Eigenvalues of Response Matrices

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Abstract

We examine the eigenvalues of the response matrix of an electrical network, specifically bounding the eigenvalues through physical interpretation and matrix analysis. Additionally, the general characteristic polynomial is considered and is evaluated in the n-star case.

Contents

1 Preliminaries

Let $G = (V, E)$ be a graph where V is the set of vertices and E is the set of edges. Let $\Omega = (G, \gamma)$ be a network where the vertices are nodes of the network and γ is a function defined on the edges of the network. V is partitioned into two disjoint sets: boundary nodes denoted by ∂G and interior nodes denoted by int G. Let v be a voltage function defined on the nodes of the network. Then a network is γ-harmonic if for all interior nodes p: $\sum_{q \sim p} \gamma_{pq}(v(p) - v(q)) = 0$ where $q \sim p$ denotes all nodes q such that an edge from p to q exists. For any electrical network Ω with a γ-harmonic function defined on its vertices, there exists a Kirchhoff matrix K defined as such:

$$K_{ij} = \begin{cases} -\gamma_{ij} & \text{if the edge } ij \text{ exists,} \\ \sum_{k \neq i} \gamma_{ik} & \text{if } i = j. \end{cases}$$

If an edge does not exist between i and j, K_{ij}=0. The row sums of K are equal to 0 as $\gamma_{ij} = \gamma_{ji}$. Thus, K is symmetric and positive semi-definite. K can be partitioned into four submatrices A, B, B^T and C.

$$\text{K=} \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

The response matrix Λ is defined by taking the Schur complement of K with respect to C, which equals: $A - BC^{-1}B^T$. Λ, like K, has row sums equal to 0, is symmetric, and is positive semi-definite. Let ϕ be a column vector where ϕ_i be the boundary voltage at boundary node i and ψ be a column vector where ψ_i be the boundary current at boundary node i. Then the response matrix maps boundary voltages to boundary currents, such that $\Lambda\phi = \psi$.

Let λ be an eigenvalue of Λ. Then $\lambda\phi$=ψ, indicating that the boundary currents are a scalar multiple of the boundary voltages. Because the net current flow into any network is 0, $\sum_i \psi_i = \lambda \sum_i \phi_i = 0$. Thus it follows that if $\lambda \neq 0$, then $\sum_i \phi_i = 0$. Also Λ is a symmetric $n \times n$ matrix, implying that there exist n eigenvalues with n linearly independent eigenvectors that form a basis of $\mathbb{R}^n$.

Lemma 1.1. *0 is an eigenvalue for all response matrices with multiplicity 1.*

Proof. Let there exist a constant column vector ϕ. Then it follows from the row sum property of Λ that $\Lambda\phi = 0$. Thus, ϕ is an eigenvector of Λ corresponding to an eigenvalue of 0. A non-constant ϕ cannot have eigenvalue 0 because there must be a current flow for the network as the current along boundary-interior edges is nonzero. Thus, 0 only exists as an eigenvalue for constant ϕ. □

Remark 1. *Lemma 1.1 bears a geometric interpretation. A constant column vector ϕ is equivalent to setting all the boundary voltages equal to a constant voltage. By the maximum/minimum principle, all interior voltages equal this constant value. This implies that no current flows through the network, which corresponds to an eigenvalue of 0.*

Lemma 1.2. *0 is the lower bound for all eigenvalues of a response matrix.*

Proof. Normalize the eigenvector ϕ such that $||\phi||_\infty = 1$. Let p be the boundary node such that $v(p) = 1$. If p is connected to n different nodes:

$$\lambda = \sum_{i}^{n} (1 - v_i) a_i$$

where v_i denotes the voltage at node i and a_i denotes the conductivity of the edge from p to i. Because $-1 \leq v_i \leq 1$ by the maximum/minimum principle, $\lambda \geq 0$. □

Remark 2. *Lemma 1.2 may also be proven using the positive semi-definite property of a response matrix, given that a positive semi-definite matrix has eigenvalues greater than or equal to 0.*

2 Bounds for the Eigenvalues of Λ

In [1], the goal was to find eigenvalues of n-star and n-lattice graphs. While determining specific eigenvalues is difficult, finding an upper bound for eigenvalues of general graphs proved to be much more feasible.

Definition 1. *A boundary spike network is a connected network such that each boundary node neighbors only one interior node.*

Definition 2. *A boundary antenna is a pair of boundary nodes that neighbor the same interior node.*

2.1 Physical Interpretations for Eigenvalues of Λ

A graph containing a boundary antenna with equal conductivities yields interesting properties for eigenvalues of its response matrix.

Theorem 2.1. *If a boundary spike network has a boundary antenna with the same conductivity a on both edges, a is an eigenvalue.*

Proof. Let $G = (V, E)$ be the graph corresponding to the boundary spike network. Let pq and pr be the edges of a boundary antenna with $\gamma(pq) = \gamma(pr) = a$, where p is an interior vertex and q and r are boundary vertices. Let ϕ be a vector such that $\phi_q = 1$, $\phi_r = -1$ and $\phi_s = 0$ for all other boundary vertices s. The function v which is 0 at all interior nodes satisfies Kirchhoff's law at all interior nodes and hence is the unique solution of the Dirichlet problem on G with boundary values ϕ. Then it is easy to see that $[\Lambda\phi]_s = 0$, $[\Lambda\phi]_q = a$ and $[\Lambda\phi]_r = -a$. Hence $\Lambda\phi = a\phi$. This proves that ϕ is an eigenvector of Λ with eigenvalue a.

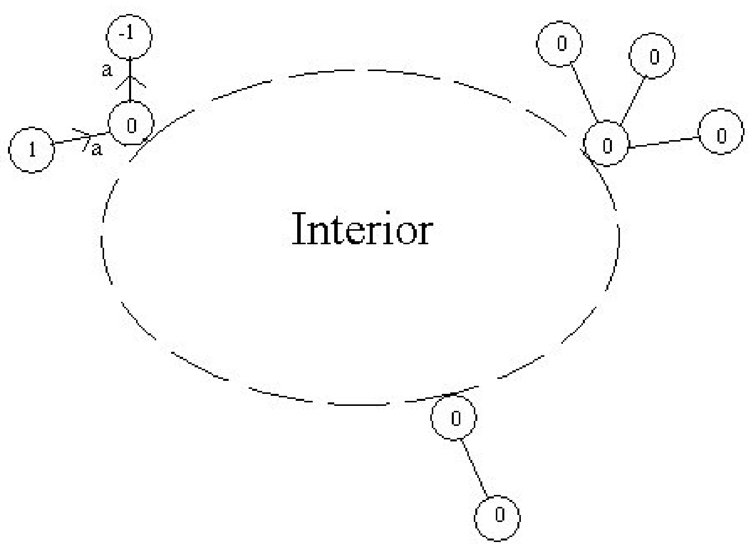

Figure 1: Assigning boundary voltages as shown in the proof of Theorem 2.1

□

Corollary 2.2. *For a boundary spike network, choose each interior node p_i such that there exists k_i boundary nodes neighboring p_i with equal conductiv-*

ities a. a is an eigenvalue with multiplicity $\sum_i (k_i - 1)$ if there exists some i for which $k_i \geq 2$.

Proof. Using the voltage assignment from Theorem 2.1 for boundary antenna pairs, $\forall j$: $e_j - e_{j+1}$ is an eigenvector if $1 \leq j \leq k_i - 1$. Thus, for each p_i, there exists $k_i - 1$ linearly independent eigenvectors all with corresponding eigenvalue a. $\square$

Due to the way we defined a boundary spike network, the eigenvalues of its response matrix are limited by its boundary-interior edge conductivities as there exists no boundary-boundary edges.

Theorem 2.3. *For a boundary spike network, let n be the number of boundary nodes and $S = \{a_1, a_2, ..., a_n\}$ be the set of conductivities for boundary-interior edges. Let a_k be the maximal element of S. For any eigenvalue λ of response matrix Λ, $0 \leq \lambda \leq a_k$.*

Proof. Assume $\lambda > a_k$. Let ϕ be an eigenvector for λ. Let $\{p_1...p_l\}$ be the set of interior nodes connected to boundary nodes and let Q_i be the set of boundary nodes connected to p_i where $q_i \in Q_i$. Let v be a γ-harmonic function on int G and $v = \phi$ for all Q_i. Therefore:

$$v(p_i) = v(q_i)(1 - \lambda/a_i)$$

where $a_i = \gamma_{p_i q_i}$. Because $\lambda > a_i$, $v(p_i)v(q_i) < 0$ for $v(q_i) \neq 0$. Because $0 \neq \phi$, there must be a q_i and q_j such that $v(q_i) > 0$ and a $v(q_j) < 0$. Thus, $v(p_i) < 0$ and $v(p_j) > 0$. Let $G = (V, E)$ be the graph corresponding to the boundary spike network. Then let $G^{'} = (V^{'}, E^{'})$ be the subgraph such that $V^{'} = V - \{Q_1, ..., Q_l\}$, $E^{'} = E - \{p_1 Q_1, ..., p_l Q_l\}$ and $\partial G^{'} = \{p_1...p_l\}$. Let $v|_{G'} = w$, which is still γ-harmonic on $G^{'}$. By the maximum/minimum principle, the maximum and minimum values of w occur on $\partial G^{'}$. Because there exists a $v(p_j) > 0$, then the maximum value of w is positive. Letting the maximum of w occur at node p_j, $v(q_j) < 0$. This implies the net current out of p_j for v in G is greater than zero. This demonstrates a contradiction since $p_i \in$ int G. Therefore, $\lambda \leq a_k$. Note that a similar argument can be made using the minimum of w.

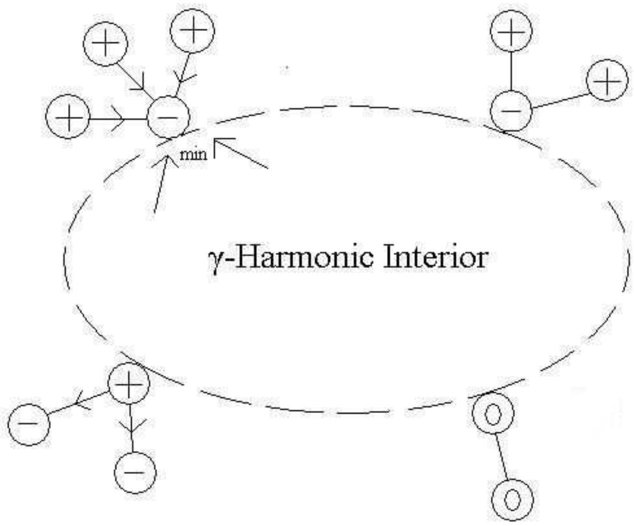

Figure 2: Contradiction shown in the proof of Theorem 2.3

$\square$

Theorem 2.4. *Suppose a_k is an eigenvalue of a boundary spike network where a_k is the maximal conductivity for all boundary-interior edges. Then there must exist a boundary antenna with equal conductivity a_k.*

Proof. Let $\lambda = a_k$ and suppose there does not exist a boundary antenna with constant conductivity a_k. Let all notation be defined as it was in the proof of Theorem 2.3. If for each p_i there exists an edge p_iq_i such that $a_i = a_k$, letting $v(p_1) = ... = v(p_l) = 0$ satisfies Ohm's Law for all edges p_iq_i. Defining $v = 0$ for all other interior nodes satisfies Kirchoff's Law and therefore is the unique Dirichlet solution of G'. Because there exists a $v(q_i) > 0$ and a $v(q_j) < 0$, there must exist a boundary antenna at p_i and p_j so γ will be harmonic at p_i and p_j. Therefore, there must be current flowing along some edge p_iq_i with conductance $a_i < a_k$. However, any boundary voltage $v(q_i)$ with edge conductance a_i has current $v(q_i)a_i$ flowing across it. This demonstrates a contradiction as it implies $\lambda \neq a_k$. Now, let there exist at least one p_i such that for each edge p_iq_i, $a_i \neq a_k$ $\forall q_i \in Q_i$. $v(q_i) \neq 0$ because if $v(p_i) = 0$, $v = 0$ for all interior nodes in G and the same contradiction as before will be reached ($\lambda \neq a_k$). Thus, knowing that $v(q_i)v(p_i) < 0$ for $v(q_i) \neq 0$, the same contradiction will be reached as in the proof of Theorem 2.3. Thus, there must exist a boundary antenna with equal conductivity a_k. □

2.2 Relating the Composition of Λ to its Eigenvalues

Consider the composition of Λ:

$$\begin{gathered}\Lambda\phi = \lambda\phi \\ (\Lambda - \lambda I)\phi = 0 \\ (A - BC^{-1}B^T - \lambda I)\phi = 0 \\ (A - \lambda I)\phi = (BC^{-1}B^T)\phi\end{gathered}$$

Using this interpretation for sub-matrices of Λ generates a means to prove the upper bound for eigenvalues in the general case. This decomposition is useful due to the properties of C^{-1} (invertible and positive semi-definite).

Notice that when $A = aI$ (a is a scalar):

$$\begin{gathered}(a - \lambda)\phi = (BC^{-1}B^T)\phi \\ \Leftrightarrow \lambda' = a - \lambda \text{ is an eigenvalue of } BC^{-1}B^T\end{gathered}$$

Theorem 2.5. *Suppose there exists a network with no boundary-boundary connections and for each $i \in \partial V$, $\sum_{i\sim k}\gamma_{ik} = a$ where $k \in int\ V$ ($A = aI$ where A is a submatrix of Kirchhoff matrix K). When the number of boundary nodes is greater than the number of interior nodes connected to the boundary, a is an eigenvalue.*

Proof. Let n be the number of boundary nodes and r be the number of interior nodes connected to the boundary.

$$\begin{gathered}rank(BC^{-1}B^T) \leq rankB \leq min[n, r] \\ Nullity(BC^{-1}B^T) = n - rank(BC^{-1}B^T)\end{gathered}$$

For $n > r$:

$$\begin{aligned}&n - rank(BC^{-1}B^T) \geq n - rankB \geq n - r > 0 \\ &\Rightarrow Nullity(BC^{-1}B^T) > 0 \\ &\Rightarrow \lambda' = 0 \text{ is an eigenvalue for } BC^{-1}B^T \\ &\Rightarrow \text{ a is an eigenvalue for } \Lambda\end{aligned}$$

$\square$

Lemma 2.6. *For all eigenvalues λ of response matrix Λ, there exists an eigenvector ϕ such that:*

$$\lambda \leq \frac{\sum_i a_{ii}\phi_i^2 + \sum_{i,j(i\neq j)} a_{ij}\phi_i\phi_j}{\sum_i \phi_i^2}$$

Proof. For any vector x:

$$x(BC^{-1}B^T)x^T = (xB)C^{-1}(xB)^T \geq 0$$

as C^{-1} is positive semi-definite. For some λ with eigenvector ϕ:

$$\begin{aligned}
&BC^{-1}B^T \text{ is positive semi-definite} \\
&\Rightarrow \phi^T BC^{-1}B^T\phi = \phi^T(A-\lambda I)\phi \geq 0 \\
&\Rightarrow \phi^T A\phi \geq \phi^T\lambda\phi \\
&\Rightarrow \sum_i \phi_i[A\phi]_i \geq \lambda\sum_i \phi_i^2
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \lambda \quad &\leq \quad \frac{\sum_{i,j} a_{ij}\phi_i\phi_j}{\sum_i \phi_i^2} \text{ (Note that } \phi \neq 0 \text{ so } \sum_i \phi_i^2 \neq 0) \\
&= \quad \frac{\sum_i a_{ii}\phi_i^2 + \sum_{i,j(i\neq j)} a_{ij}\phi_i\phi_j}{\sum_i \phi_i^2}
\end{aligned}$$

Choosing the largest eigenvalue shows that for all λ, there exists an eigenvector ϕ such that:

$$\lambda \leq \frac{\sum_i a_{ii}\phi_i^2 + \sum_{i,j(i\neq j)} a_{ij}\phi_i\phi_j}{\sum_i \phi_i^2}$$

$\square$

Theorem 2.7. *Let Λ be the response matrix of any network with no boundary-boundary connections (A is a diagonal submatrix of Kirchhoff matrix K). For any eigenvalue λ of Λ, $\lambda \leq max_i\{a_{ii}\}$.*

Proof. By Lemma 2.6:

$$\begin{aligned}
\lambda &\leq \frac{\sum_i a_{ii}\phi_i^2 + \sum_{i,j(i\neq j)} a_{ij}\phi_i\phi_j}{\sum_i \phi_i^2} \\
&= \frac{\sum_i a_{ii}\phi_i^2}{\sum_i \phi_i^2} \text{ as } a_{ij} = 0 \text{ for } i \neq j \\
&\leq \frac{max_i\{a_{ii}\}\sum_i \phi_i^2}{\sum_i \phi_i^2} \\
&= max_i\{a_{ii}\}
\end{aligned}$$

□

Now the foundation has been built to prove the upper bound for eigenvalues of general response matrices. First, the following lemma must be shown:

Lemma 2.8. *Let t_{ij} be the ijth entry of an $n \times n$ symmetric matrix T such that $t_{ij}t_{kl} \geq 0$ for $i \neq j$ and $k \neq l$. Letting x be a vector in $\mathbb{R}^n$:*

$$|\sum_{i,j(i\neq j)} t_{ij}x_ix_j| \leq |\sum_{i,j(i\neq j)} t_{ij}x_i^2|$$

Proof.

$$\begin{aligned}
&\forall x_i, x_j : (|x_i - x_j|)^2 \geq 0 \\
&\Rightarrow x_i^2 + x_j^2 \geq 2|x_i||x_j| \\
&\Rightarrow \sum_{i\neq j} |t_{ij}|(x_i^2 + x_j^2) \geq 2\sum_{i\neq j} |t_{ij}||x_i||x_j| \\
&\Rightarrow \sum_{i<j} |t_{ij}|(x_i^2 + x_j^2) \geq \sum_{i\neq j} |t_{ij}||x_i||x_j| \\
&\Rightarrow \sum_{i<j} 2|t_{ij}|x_i^2 \geq \sum_{i\neq j} |t_{ij}||x_i||x_j| \\
&\Rightarrow \sum_{i\neq j} |t_{ij}|x_i^2 \geq \sum_{i\neq j} |t_{ij}||x_i||x_j| \\
&\Rightarrow \sum_{i\neq j} |t_{ij}x_i^2| \geq \sum_{i\neq j} |t_{ij}x_ix_j| \\
&\Rightarrow |\sum_{i\neq j} t_{ij}x_i^2| \geq |\sum_{i\neq j} t_{ij}x_ix_j|
\end{aligned}$$

□

Theorem 2.9. *Let a_{ij} be the ijth entry of A corresponding to a Kirchhoff matrix K. For the response matrix Λ derived from K, $\lambda \leq max_i\ [a_{ii} - \sum_{i \neq j} a_{ij}]$ where λ is any eigenvalue of Λ.*

Proof. By Lemma 2.6:

$$\begin{aligned}
\lambda &\leq \frac{\sum_i a_{ii}\phi_i^2 + \sum_{i,j(i\neq j)} a_{ij}\phi_i\phi_j}{\sum_i \phi_i^2} \\
&\leq \frac{\sum_i a_{ii}\phi_i^2 - \sum_{i,j(i\neq j)} a_{ij}\phi_i^2}{\sum_i \phi_i^2} \text{ by Lemma 2.8} \\
&\leq \frac{max_i\{\sum_i a_{ii} - \sum_{i\neq j} a_{ij}\}\sum_i \phi_i^2}{\sum_i \phi_i^2} \\
&= max_i\{\sum_i a_{ii} - \sum_{i\neq j} a_{ij}\}
\end{aligned}$$

□

Theorem 2.10. *Let λ be an eigenvalue of Λ and μ be an eigenvalue of Kirchhoff submatrix A. Then $\lambda \leq max\{\mu\}$. If $\lambda < min_i\{a_{ii} + \sum_{i\neq j} a_{ij}\}$, $\lambda < max\{\mu\}$.*

Proof. By Lemma 2.6:

$$\begin{aligned}
\lambda &\leq \frac{\sum_i a_{ii}\phi_i^2 + \sum_{i,j(i\neq j)} a_{ij}\phi_i\phi_j}{\sum_i \phi_i^2} \\
&= \frac{\sum_{i,j} a_{ij}\phi_i\phi_j}{\sum_i \phi_i^2} \leq max\{\mu\} \text{ by properties of the Rayleigh Quotient}
\end{aligned}$$

If $\lambda < min_i\{a_{ii} + \sum_{i\neq j} a_{ij}\}$, then $\forall a_{ii} \in A : a_{ii} - \lambda > -\sum_{j(i\neq j)} a_{ij}$.

$$\begin{aligned}
\phi^T B C^{-1} B^T \phi &= \phi^T (A - \lambda I)\phi \\
&= \sum_i \phi_i [(A - \lambda I)\phi]_i \\
&= \sum_i \phi_i [\sum_j (a_{ij} - \lambda\delta_{ij})\phi_j] \\
&= \sum_i [(a_{ii} - \lambda)\phi_i^2 + \phi_i \sum_{j(i\neq j)} a_{ij}\phi_j] \\
&> -\sum_{i,j(i\neq j} a_{ij}\phi_i^2 + \sum_{i,j(i\neq j)} a_{ij}\phi_i\phi_j \geq 0 \text{ by Lemma 2.8}
\end{aligned}$$

$$\Rightarrow \quad \phi^T A\phi > \phi^T \lambda\phi$$
$$\Rightarrow \quad \lambda < \frac{\sum_{i,j} a_{ij}\phi_i\phi_j}{\sum_i \phi_i^2} \leq max\{\mu\} \text{ by properties of the Rayleigh Quotient}$$

□

3 Characteristic Polynomial of Λ

When considering the characteristic polynomial of Λ, a useful relationship between Λ and K, the Kirchhoff matrix, can be derived as follows:

$$\begin{aligned} 0 &= det(\Lambda - \lambda I) \\ &= det((A - BC^{-1}B^T) - \lambda I) \\ &= det((A - \lambda I) - BC^{-1}B^T) \\ &= det(K^{'}/C) \\ &= det\ K^{'}/det\ C \end{aligned}$$

where

$$K^{'} = \begin{pmatrix} A - \lambda I & B \\ B^T & C \end{pmatrix}$$

and $K^{'}/C$ denotes the Schur complement of $K^{'}$ with respect to C.

Because det $C \neq 0$:

$$det(\Lambda - \lambda I) = 0 \Leftrightarrow detK^{'} = 0$$

Remark 3. *Letting $(A - \lambda I)$ be an $n \times n$ matrix:*

$$detK^{'} = \sum_k (-1)^k \lambda^k (\sum_M det(M_{n-k}))$$

where M_{n-k} are minors of $K^{'}$ through removal of k rows and columns, leaving C intact.

The proof of this theorem is similar to the proof of the general characteristic polynomial in [2].

3.1 The Characteristic Polynomial for Λ of an n-star

Theorem 3.1. *For an n-star network where $\{\gamma_1 ... \gamma_n\}$ are the edge conductivities, its characteristic polynomial is of the form:*

$$(-\lambda)^n(\textstyle\sum_i \gamma_i) + (-\lambda)^{n-1}(2\sum_{i<j}\gamma_i\gamma_j) + (-\lambda)^{n-2}(3\sum_{i<j<k}\gamma_i\gamma_j\gamma_k) + ... + (-\lambda)(n\gamma_1\gamma_2...\gamma_n)$$
$$= \textstyle\sum_0^{n-1}[(-\lambda)^{n-k}(k+1)\sum_{a_1<a_2<...<a_{k+1}}^n \gamma_{a_1}\gamma_{a_2}...\gamma_{a_{k+1}}]$$

Proof. A network is an n-star $\Rightarrow$

$$detK' = det\begin{pmatrix} K_{ii}-\lambda & K_{i,n+1} \\ K_{n+1,i} & \sum_i^n \gamma_i \end{pmatrix}$$

where $K_{ii} = -K_{n+1,i} = -K_{i,n+1} = \gamma_i$.

$$\begin{aligned} &= \sum_i^n[\gamma_i \prod_i^n(\gamma_i-\lambda)] - \sum_i[\gamma_i^2 \prod_{j\neq i}^n(\gamma_j-\lambda)] \\ &= \sum_i^n[(\gamma_i(\gamma_i-\lambda)-\gamma_i^2)\prod_{j\neq i}^n(\gamma_j-\lambda)] \\ &= \sum_i^n[-\gamma_i\lambda\prod_{j\neq i}^n(\gamma_j-\lambda)] = 0 \end{aligned}$$

Choosing $n-k$ entries of λ letting $i = a_1$:

$$\begin{aligned} &\sum_{a_1}^n[-\gamma_{a_1}\lambda \sum_{a_2\neq...\neq a_{k+1}\neq a_1}^n \gamma_{a_2}...\gamma_{a_{k+1}}(-\lambda)^{n-k-1}] \\ &= (-\lambda)^{n-k}\sum_{a_1\neq a_2\neq...\neq a_{k+1}}^n \gamma_{a_1}\gamma_{a_2}...\gamma_{a_{k+1}} \\ &= (-\lambda)^{n-k}(k+1)\sum_{a_1<a_2...<a_{k+1}}^n \gamma_{a_1}\gamma_{a_2}...\gamma_{a_{k+1}} \end{aligned}$$

Summing over all $n-k$ entries for $0 \leq k \leq n-1$ yields:

$$\sum_0^{n-1}[(-\lambda)^{n-k}(k+1)\sum_{a_1<a_2<...<a_{k+1}}^n \gamma_{a_1}\gamma_{a_2}...\gamma_{a_{k+1}}]$$

□

Remark 4. *The characteristic polynomial of the n-star can also be derived through summation of minors for K'.*

Theorem 3.2. *Suppose $\{\gamma_1, ...\gamma_n\}$ is the set of edge conductivities for an n-star network. If $\{\lambda_1, ...\lambda_{n-1}\}$ is the set of nonzero eigenvalues for Λ, then $\gamma_1 < \lambda_1 < \gamma_2 < \lambda_2 < ... < \lambda_{n-1} < \gamma_n$. But $\forall(1 \leq k \leq n-1)$: if $\gamma_k = \gamma_{k+1}$, then $\lambda_k = \gamma_k$.*

Proof. Consider the characteristic polynomial for an n-star:

$$(-\lambda)^n(\sum_i \gamma_i) + (-\lambda)^{n-1}(2\sum_{i<j}\gamma_i\gamma_j) + ... + (-\lambda)(n\gamma_1\gamma_2...\gamma_n) = 0 \qquad (1)$$

$\lambda \neq 0 \Rightarrow$

$$(-\lambda)^{n-1}(\sum_i \gamma_i) + (-\lambda)^{n-2}(2\sum_{i<j}\gamma_i\gamma_j) + ... + (n\gamma_1\gamma_2...\gamma_n) = 0 \qquad (2)$$

Letting $\rho = 1/\lambda \Rightarrow$

$$(\sum_i \gamma_i) + (-\rho)(2\sum_{i<j}\gamma_i\gamma_j) + ... + (-\rho)^{n-1}(n\gamma_1\gamma_2...\gamma_n) = 0 \qquad (3)$$

Integrating in terms of $\rho \Rightarrow$

$$-1 - (-\rho)(\sum_i \gamma_i) - (-\rho)^2(\sum_{i<j}\gamma_i\gamma_j) + ... - (-\rho)^n(\gamma_1\gamma_2...\gamma_n) = 0 \qquad (4)$$

Substituting λ and multiplying by -1 $\Rightarrow$

$$(-\lambda)^n + (-\lambda)^{n-1}(\sum_i \gamma_i) + (-\lambda)^{n-2}(\sum_{i<j}\gamma_i\gamma_j) + ... + (\gamma_1\gamma_2...\gamma_n) \qquad (5)$$

$$= (\lambda - \gamma_1)(\lambda - \gamma_2)...(\lambda - \gamma_n) = 0$$

Thus the roots of (5) are $\gamma_1...\gamma_n$, which implies the roots of (4) are $1/\gamma_1...1/\gamma_n$. Because (3) is the derivative of (4) with respect to ρ, the roots of (3) lie between the roots of (4) unless $\gamma_k = \gamma_{k+1}$ wherein $\rho_k = 1/\gamma_k$. Therefore:

$$1/\gamma_1 < \rho_1 < 1/\gamma_2 < ... < \rho_{n-1} < 1/\gamma_n$$

By (2):

$$\gamma_1 > \lambda_1 > \gamma_2 > ... > \lambda_{n-1} > \gamma_n$$

Hence $\{\lambda_1, ..., \lambda_n\}$ and 0 are roots of (1), the characteristic polynomial for an n-star. Note that if $\gamma_k = \gamma_{k+1}$, then $\lambda_k = \gamma_k$. □

4 Future Research

There is much research still to be done regarding eigenvalues of the response matrix. Future problems include:

- Further bounding the eigenvalues for specific cases
- Relating the eigenvalues of K and its submatrices to eigenvalues of Λ
- Finding general criteria for the existence of certain eigenvalues
- Relating the minimal path between boundary nodes to eigenvalue bounds
- Using eigenvalues to characterize the eigenvectors
- Further evaluating the characteristic polynomial for specific cases

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