ENUMERATING SPANNING TREES WITH DETERMINANTS

JEFFREY BURKERT

1. Introduction

The work presented in this paper was motivated by a section of Matt Lewandowski's paper [1] on the connection between determinants and tree diagrams. In this section we will present notation along with the key theorem of his paper. Consider a network $\Gamma = (G, \gamma)$ where $G = (V, E)$, Where V is the vertex set of the graph G and E is the edge set. The function γ is a function on E. V can be separated into two sets: ∂V and $intV$, termed the boundary and the interior of the network repectively. We index the vertices starting with the boundary nodes, so that if there are $|\partial V| = m$ and $|intV| = n$ then the vertices are indexed $v_1, v_2, ..., v_m, v_{m+1}, ..., v_{m+n}$ so that the first k correspond to boundary vertices and the remaining verices are interior. The Kirchhoff matrix K of a network is defined so that $k_{ij} = -\gamma_{ij}$ when ij, but $k_{ii} = \sum_{j=1}^{m+n} \gamma_{ij}$. We say that K $= \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$, where A corresponds to a $|\partial V| x |\partial V|$ block, thus A represents the boundary to boundary connections and C represents the interior to interior connections.

We call a subset of the edges of G a Tree Diagram T if and only if the components of T form trees that span G such that each component contains exactly one boundary node. Let Ω be the set of all Tree Diagrams on G. We conclude with Lewandowski's result.

Theorem 1.1 (Lewandowski's Theorem)**.**

$$detC = \sum_{T \in \Omega} \prod_{e \in T} \gamma(e)$$

2. Enumerating Spanning Trees

In Lewandowski's paper, he applies the theorem above to enumerate the spanning tree of a given graph.

Corollary 2.1 (Kirchhoff's Matrix Tree Theorem)**.** *Given a graph G with vertex set V and edge set E, define a $\gamma(e) = 1$ on all $e \in E$. Promote one vertex to be a boundary node and call the new graph G'. Let $K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ the Kirchhoff matrix of G'. Then det C counts the number of spanning trees of C.*

The paper proceeds to consider the question of counting spanning trees with specified edge sets and succeeds to give formulas for 1,2, or 3 edges. We proceed to give a general formula, notice we are utilizing the notation above. Furthermore, let $G(e_1, e_2..., e_n)$ be the G corresponding with removing the edges $e_1, e_2..., e_n$ and $C(e_1, e_2..., e_n)$ corresponding to the interior to interior portion of the Kirkhoff matrix ($\gamma(e) = 1$) associated with $G(e_1, e_2..., e_n)$. Let $S(e_1, e_2..., e_n)$ be the set of spanning trees containing $e_1, e_2..., e_n$.

Theorem 2.2. *let* $E^* = e_1, e_2, ...e_n$ *be the edges contained in the spanning tree. Then*

$$|S(E^*)| = detC - \sum_{e \in E^*} detC(e) + \sum_{e_i, e_j \in E^*} detC(e_i, e_j) \ldots \pm detC(E^*)$$

Note that the last $\pm$ *depends on the number of intervening terms.*

Proof. Let $A_n = \{T \in S | e_n \notin T\}$. Notice that $|S| - |\bigcup A_n|$ is precisely that quantity we are trying to count as we are subtracting off all the trees that are missing at least one edge in our set. Notice that for any set $\bar{E} = e_{a1}, e_{a2}, ..., e_{ak}$, $|\bigcap_{e_i \in \bar{E}} A_i| = detC(\bar{E})$. This is a direct result of Kirchoff's Tree Theorem. Using the Principle of Inclusion Exclusion we have:

$$\begin{aligned}
|S(E^*)| &= |S| - |\bigcup A_n| \\
&= |S| - \sum_{e_i \in E^*} |A_i| + \sum_{e_i, e_j \in E^*} |A_i \bigcap A_j| - \ldots \pm |\bigcap_{e_i \in E^*} A_i| \\
&= detC - \sum_{e \in E^*} detC(e) + \sum_{e_i, e_j \in E^*} detC(e_i, e_j) \ldots \pm detC(E^*)
\end{aligned}$$

□

References

[1] Matt Lewandowski. "Determinant of A Principle Proper Submatrix of the Kirchhoff Matrix." 2008.

Harvey Mudd College, 340 E Foothill Blvd, Claremont, CA 91711
E-mail address: jeffrey.burkert@gmail.com