-x^{-j}}(x^k-x^{-k})\right]_{j=1}^{1} = \frac{x+x^{-1}}{x-x^{-1}}(x^k-x^{-k}). \tag{5.4}$$

As an alternative base case, we can extend the recurrence relation (5.2) through $n = 1$, so that comparing with (5.4) gives the simpler condition

$$\chi_{-1}(P_k) = 0. \tag{5.5}$$

In order to guarantee that the recurrence relation (5.3) holds, we will impose a similar condition on the P_k themselves. To wit, let $\sigma : \mathbb{Q}(x)[y, y^{-1}] \to \mathbb{Q}(x)[y, y^{-1}]$ be defined by fixing $\mathbb{Q}(x)$ and sending y to $x^{-1}y$, so that $\chi_{n-m} = \chi_n \circ \sigma^m$. Then we can impose a condition on P_k, namely:

$$P_k = \left(\frac{y+y^{-1}}{y-y^{-1}}(x^k-x^{-k})\right)\sigma(P_k) + \sigma^2(P_k), \tag{5.6}$$

so that when we evaluate with a χ_n homomorphism, we obtain equation (5.3). It will be our goal to find polynomials P_k in $\mathbb{Q}(x)[y, y^{-1}]$ that satisfy both (5.6) and (5.5). (In fact, we shall see that (5.6) actually implies (5.5).) By inspecting the forms of $P_1 = xy - x^{-1}y^{-1}$

and $P_2 = x^2y^2 - 2 + x^{-2}y^{-2}$ from the beginning of this section, we can guess as an ansatz that the general form of P_k will be

$$P_k(x) = \sum_{j=-k}^{k} a_j y^j, \tag{5.7}$$

where for each j, $a_j \in \mathbb{Q}(x)$, i.e is a rational function of x. (Note that each a_j depends on k, which we suppress in the following for clarity.) Equation (5.6) becomes

$$\sum_{j=-k}^{k} a_j y^j = \left(\frac{y+y^{-1}}{y-y^{-1}}(x^k - x^{-k})\right) \sum_{j=-k}^{k} a_j x^{-j} y^j + \sum_{j=-k}^{k} a_j x^{-2j} y^j.$$

Manipulating yields

$$\sum_{j=-k}^{k} a_j(1-x^{-2j})y^j = \left(\frac{y+y^{-1}}{y-y^{-1}}(x^k - x^{-k})\right) \sum_{j=-k}^{k} a_j x^{-j} y^j$$

$$\sum_{j=-k}^{k} a_j(1-x^{-2j})y^j(y-y^{-1}) = \sum_{j=-k}^{k} (x^k - x^{-k}) a_j x^{-j} y^j (y+y^{-1})$$

$$\sum_{m=-k-1}^{k+1} (a_{m-1}(1-x^{-2(m-1)}) - a_{m+1}(1-x^{-2(m+1)}))y^m \tag{5.8}$$

$$= \sum_{m=-k-1}^{k+1} (x^k - x^{-k})(a_{m-1}x^{-(m-1)} + a_{m+1}x^{-(m+1)})y^m, \tag{5.9}$$

where we define $a_j = 0$ for $|j| > k$ and re-index. Comparing the coefficients of y^m in 5.8 and 5.9, we have

$$\begin{aligned}
(a_{m-1}(1-x^{-2(m-1)}) - a_{m+1}(1-x^{-2(m+1)})) &= (x^k - x^{-k})(a_{m-1}x^{-(m-1)} + a_{m+1}x^{-(m+1)}) \\
a_{m-1}(1-x^{-2(m-1)} - x^{k-m+1} + x^{-k-m+1}) &= a_{m+1}(1-x^{-2(m+1)} + x^{k-m-1} - x^{-k-m-1}) \\
(a_{m-1})(x^{-m-k+1})(1-x^{k-m+1})(x^{k+m-1}+1) &= (a_{m+1})(x^{-m-k-1})(1+x^{k-m-1})(x^{m+k+1}-1).
\end{aligned}$$

The above coefficient of a_{m-1} is nonzero iff $k - m + 1 \neq 0$, i.e. $m - 1 \neq k$. Hence we cannot escape setting a_{k-1} proportional to $a_{k+1} = 0$, and hence also setting a_{k-3} proportional to $a_{k-1} = 0$, and so on; thus $a_m = 0$ unless $k - m$ is a multiple of 2. On the other hand, a_k need not be zero though a_{k+2} is, and the above relation fixes the ratio of a_{m+1} to a_{m-1} for all $-k < m < k$. Thus we have proved:

Lemma 5.1. *Suppose P_k is a polynomial in $Q(x)[y, y^{-1}]$ with coefficients a_m as in* (5.7)*. Then P_k satisfies the relation* (5.6) *iff the coefficients $a_m = 0$ whenever m is not congruent to k modulo 2, and if $|m| < k$ the coefficients also satisfy:*

$$\frac{a_{m+1}}{a_{m-1}} = (-x^2)\frac{(x^{k-m+1}-1)(x^{k+m-1}+1)}{(1+x^{k-m-1})(x^{k+m+1}-1)}.$$

We also have a simple relation between a_m and a_{-m}, as shown below:

Lemma 5.2. *Suppose P_k and a_m are as in Lemma 5.1, and that a_m is nonzero (and hence necessarily a_{-m} is as well). Then $a_m/a_{-m} = (-x^2)^m$.*

Proof. We use induction on m, but there are two base cases: $m = 0$ or $m = 1$, depending on whether k is even or odd. If k is even and $m = 0$, then we have

$$\frac{a_0}{a_0} = 1 = (-x^2)^0.$$

If k is odd and $m = 1$, then

$$\frac{a_1}{a_{-1}} = (-x^2)\frac{(x^{k+1}-1)(x^{k-1}+1)}{(1+x^{k-1})(x^{k+1}-1)} = (-x^2)^1.$$

Now suppose we know that $a_{m-1}/a_{-m+1} = (-x^2)^{m-1}$; we will show that $a_{m+1}/a_{-m-1} = (-x^2)^{m+1}$. In particular:

$$\begin{aligned}\frac{a_{m+1}}{a_{-m-1}} &= \frac{a_{m+1}}{a_{m-1}}\frac{a_{m-1}}{a_{-m+1}}\frac{a_{-m+1}}{a_{-m-1}}\\ &= (-x^2)\frac{(x^{k-m+1}-1)(x^{k+m-1}+1)}{(1+x^{k-m-1})(x^{k+m+1}-1)}(-x^2)^{m-1}(-x^2)\frac{(x^{k+m+1}-1)(x^{k-m-1}+1)}{(1+x^{k+m-1})(x^{k-m+1}-1)}\\ &= (-x^2)^{m+1}.\end{aligned}$$

Hence $a_m/a_{-m} = (-x^2)^m$ for any such m. ☐

Lemma 5.3. *Let k be odd, and suppose P_k satisfies the recurrence relation* (5.6)*. Then P_k also satisfies the base case condition* (5.5)*:*

$$\chi_{-1}(P_k) = 0.$$

Proof. We have

$$\begin{aligned}\chi_{-1}(P_k) &= \sum_{-k\le j\le k} a_j x^{-j}\\ &= \sum_{0<j\le k} a_j x^{-j} + a_{-j}x^j\\ &= \sum_{0<j\le k} a_j x^{-j} + a_j(-x^2)^{-j}x^j \text{ by Lemma 5.2.}\\ &= \sum_{0<j\le k} a_j x^{-j} - a_j x^{-j} = 0,\end{aligned}$$

since if j is even, we have $a_j = 0$, and if j is odd, we have $(-x^2)^{-j}x^j = -x^{-j}$. ☐

The implication in Lemma 5.3 holds for even k as well, but the proof is much more involved.

Lemma 5.4. *Let k be even, and suppose P_k satisfies the recurrence relation* (5.6)*. Then P_k also satisfies the base case condition* (5.5)*:*

$$\chi_{-1}(P_k) = 0.$$

Proof. Let $k = 2\ell$. Now the condition (5.6) on P_k only fixes the coefficients a_m up to an overall factor from $\mathbb{Q}(x)$, so we can, without loss of generality, set $a_0 = 1$ for ease of computation. Then employing similar manipulations to those in the proof of Lemma 5.3

give:

$$\begin{aligned}
\chi_{-1}(P_k) &= \sum_{i=-k}^{k} a_i x^{-i} \\
&= \sum_{\substack{-k\le i\le k \\ i\,even}} a_i x^{-i} \text{ since } a_i = 0 \text{ if } i \text{ is odd} \\
&= \sum_{-\ell\le j\le \ell} a_{2j} x^{-2j} \\
&= 1 + \sum_{j=1}^{\ell} a_{2i} x^{-2j} + a_{-2j} x^{2j} \\
&= 1 + \sum_{j=1}^{\ell} a_{2j} x^{-2j} + a_{2j}(-x^2)^{-2j} x^{2j} \text{ by Lemma 5.2} \\
&= 1 + \sum_{j=1}^{\ell} 2a_{2j} x^{-2j}.
\end{aligned}$$

So our problem reduces to showing

$$-1 = 2\sum_{j=1}^{\ell} a_{2j} x^{-2j}.$$

By the recurrence relation in Lemma 5.1 for the a_i, we have

$$\begin{aligned}
2\sum_{j=1}^{\ell} x^{-2j} a_{2j} &= 2\sum_{j=1}^{\ell} x^{-2j} \prod_{0\le r<j} \frac{a_{(2r+1)+1}}{a_{(2r+1)-1}} \\
&= 2\sum_{j=1}^{\ell} x^{-2j} \prod_{0\le r<j} -x^2 \frac{(x^{2\ell-2r}-1)(x^{2\ell+2r}+1)}{(x^{2\ell-2r-2}+1)(x^{2\ell+2r+2}-1)} \\
&= 2\sum_{j=1}^{\ell} x^{-2j} (-x^2)^j \prod_{0\le r<j} \frac{(x^{2\ell-2r}-1)(x^{2\ell+2r}+1)}{(x^{2\ell-2r-2}+1)(x^{2\ell+2r+2}-1)} \\
&= 2\sum_{j=1}^{\ell} (-1)^j \prod_{0\le r<j} \frac{(x^{2\ell-2r}-1)(x^{2\ell+2r}+1)}{(x^{2\ell-2r-2}+1)(x^{2\ell+2r+2}-1)}.
\end{aligned}$$

In order to evaluate this, we will prove the following identity for arbitrary $0 \le s \le \ell$:
(5.10)

$$\sum_{j=s}^{\ell} (-1)^{j-s} \prod_{s\le r<j} \frac{(x^{2\ell-2r}-1)(x^{2\ell+2r}+1)}{(x^{2\ell-2r-2}+1)(x^{2\ell+2r+2}-1)} = \frac{(x^{2\ell-2s}+1)(x^{2\ell+2s}-1)}{2(x^{4\ell}-1)},$$

or, for the sake of brevity:

$$\mathrm{LHS}(s) = \mathrm{RHS}(s).$$

We will induct on s, but in reverse: the base case will be $s = \ell$, and for the induction step we will assume that the identity holds at $s+1$ and show it holds for s. The base case is simple: when $s = \ell$, the sum on the left-hand side of (5.10) consists of only one term, namely an empty product, so $\mathrm{LHS}(\ell) = 1$. The right-hand side, in turn, evaluates to

$$\mathrm{RHS}(\ell) = \frac{(x^{2\ell-2\ell}+1)(x^{2\ell+2\ell}-1)}{2(x^{4\ell}-1)} = \frac{(1+1)(x^{4\ell}-1)}{2(x^{4\ell}-1)} = 1,$$

so the identity (5.10) holds when $s = \ell$.

For the induction step, we now assume that the identity (5.10) holds at $s+1$. Then we can write:

$$\begin{aligned}\mathrm{LHS}(s) &= \sum_{j=s}^{\ell}(-1)^{j-s}\prod_{s\le r<j}\frac{(x^{2\ell-2r}-1)(x^{2\ell+2r}+1)}{(x^{2\ell-2r-2}+1)(x^{2\ell+2r+2}-1)} \\ &= 1 + \sum_{j=s+1}^{\ell}(-1)^{j-s}\prod_{s\le r<j}\frac{(x^{2\ell-2r}-1)(x^{2\ell+2r}+1)}{(x^{2\ell-2r-2}+1)(x^{2\ell+2r+2}-1)},\end{aligned}$$

since the $j = s$ term is still just an empty product. Now we can factor out a (-1) and the $r = s$ factor from each term in the sum to obtain:

$$\mathrm{LHS}(s) = 1 - \frac{(x^{2\ell-2s}-1)(x^{2\ell+2s}+1)}{(x^{2\ell-2s-2}+1)(x^{2\ell+2s+2}-1)}\sum_{j=s+1}^{\ell}(-1)^{j-(s+1)}\prod_{s+1\le r<j}\frac{(x^{2\ell-2r}-1)(x^{2\ell+2r}+1)}{(x^{2\ell-2r-2}+1)(x^{2\ell+2r+2}-1)}.$$

But this sum of products is now just $\mathrm{LHS}(s+1)$, so we can apply the induction hypothesis:

$$\begin{aligned}\mathrm{LHS}(s) &= 1 - \frac{(x^{2\ell-2s}-1)(x^{2\ell+2s}+1)}{(x^{2\ell-2s-2}+1)(x^{2\ell+2s+2}-1)}\mathrm{LHS}(s+1) \\ &= 1 - \frac{(x^{2\ell-2s}-1)(x^{2\ell+2s}+1)}{(x^{2\ell-2s-2}+1)(x^{2\ell+2s+2}-1)}\left(\frac{(x^{2\ell-2(s+1)}+1)(x^{2\ell+2(s+1)}-1)}{2(x^{4\ell}-1)}\right) \\ &= 1 - \frac{(x^{2\ell-2s}-1)(x^{2\ell+2s}+1)}{2(x^{4\ell}-1)} = \frac{2(x^{4\ell}-1)-(x^{2\ell-2s}-1)(x^{2\ell+2s}+1)}{2(x^{4\ell}-1)}.\end{aligned}$$

We can simplify the numerator, obtaining:

$$\begin{aligned}\mathrm{LHS}(s) &= \frac{2x^{4\ell}-2-x^{4\ell}-x^{2\ell-2s}+x^{2\ell+2s}+1}{2(x^{4\ell}-1)} \\ &= \frac{x^{4\ell}-x^{2\ell-2s}+x^{2\ell+2s}-1}{2(x^{4\ell}-1)} \\ &= \frac{(x^{2\ell-2s}+1)(x^{2\ell+2s}-1)}{2(x^{4\ell}-1)} \\ &= \mathrm{RHS}(s).\end{aligned}$$

Hence (5.10) holds for all s, and in particular at $s = 1$, so we find:

$$\sum_{j=1}^{\ell}(-1)^{j-1}\prod_{1\le r<j}\frac{(x^{2\ell-2r}-1)(x^{2\ell+2r}+1)}{(x^{2\ell-2r-2}+1)(x^{2\ell+2r+2}-1)} = \frac{(x^{2\ell-2}+1)(x^{2\ell+2}-1)}{2(x^{4\ell}-1)}.$$

Now we are in a position to evaluate our original sum:

$$\begin{aligned}2\sum_{j=1}^{\ell}a_{2j}x^{-2j} &= 2\sum_{j=1}^{\ell}(-1)^j\prod_{0\le r<j}\frac{(x^{2\ell-2r}-1)(x^{2\ell+2r}+1)}{(x^{2\ell-2r-2}+1)(x^{2\ell+2r+2}-1)} \\ &= -2\frac{(x^{2\ell}-1)(x^{2\ell}+1)}{(x^{2\ell-2}+1)(x^{2\ell+2}-1)}\sum_{j=1}^{\ell}(-1)^{j-1}\prod_{1\le r<j}\frac{(x^{2\ell-2r}-1)(x^{2\ell+2r}+1)}{(x^{2\ell-2r-2}+1)(x^{2\ell+2r+2}-1)},\end{aligned}$$

where we have factored out the $r = 0$ term common to all summands,

$$\begin{aligned} &= -2\frac{(x^{2\ell}-1)(x^{2\ell}+1)}{(x^{2\ell-2}+1)(x^{2\ell+2}-1)}\frac{(x^{2\ell-2}+1)(x^{2\ell+2}-1)}{2(x^{4\ell}-1)} \\ &= -1. \end{aligned}$$

Hence $\chi_{-1}(P_k) = 1 + 2\sum_{j=1}^{\ell} a_{2j}x^{-2j} = 0$ as we desired. □

Lemma 5.5. *For all k, if k is even, then setting $a_0 = 1$ and if k is odd, setting $a_1 = x$ yields*

$$\chi_0(P_k) \neq 0.$$

Proof. Suppose for contradiction that $\chi_0(P_k) = 0$. Then since $\chi_{-1}(P_k) = 0$ as well, and since the $\chi_n(P_k)$ satisfy relation (5.3), we have $\chi_n(P_k) = 0$ for all n. Then we have $y^k \cdot P_k$ is a polynomial in $\mathbb{Q}(x)[y]$ which has roots x^n for all n. Since $y^k P_k$ has infinitely many roots, $y^k P_k = 0$. But since $a_0 \neq 0$ for k even and $a_1 \neq 0$ for k odd, we have $y^k P_k \neq 0$. This is a contradiction. Hence $\chi_0(P_k) \neq 0$. □

Now we know that all that is necessary for P_k to satisfy both the recurrence relation (5.6) and the base case (5.5) is for the a_m to satisfy the relations found in Lemma 5.1. Next we have a pair of lemmas to show that we don't accidentally divide by zero when evaluating with the χ_n homomorphisms.

Lemma 5.6. *Suppose $P_k \neq 0$ satisfies the recurrence relation* (5.6)*. Then for $n \geq 0$, we have $\chi_n(P_k) \neq 0$, and for $n \geq 1$*

$$\frac{\chi_n(P_k)}{\chi_{n-1}(P_k)} = \left[\frac{x^j + x^{-j}}{x^j - x^{-j}}(x^k - x^{-k})\right]_{j=1}^{n}.$$

Proof. The proof proceeds by induction. For the base case $n = 0$, we know from Lemma 5.5 that $\chi_0(P_k) \neq 0$. Also, we use the fact that $\chi_{-1}(P_k) = 0$, so that by 5.3

$$\begin{aligned} \frac{\chi_1(P_k)}{\chi_0(P_k)} &= \frac{x + x^{-1}}{x - x^{-1}}(x^k - x^{-k}) + \frac{\chi_{-1}(P_k)}{\chi_0(P_k)} \\ &= \frac{x + x^{-1}}{x - x^{-1}}(x^k - x^{-k}) = \left[\frac{x^j + x^{-j}}{x^j - x^{-j}}(x^k - x^{-k})\right]_{j=1}^{1}. \end{aligned}$$

For the inductive step, we assume that $\chi_{n-1}(P_k) \neq 0$ and that

$$\frac{\chi_{n-1}(P_k)}{\chi_{n-2}(P_k)} = \left[\frac{x^j + x^{-j}}{x^j - x^{-j}}(x^k - x^{-k})\right]_{j=1}^{n-1}.$$

Then we have:

$$\begin{aligned} \chi_n(P_k) &= \left(\frac{x^n + x^{-n}}{x^n - x^{-n}}(x^k - x^{-k})\right)\chi_{n-1}(P_k) + \chi_{n-2}(P_k) \\ \frac{\chi_n(P_k)}{\chi_{n-1}(P_k)} &= \frac{x^n + x^{-n}}{x^n - x^{-n}}(x^k - x^{-k}) + \frac{1}{\frac{\chi_{n-1}(P_k)}{\chi_{n-2}(P_k)}} \\ &= \frac{x^n + x^{-n}}{x^n - x^{-n}}(x^k - x^{-k}) + \frac{1}{[\frac{x^j + x^{-j}}{x^j - x^{-j}}(x^k - x^{-k})]_{j=1}^{n-1}} \\ &= \left[\frac{x^j + x^{-j}}{x^j - x^{-j}}(x^k - x^{-k})\right]_{j=1}^{n}. \end{aligned}$$

Note that if we evaluate the continued fraction, setting x to be a real number greater than 1, all the partial quotients, $\frac{x^j+x^{-j}}{x^j-x^{-j}}(x^k - x^{-k})$, are positive. Consequently the whole continued fraction is positive. Hence the expression with unevaluated x is a nonzero element of $\mathbb{Q}(x)$, so $\chi_n(P_k)$ cannot be zero either. $\square$

Now we have almost reached our goal. Next is a lemma which shows how nicely our polynomials behave when we evaluate them at ζ:

Lemma 5.7. *Let ζ be a $(2n+1)$st root of unity, and P_k satisfy the recurrence relation* (5.6)*. Then we have*

$$\chi_n(P_k)(\zeta) = (-1)^k \chi_{n-1}(P_k)(\zeta).$$

Proof. We have

$$\begin{aligned}
\chi_n(P_k)(\zeta) &= \sum_{j \equiv k \mod 2} a_j(\zeta)\zeta^{nj} \\
&= \sum_{j \equiv k \mod 2} a_j(\zeta)\zeta^{-(n+1)j} \text{ since } \zeta^n = \zeta^{-(n+1)} \\
&= \sum_{j \equiv k \mod 2} a_{-j}(\zeta)(-\zeta^2)^j\zeta^{-(n+1)j} \text{ by Lemma 5.2} \\
&= (-1)^k \sum_{j \equiv k \mod 2} a_{-j}(\zeta)\zeta^{-(n-1)j} \\
&= (-1)^k \sum_{j \equiv k \mod 2} a_j(\zeta)\zeta^{(n-1)j} \\
&= (-1)^k \chi_{n-1}(P_k)(\zeta).
\end{aligned}$$

$\square$

Main Theorem. *Let $n \geq k \geq 1$, and let ζ be a primitive $(2n+1)$st root of unity. Then we have*

$$\left[\frac{\zeta^j + \zeta^{-j}}{\zeta^j - \zeta^{-j}}(\zeta^k - \zeta^{-k})\right]_{j=1}^n = (-1)^k.$$

Proof. Define the Laurent polynomial $P_k \in \mathbb{Q}(x)[y, y^{-1}]$ as above, by stipulating that its coefficients satisfy the relation in Lemma 5.1, and for definiteness fixing $a_0 = 1$ or $a_1 = x$ if k is even or odd, respectively. We have $\chi_n(P_k) \neq 0$ by Lemma 5.6, but it could still transpire that $\chi_n(P_k)(\zeta) = 0$ or ∞. In other words ζ might be a zero of the numerator or denominator of $\chi_n(P_k)$. In this case, (see, for example, [2]), we know that the minimal polynomial of ζ, which is the cyclotomic polynomial

$$\Phi_{2n+1}(x) = \prod_{\zeta' \text{ a primitive } (2n+1)\text{st root of unity}} (x - \zeta'),$$

divides the numerator or denominator of $\chi_n(P_k)$, so for example if ζ is a zero of the numerator, we can replace P_k with $\widetilde{P}_k = P_k/\Phi_{2n+1}(x)^t$, where t is the multiplicity of Φ_{2n+1} as a factor of the numerator of $\chi_n(P_k)$, and preserve all of the crucial properties of P_k. such as the relations in Lemma 5.1. Then $\widetilde{P}_k$ satisfies (5.6) and $\chi_n(\widetilde{P}_k)(\zeta) \neq 0, \infty$ and by Lemma 5.7, $\chi_{n-1}(\widetilde{P}_k)(\zeta) \neq 0, \infty$. Thus we have:

$$\begin{aligned}
\left[\frac{\zeta^j + \zeta^{-j}}{\zeta^j - \zeta^{-j}}(\zeta^k - \zeta^{-k})\right]_{j=1}^n &= \frac{\chi_n(\widetilde{P}_k)(\zeta)}{\chi_{n-1}(\widetilde{P}_k)(\zeta)} \text{ by Lemma 5.6} \\
&= (-1)^k \text{ by Lemma 5.7.}
\end{aligned}$$

$\square$

6. THE NETWORK AND ITS KIRCHHOFF MATRIX

In this section we display an example network and its Kirchhoff matrix. Our example is a six-layer network with central vertex and nine boundary to boundary edges. The conductivities, beginning at the boundary-to-boundary edges are,

$$c_6 = \cot\left(\frac{6\pi}{13}\right),\ c_5 = \tan\left(\frac{5\pi}{13}\right),\ c_4 = \cot\left(\frac{4\pi}{13}\right),$$
$$c_3 = \tan\left(\frac{3\pi}{13}\right),\ c_2 = \cot\left(\frac{2\pi}{13}\right),\ c_1 = \tan\left(\frac{\pi}{13}\right).$$

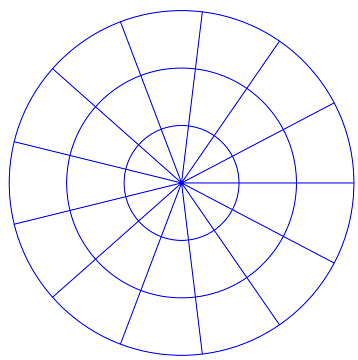

FIGURE 2. Six Layers, Thirteen Rays, Central Vertex

The Kirchhoff matrix in block form is

$$K = \begin{bmatrix} c_6L + c_5I & -c_5I & 0 & 0 \\ -c_5I & c_5I + c_4L + c_3I & -c_3I & 0 \\ 0 & -c_3I & c_3I + c_2L + c_1I & -c_1E \\ 0^T & 0^T & -c_1E^T & 13c_1 \end{bmatrix}.$$

In this formula, I is the 13×13 identity matrix; in column three, 0 is the 13×13 matrix of all 0's; in column four, 0 is the 13×1 column matrix of all 0's; in column four, E is the 13×1 column matrix of all 1's; and L is the 13×13 matrix defined in equation 2.1.

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