AGREEMENT PROTOCOLS AND NETWORK IDENTIFICATION

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Abstract. In this paper we introduce a new inverse problem. An agreement protocol can also be modeled using the Laplacian operatator. Using an input output mapping, we hope to indentify the graph underlying the network. We assume constant conductivity of 1 on all edges.

1. Introduction

Let Γ be a graph with n nodes.

1.1. The Forward Problem.

Suppose we number the nodes of Γ. Now consider a vertex function $v_j(t)$, where j indicates the vertex, and t the time. With each time step, information gets passed between the vertices in the following fashion: Suppose i has r neighbors $j, k, ..., o$. Then the time derivative of $v_i(t)$ is

$$v(\dot{_i(t)}) = rv_i(t) - v_j(t) - v_k(t) ... - v_o(t)$$

. The equation that models the time derivative is then clearly

$$[v\dot{(t)}] = -L[v(t)]$$

where L is the laplacian for the graph: the adjacency matrix for the graph with the diagonal such that the row and column sums are zero. Over time, the initial states of all of the vertices will tend towards one value.

1.2. Our Inverse Problem.

We are going to pick a certain set R of r nodes at which to excite the network- by sending a current $[u(t)] = [u_1(t)...u_r(t)]$ along them, if we think of this as an electrical network. Then, at another, not necessarily disjoint, set S of s nodes, we will read output measurements $[y] = C[x(t)]$ where $[y] = [y_1 \dots y_s]$. Our system is now

$$P = \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

B, C are simply matrices of 0's and 1's which pick out and make u the right size, pick out the vertices in x at which measurements are taken.
We have a black box engineering method called system identification. This method yields an approximation of the original system $P = \{A, B, C\} \rightarrow P\prime = \{\tilde{A}, tildeB, tildeC\}$.

Date: August 11, 2009.

2. Tackling the Inverse Problem

2.1. **What information can we get directly from the measurements?** The relation between the two systems is a transformation T so that

$$\begin{array}{l} \tilde{A} = TAT^{-1} \\ \tilde{B} = TB \\ \tilde{C} = CT^{-1} \end{array} .$$

Thus we know that $CAB = \tilde{C}\tilde{A}\tilde{B}$. Furthermore, because of the structure of B, C, the product yields some entries in the Laplacian A, which fully characterizes the graph.
Our goal is to find A, knowing $B, C, P\prime$.
A can be characterized in a number of ways. We know that it is the Kirchoff matrix for a simple connected graph, so it has all of those poperties. Furthermore, A has only integer entries.
The next step we can take is to adopt the convention of number the boundary nodes first. This allows for a natural block partition of the Laplacian, and of the transformation T.

Remark 1. *Another convention which we adopt for simplicity is that $C = B^T$. This means that we measure input and output at the same nodes- and it makes it easier to think of there simply being one boundary. This seems like a very simple convention, but it actually gives us some immediate insight into the problem. Notice now that* $B = \begin{bmatrix} I_{m\times m} \\ 0 \end{bmatrix}$.

Now we can think of

$$[A] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

$A_{12} = A_{21}^T$ A_{11} is $m \times m$. We can partition T the same way, but T does not have any explicit properties except for invertability; it is not necessarily a symmetric matrix.
It is now obvious that the product $BAC = A_{11}$. Thus, we have the fixed boundary of our unknown graph, and also some access into the interior. Specifically, we know how many interior vertices are connected to each boundary vertex. This is good, since it will allow us to identify boundary antennae- but it will not allow us to differentiate between a spike and an antannae of any number of spikes.
We then observe that we have not used $\tilde{B}, \tilde{C}$ for any information. It is interesting to note that before we adopted the convention of numbering the boundary vertices first, we had no idea what they meant. Now, it is totally obvious. Partition T in the usual manner, and use block mutliplication to see the following.

$$\tilde{B} = TB = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} I_{m\times m} \\ 0 \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix}$$

$$\tilde{C} = CT^{-1} = \begin{bmatrix} I_{m\times m} & 0 \end{bmatrix} \begin{bmatrix} T_{11}^{-1} & T_{12}^{-1} \\ T_{21}^{-1} & T_{22}^{-1} \end{bmatrix} = \begin{bmatrix} T_{11}^{-1} & T_{12}^{-1} \end{bmatrix}$$

We now have the explicit relationship between parts of T and the boundary. One of our intial attempts to construct T and A from $\tilde{A}$ involved breaking T up into elementary matrices, and trying to equate those to physical transformations of the graph. This does not turn out to be a viable method, because most elementary

operations on a matrix do not preserve many properties of the Kirchoff matrix.
Our next observation is that A and $\tilde{A}$ have the same eigenvalues, because they are symmetric matrices. This means that we know the trace of A. Since we also know A_{11}, we know the trace of A_{22}, which is the difference of the two.

2.2. **Degree Based Graph Construction.** Since we now have the sum of the unknown diagonal entries (there are $n - m$ of them), we can can generate all of the sets of cardinality $n - m$ of integers > 0 whose sum is $tr(A_{22})$. Then we can begin to eliminate some of these sets as possible for the diagonal of the Kirchhoff matrix. The diagonal of A is the degree sequence for the graph. There can be more than one graph associated to each degree sequence, but we can limit them somewhat since we have fixed the boundary of the graph. We can further eliminate many of these because they will have the wrong eigenvalues, even though they have the correct trace.
Now we will describe the algorithm through which we can build all possible graphs with a given degree sequence. This is outlines in the paper "Degree Based Graph Construction" by Kim, Toroczkai, Erdos, Miklos, and Szekely, avaialbe on the arXiv.

3. Some interesting things from Combinatorial/Algebraic Graph Theory

In this section the author will talk about her attempt to read section 5 of C.D. Godsil's paper "Tools from Linear Algebra" which appeared as Chapter 31 in the book "Handbook for Combinatorics." The paper was very interesting, but we had some trouble reading it due to numerous misprints and mistakes. The paper can be found at http://quoll.uwaterloo.ca/pstuff/tools.pdf.

4. Interpretting the characteristic polynomial of the adjacency matrix of a graph

Here we have anumber of goals. First, we wish to see what information can be read off from the coefficients of the characteristic polynomail of an adjacency matrix; this had been done before, but we derived this independantly.

5. Bibliography

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