Victor Klee

UNSOLVED PROBLEMS IN INTUITIVE GEOMETRY

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Reproduced with comments by B. Grünbaum

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An Introduction to Vic Klee's 1960 collection of

"Unsolved Problems in Intuitive Geometry"

One aspect of Klee's mathematical activity which will be influential for a long time are the many open problems that he proposed and popularized in many of his papers and collections of problems. The best known of the collections is the book *"Old and New Unsolved Problems in Plane Geometry and Number Theory"*, coauthored by Stan Wagon [KW91]. This work continues to be listed as providing both historical background and references for the many problems that still – after almost twenty years – elicit enthusiasm and new results. The simplest approach to information about these papers is the "Citations" facility of the MathSciNet; as of this writing, 23 mentions are listed.

Long before the book, Klee discussed many of these problems in the "Research Problems" column published in the late 1960's and early 1970's in the American Mathematical Monthly. The regularly appearing column was inspired by Hugo Hadwiger's articles on unsolved problems that appeared from time to time in the journal "Elemente der Mathematik".

However, before all these publications, Klee assembled in 1960 a collection of problems that was never published but was informally distributed in the form of (purple) mimeographed pages. The "Unsolved problems in intuitive geometry" must have had a reasonably wide distribution since quite a few publications refer to it directly or indirectly. While I cannot estimate the precise number of mentions of the "Unsolved problems" in the literature, it is among the references of my paper [Grü63], as well as of the much later works [MS05] and [Sol05] in 2005. This fact is one of the reasons for the present attempt to make the collection more widely and more formally available. In other cases one can surmise that the indirect attribution is to the "Unsolved problems"; for example, Hiriart-Urruty [HU07] in 2007 states that it [Problem K-4] "was clearly stated by Klee (circa 1961)". Even the Math. Reviews account of [HU05] states that the problem was "posed by V. Klee in 1961". Fang *et al.* [FSL09] in 2009 mention the problem as "dating back to the 1960s". Another reason to reissue the "Unsolved problems" is the wide sweep of the problems considered in the collection – much wider than in the collections that Klee published later.

As Klee states at the start of the "Unsolved Problems in Intuitive Geometry", this was to be his contribution to a book planned to be coauthored by Erdös, Fejes-Tóth, Hadwiger and Klee. Much later, Soifer [Soi09] mentions in 2009 this "great book-to-be [that] never materialized".

About mid-May 2010 it occurred to me that it might be appropriate to have the fifty-years old collection made available to participants at the "100 Years in Seattle" conference. The organizers were very supportive, and quickly made possible the preparation of a LaTeXversion of the existing notes. For this my heartfelt thanks go to my colleagues James Morrow, Isabella Novik, and Rekha Thomas. The only original copy that I could lay my hands on was the one I have had since 1960; its purple ink was quite faded in places, but it was still possible to prepare a few copies. One of the copies was used by the undergraduate student Mark Bun to type the problems; my thanks to him for a job done exceedingly well and for several comments and suggestions that improved the quality of the text. Problem K-20 does not appear in my copy; I do not know whether it somehow got lost, or whether Klee never got to write it down.

It was obvious to me that the "Unsolved problems" from 1960 should be accompanied by some reporting on what happened to them in the half-century since they were written. For lack of a better alternative I decided to try to do the job of writing for each problem a short guide to the developments that have occurred. This turned out to be more complicated than I had expected. Some of the problems were clearly out of my field, and I was happy to have my colleagues Isaac Namioka and Jack Segal point out some relevant literature. For many of the other problems the "Citations" feature of the MathSciNet mentioned above enabled me to get hold of a lot of the publications since 2000 that have bearing on the problems. One way or another I was able to prepare some "Comments" for each of the problems. It is my hope that these may be useful to some readers; I also hope that any omissions and errors will not only be brought to my attention, but also forgiven. Any other suggestions for better updating the "Unsolved Problems" will be greatly appreciated as well.

The organization of the whole is quite simple. The problems are reproduced exactly as Klee wrote them (I suspect that he even typed them himself), with the following exceptions:

- Obvious typos were corrected.
- Several incomplete references were completed.
- The software used to digitize renumbered the references. (Although it probably would be possible to overcome the tyranny of the software and retain the original numbering, there seemed to be little point in doing so.)
- A "Contents" list was added.

As in the original, each problem starts on a new page, and my comments are inserted on a separate page (or pages), each with a separate list of references.

* * * * *

These comments should be considered as a small token of appreciation of the many years of friendship and support that Vic Klee extended to me. Would that he could be with us today.

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UNSOLVED PROBLEMS IN INTUITIVE GEOMETRY

The problems which follow are among those being considered for inclusion in a book on "Ungeloste Probleme der anschauliche Geometrie," now being written by H. Hadwiger, L. Fejes-Tóth, P. Erdös, and V. Klee (to be published by Birkhäuser in Basel). The discussion as given is not in its final form, and your comments are welcomed (also solutions, of course)!

V. Klee
October 1960

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K1 Double Normals of Convex Bodies

PROBLEM: Must a convex body in E^n admit at least n doubly normal chords?

By a chord of a convex body C we mean a line segment determined by two boundary points p and q of C. The chord is said to be normal (at p) provided the body lies on one side of the hyperplane which passes through p and is perpendicular to the chord; the chord is doubly normal provided it is normal at both p and q. It is known that if C is a convex body of constant width, then every normal chord of C is doubly normal, and in fact this property characterizes the convex bodies of constant width. But if $0 < a_1 < \cdots < a_n$ and C is the ellipsoid in E^n determined by the inequality

$$\left(\frac{x_1}{a_1}\right)^2 + \cdots + \left(\frac{x_n}{a_n}\right)^2 \leq 1,$$

then the n axes of C are the only doubly normal chords. These considerations lead naturally to the question above. When the bounding surface of the convex body is subjected to various regulatory conditions, existence of at least n doubly normal chords follows from the results of Marston Morse [2] or Lyusternik and Schnirelmann [1]. In the general case, it is easy to produce two doubly normal chords. First, if p and q are points of C whose distance is the maximum possible for such points, then the chord pq is easily seen to be doubly normal. Secondly, let H and J be a pair of parallel supporting hyperplanes of C whose distance apart is a minimum for such pairs, and let u and v be points of $H \cap C$ and $J \cap C$ respectively whose distance is a minimum for such points. Then the chord uv is doubly normal (and if $uv = pq$, then C is of constant width and admits infinitely many doubly normal chords).

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[2] Marston Morse, *Calculus of variations in the large*, New York, 1934.

Comments by Grünbaum (K1)

For $n = 3$ an unconvincing proof of the existence of three double normals is presented in the Russian original of Lyusternik [Lyu56], published in 1956, before Klee's collection; in the two translated versions the assertion is even less believable. Lyusternik's statement that in the n-dimensional case there are at least n double normals is supported only by a general reference to the Lyusternik-Shnirel'man paper [LS47]. Soltan [Sol05] states that Klee and Lyusternik are sources of the problem, and that Lyusternik mentions the affirmative solution as a known fact but without any reference or proof.

An affirmative solution of the problem is contained in the paper by Kuiper [Kui64] in 1964. The proof involves quite heavy topological machinery. Details of the proof are way beyond my ken.

After 1960, Vic may have become aware of Lyustenik's claim, and as a consequence of that, or of Kuiper's affirmative solution, he did not mention the question in any of his later collections of problems.

One of the generalizations of the problem treats it as the case $k = 2$ of the question how many distinct k-gonal billiard paths exist in a domain bounded by a smooth curve. Detailed references to this generalization can be found in the survey [Kar08] by Karasev.

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K2 Line Segments in the Boundary of a Convex Body

PROBLEM: Is there a convex body C in E^n such that every direction in E^n is realized by some line segment in the boundary of C?

Let S denote the unit sphere $\{x \in E^n : ||x|| = 1\}$, and let D_C be the set of all $s \in S$ for which the segment $[0, s]$ is parallel to some segment in the boundary of C. Then D_C is an F_σ set in S. When $n = 2$, D_C is countable. It seems "intuitively obvious" that for arbitrary n, D_C must be of the first category in S, and the $(n-1)$-dimensional measure of D_C is equal to zero. For $n = 3$, this has been proved by McMinn [1].

References

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Comments by Grünbaum (K2)

This problem, or slight variants of it, were proposed by Klee in several venues. The first of these that I am aware of is Klee [Kle57] in 1957. This led to the negative answer for $n = 3$ by McMinn [McM60] in 1960, mentioned in Klee's explanations for problem K-2. Besicovitch [Bes63] provided an alternative proof of McMinn's result. The same situation is described in [Kle69] in 1969, where some additional facts are mentioned. However, in 1970 the non-existence of such C was established in full generality by Ewald, Larman and Rogers [ELR70]. In that paper they mention that W. D. Pepe proved the case of $n = 4$ in a paper to appear in the Proc. Amer. Math. Soc. Pepe's result is also mentioned in the report by Guy and Klee [GK71]. However, it seems that Pepe withdrew his paper on learning that Ewald *et al.* had established the general case. The report [GK71] mentions the complete solution of problem K-2 by Ewald *et al.* A refinement of the result of [ELR70] is contained in the paper [LR71] by Larman and Rogers. Another solution of the general case is contained in the recent paper [PZ07] by Pavlica and Zajcek.

Not mentioned in any of these papers is that the topic has a prehistory – albeit one that probably deserves to be forgotten. In [Fuj16], published in 1916, M. Fujiwara claims to prove the following theorem:

> The number of edge-lines of a closed convex surface is at most denumerably infinite. (In the original: "Die Anzahl der Kantenlinien einer geschlossenen konvexen Fläche in höhstens abzählbar unendlich.")

This is obviously wrong: A cone over a circular basis has continuum many edges (or "edge-lines"). But an even more strange aspect of this is that in his review [Rad] of [Fuj16] in the Jahrbuch über die Fortschritte der Mathematik (one of the journals that in its time served as the Math. Reviews), the eminent mathematician Hans Rademacher says only:

> ... The author attempts ... to prove that the edge-lines of a convex body can be present only in a denumerable quantity. (In the original: "... so bemüht sich der Verf. ... zu beweisen, dass die Kantenlinien eines konvexen Körpers nur in abzählbarer Menge vorhanden sein können.")

It is possible that a century ago this was a polite way of saying that something is fishy in a paper – but it certainly seems that a stronger statement would have been appropriate.

(I am somewhat allergic to such understatements, due to personal experience. As a starting PhD student in Jerusalem in the mid-1950's, I read the review [Bar] of a paper by Michal [Mic54]. The part of the review that captured my attention is: "Of considerable interest is the following theorem: The modulus of a homogeneous polynomial in a complex Banach space is equal to the modulus of its polar form. In the case of a real Banach space, this theorem is not true, but bounds relating these moduli are given. The reviewer regrets that he must report his inability to establish the last inequality in the fundamental Lemma 1 and a similar inequality in Lemma 5." As Michal's paper was not available in Jerusalem at the time, I naively assumed that the theorem had been proved, and that the reviewer just lacked the ability to follow the proof. This had two very bad consequences for me. On the one hand, I spent lots of time and effort trying to prove the result by myself – with no success. Clearly, this was bad for my morale. On the other hand, I derived from the theorem several interesting consequences. But it all collapsed when the widow of Prof. Michal graciously responded to my request to him for a copy of the paper. On reading it I saw that there is an actual error in his proof. Then it turned out to be easy to find examples that show that the relation between the polynomial form and the polarized form is the same in the complex case as in the real case, see [Grü57]. All this had one salutary consequence: I became convinced that if I have the appropriate knowledge of the background material but encounter a claim that I cannot verify – then I should suspect that the claim is unjustified, or possibly wrong. This has served me well ever since.)

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K3 Commuting Continuous Maps on an Interval

PROBLEM: Suppose I is a closed interval of real numbers, and f and g are commuting continuous maps of I into itself. Must they have a common fixed point?

A subset of E^n is said to have the fixed-point property provided each continuous map of the set into itself leaves at least one point invariant. This property has been much studied, and the fixed-point property has been established for important classes of sets. It is possessed, for example, by every bounded closed convex subset of E^n, and by every "tree".[1]

Now if a set X has the fixed-point property, it is natural (and for certain applications important) to consider a family F which consists of two or more continuous maps of X into itself, and to search for conditions on F which ensure the existence of a fixed point common to all members of F – that is, a point p of X such that $fp = p$ for all members f of F. Several results of this sort involve commutativity of the family F, this meaning that $f(gx) = g(fx)$ for all x in X and f, g in F. Especially useful is the theorem of Markov and Kakutani [1] to the effect that if X is a bounded closed convex set in E^n and F a commutative family of affine maps of X into itself, then there is a common fixed point. Isbell [2] has observed that if F is a commutative family of homeomorphisms of a tree onto itself, then there is a common fixed point. All pairs of commuting polynomials were determined by Ritt [3], and his result has enabled Isbell to prove that if two commuting polynomials map an interval into itself, they must have a common fixed point in that interval. (Thus the problem stated above is solved affirmatively for the case in which f and g are both polynomial functions.)

Aside from the results just mentioned, very little is known about common fixed points of commutative families of continuous maps. For example (as remarked by Isbell), there seems to be no known example of a set X and a commutative family of continuous maps of X into X such that X has the fixed-point property but there is no fixed point common to all the members of F. (However, existence of such an example seems highly probable.) On the other hand, there are open problems even for very special sets, the simplest being that stated above (due to Isbell [2]).

The notion of a common fixed point is closely related to that of a coincidence, this being a point x at which all members of F have the same value – $fx = gx$ for all f, g in F. A common fixed point for the family F is the same as a coincidence for the family F' which is obtained from F by addition of the identity map. It is easy to prove that two commuting continuous maps of the interval I into itself must have a coincidence; for three commuting maps on I, existence of a coincidence is not known, as it would imply an affirmative answer to the problem K3. For the triod, it is not known even whether two commuting continuous maps must have a coincidence.[2]

[1] A tree is a compact locally connected set in which each two points are joined by a unique arc. In particular, if a set is formed from a finite set of line segments subject to the following conditions, it will be a tree: (i) distinct segments are disjoint or intersect in an endpoint of both; (ii) if p and q are endpoints of segments, there are segments $[x_1, y_1], \ldots, [x_m, y_m]$ such that $x_1 = p, y_m = q$, and always $y_i = x_{i+1}$; (iii) there are not distinct segments $[x_1, y_1], \ldots, [x_m, y_m]$ such that always $y_i = x_{i+1}$, and also $y_m = x_1$

[2] (Probably not to be included in the book, but perhaps of interest to you.) This may be connected with the famous problem as to whether the fixed-point property is possessed by all plane continua which do not separate the plane. Dyer has observed that if the triod admits two commuting continuous maps without coincidence, they can be used to construct a certain inverse limit L of triods such that L lacks the fixed-point property. It seems reasonable to hope that L can be topologically embedded in the plane (though this has not been settled), and is known that if so embedded, it would not separate the plane. Thus plane embeddability of L (and, of course, existence of f and g as described) would provide a negative

answer to a question of long standing. (It should be remarked, however, that by a result of Bing, not every inverse limit of triods can be embedded in the plane.)

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Comments by Grünbaum (K3)

Klee correctly quotes Isbell [Isb57] as the first mention of the problem in print, in 1957. In fact Isbell asked a more general question, but the negative answer follows from the one for the interval I. However, according to Boyce [Boy69] the question was posed – but apparently not published – earlier, by Eldon Dyer in 1954, A. L. Shields in 1955, and Lester Dubins in 1956. In any case, the problem was solved quite soon, in 1967, independently by W. M. Boyce [Boy67] and J. P. Huneke [Hun67,Hun68]. Both published detailed proofs (in 1969 in the same journal [Boy69], [Hun69]) of the negative answer, by constructing examples of pairs of commuting functions that map I into itself but have no common fixed point. Since then much effort has been devoted to the question of determining pairs of commuting functions that do have common fixed points. A detailed history of both aspects of the problem can be found in the survey [McD09] by McDowell, which contains new results as well, and an extensive bibliography.

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K4 Farthest and Nearest Points in Hilbert Space

PROBLEM: Suppose X is a subset of Hilbert space H such that each point of H admits a unique farthest point in X. Must X consist of a single point?

PROBLEM: Suppose Y is a subset of Hilbert space H such that each point of H admits a unique nearest point in Y. Must Y be convex?

Hilbert space is an infinite-dimensional analogue of Euclidean n-space. Just as a point of E^n is an ordered n-tuple $x = (x_1, \dots, x_n)$ of real numbers whose distance from the origin $(0, \dots, 0)$ is equal to $(x_1^2 + \cdots + x_n^2)^{1/2}$, so a point of Hilbert space is an infinite sequence $x = (x_1, x_2, \dots)$ of real numbers for which $\sum_1^\infty x_i^2 < \infty$. The last condition is imposed so that the distance of x from the origin $(0, 0, \dots)$ can be defined as in the finite-dimensional case; more generally, the distance between two points x and y of H is defined to be the number

$$\left(\sum_1^\infty (x_i - y_i)^2 \right)^{1/2},$$

entirely in analogy with the finite-dimensional case. It was observed by Motzkin, Straus, and Valentine [3] that if a subset of E^n is such that each point of E^n admits a unique farthest point in the set, then the set consists of but a single point. The first problem above asks whether this theorem is valid in Hilbert space. Klee [2] has established an affirmative answer under the additional assumption that X is compact. The paper [3] contains some unsolved problems on farthest points in E^n.

Addition of points (or vectors) in E^n is defined coordinate-wise, as is multiplication of points by scalars. The same procedure applies in Hilbert space: if $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$, are points of H and r is a real number, then $x + y = (x_1 + y_1, x_2 + y_2, \dots)$ and $rx = (rx_1, rx_2, \dots)$. Thus the definitions of line segment and of convexity can be extended to H: a subset C of H is convex provided it includes each line segment which joints two of its points – that is, provided $rx + (1 - r)y$ is a point of C whenever x and y are points of C and $0 \leq r \leq 1$. A theorem of Motzkin, extended by Jessen [1] and others, asserts that if a subset of E^n is such that each point of E^n admits a unique nearest point in the set, then the set is closed and convex (and in fact, this property characterizes the closed convex sets). The second problem above asks whether the theorem is valid in Hilbert space. Klee [2] has established an affirmative answer under the additional assumption that Y is weakly compact. He also describes a close connection between the two problems on H stated here.

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Comments by Grünbaum (K4)

This contains a pair of problems, relabeled here, which have had different fates.

PROBLEM: Suppose X is a subset of Hilbert space H such that each point of H admits a unique farthest point in X. Must X consist of a single point?

PROBLEM: Suppose Y is a subset of Hilbert space H such that each point of H admits a unique nearest point in Y. Must Y be convex?

Sets X with the property assumed in the first problem are often called Klee sets, while the sets Y with the property assumed in the second problem are called Chebyshev sets. Several *restricted* versions of the first problem have been solved in the affirmative. Already Klee [Kle61] has solved the problem assuming that X is compact. For other cases see Fang *et al.* [FSL09] in 2009 and its references. Also, several variants of the first problem have been explored, such as unsymmetric norms in Minkowski spaces (Asplund [Asp67] in 1967), or distance functions other than those based on norms (Bauschke *et al.* [BWYY09a] in 2009, and the references given there). However, the problem as posed by Klee is still – after half a century – open. This is explicitly mentioned by Hiriart-Urruty [HU07] in 2007 and by Bauschke *et al.* [BWYY09b] in 2009, among others.

Chebyshev sets in (finite-dimensional) Euclidean spaces have been characterized long ago as those that are closed and convex (Bunt [Bun34] in 1934, more easily accessible in Motzkin [Mot35b], [Mot35a] in 1935). Efimov and Steckin [ES59] in 1959, Klee [Kle61] in 1961, and many others solved the second problem under various additional assumptions, see [BWYY09a]; the problem was tackled in Banach spaces as well. Hiriart-Urruty [HU07] states that (circa 1961) Klee conjectured that the answer is negative for Hilbert space.

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K5 Neighboring Families of Convex Polyhedra

PROBLEM: For each pair of integers k and n with $k-1 \geq n \geq 3$, determine the maximum possible number $F(k,n)$ of neighboring convex polyhedra in E^n, each having at most k vertices.

Two convex bodies in E^n are said to be neighboring provided their intersection is exactly $(n-1)$-dimensional; a family of convex bodies is said to be neighboring provided each two of its members are neighboring. By convex polyhedron having at most k vertices we mean a convex body which is the convex hull of k or fewer points.

It is known that a neighboring family of plane convex bodies has at most four members, and that $F(k,2) = 4$ for all $k \geq 3$; on the other hand, E^3 contains an infinite neighboring family of convex polyhedra.[1] The problem above asks what bounds can be placed on the size of a neighboring family when the polyhedra are limited as to number of vertices. Bagemihl [1] proposed the problem of determining $F(4,3)$ – that is, the maximum possible neighboring tetrahedra in E^3. He proved that $8 \leq F(4,3) \leq 17$, and conjectured that $F(4,3) = 8$. An elaboration of his argument shows that $F(k,n)$ is always finite.

Another unsolved problem is to determine for various convex polyhedra P the maximum possible number of neighboring affine images of P.

[1] This result was first proved in 1905 by Tietze, who provides in [5] an interesting elementary discussion of neighboring domains in E^2 and E^3. A simpler proof was given by Besicovitch [2], and refinements by Rado [4] and Eggleston [3].

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Comments by Grünbaum (K5)

A more detailed account of the problem and its history (with several additional references) appears in Klee's column [Kle69] from 1969. Among others, he mentions there that Bagemihl's problem of determining the value of $F(4,3)$ was partially answered by Baston [Bas65] in 1965, who proved that $F(4,3) \leq 9$. The complete solution of that problem, namely that $F(4,3) = 8$, was announced by Zaks [Zak86] in 1986, with full details provided in [Zak91] in 1991.

The related question about the maximum number $f(d,k)$ of convex d-polytopes in a neighborly family in E^d, each of which has at most k facets, was considered by Zaks [Zak79] in 1979. It was conjectured by Baston that $f(d,d+1) = F(d+1,d) = 2^d$, and repeatedly by others. Zaks [Zak79] proved that $f(d,d+1) \leq \frac{2}{3}(d+1)!$ and $f(d,k) \leq 2k!$ for $d \geq 3$ and $k \geq d+1$. The former estimate was improved by Perles [Per84] in 1981 to $f(d,d+1) \leq 2^{d+1}$; a minor improvement to $f(d,d+1) \leq 2^{d+1} - 1$, by M. Markert (unpublished) is mentioned in Simon [Sim91] in 1991. In the other direction, Zaks [Zak81] showed in 1981 that $f(d,d+1) \geq 2^d$.

Several related questions are discussed in a number of other publications of Joseph Zaks, as well as in [Sim91].

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K6 Generation of Convex Borel Sets

PROBLEM: For a family S of subsets of E^n, let S_1 be the family of all sets expressible as the union of an increasing sequence of members of S, S_2 the family of all sets expressible as the intersection of a decreasing sequence of members of $S_1, \ldots, S_{2n+1}$ the family of all unions of increasing sequences from S_{2n}, S_{2n+2} the family of all intersections of decreasing sequences from $S_{2n+1}, \ldots$. Let F denote the family of all closed subsets of E^n, and K the family of all closed convex subsets of E^n. Must each convex member of F_2 be a member of K_4? More generally, determine for each m and i the smallest integer t such that every m-dimensional convex member of F_i is a member of K_t. If, for some m and i, such a t fails to exist, is it at least true that every m-dimensional convex member of F_i is a member of some K_j?

In topology and measure theory, one often meets the above method of generating, from a family S of sets, an increasing sequence of families, $S \subset S_1 \subset S_2 \ldots$. Observe that always $K_i \subset F_i$, and all members of K_i are convex. Thus the above questions ask to what extent the convex members of F_i can be generated without leaving the realm of convexity. It is easily seen that every convex member of F_1 is a member of K_1, and Klee [1] has proved that every two-dimensional member of F_i is a member of K_{i+2}. For higher dimensions, only partial results are known when $i \geq 1$ – for example, if C is a convex member of F_i which intersects no boundary segment of C (or, in particular, if the boundary of C contains no segment!), then C is a member of K_{i+2} [1].

The above generation process may be extended by defining the family BS to be the smallest family of sets which contains S, includes the union of each increasing sequence of its members, and includes the intersection of each decreasing sequence of its members. (The family BF is the family of all Borelian subsets of E^n.) It can be verified that $BK \subset BF$ and each member of BK is convex. Hence the additional problem: Is every convex member of BF a member of BK? An affirmative answer is known for the two-dimensional case, but the three-dimensional problem is open.

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Comments by Grünbaum (K6)

This problem goes back to Klee's paper [Kle51] from 1951, while in [Kle53] partial results are obtained (for the 2-dimensional case). Klee returned to the problem several times for the case of Borel sets, where the question is whether all convex Borel sets can be generated from closed convex sets [Kle69]. Larman [Lar69] showed in 1969 that convex Borel sets in E^3 that satisfy certain conditions can be generated in this way, while Klee conjectured in [Kle67] that this does not hold for all convex Borel sets in E^3. However, Preiss [Pre73] showed in 1973 that every convex Borel set in any finite-dimensional Banach space may be generated by iteration of countable increasing unions and countable decreasing intersections starting from compact convex sets. Holický [Hol74] found that some of the results of Preiss do not transfer to the infinite-dimensional case. It seems that this particular problem did not attract any additional research.

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K7 Extreme Points and Exposed Points of a Convex Body

PROBLEM: If C is a convex body in E^n, must the boundary of C contain a sequence $J_1, J_2, \ldots,$ of closed sets satisfying the following condition:
Each nonexposed extreme point p of C lies in some set J_i; further, for each J_i containing p there are exposed points arbitrarily close to p which lie outside J_i.

A point p of C is extreme provided it is not the midpoint of any line segment in C, and exposed provided C lies on one side of a hyperplane which intersects C only at p. Thus an exposed point is a special sort of extreme point. If there are no line segments in the boundary of C, then every boundary point is exposed. If C is a convex polyhedron (that is, the convex hull of a finite set), then every extreme point is exposed. But if C is a closed metric neighborhood of a convex body X whose boundary contains a line segment, or C is the convex hull of a smooth[1] convex body Y and a convex body Z not containing Y, then there are extreme points x of C which are not exposed.

The examples suggest that even though not all extreme points are exposed, perhaps in some sense "almost all" of them are exposed. A theorem of Straszewicz [3] asserts that the set $\exp C$ of exposed points must be dense in the set $\operatorname{ex} C$ of extreme points. This implies that $\exp C = \operatorname{ex} C$ when $\operatorname{ex} C$ is finite (a result stated above in a different way) and that $\exp C$ is infinite if $\operatorname{ex} C$ is infinite. On the other hand, Klee [2] has described a two-dimensional convex body A for which the set $\operatorname{ex} A \sim \exp A$ is dense in the set $\operatorname{ex} A$, and a three-dimensional convex body C with boundary B for which each of the three sets $B \sim \operatorname{ex} C$, $\operatorname{ex} C \sim \exp C$, and $\exp C$ is dense in B. Of course none of these facts has to do with "almost all" extreme points. (Although the set of rational numbers is dense in the real line, there is no reasonable sense in which "almost all" real numbers are rational!)

To explain the significance of Problem K7, let us merely state that an affirmative solution amounts to proving that the set of nonexposed extreme points is of the first category relative to the set of all extreme points, and since the latter is the intersection of a sequence of open sets (easily verified), this shows that in a sense commonly employed in topology and analysis, "almost all" the extreme points are exposed. (For an exposition of the topological definitions and theorems involved here, see [1].) Affirmative solution of Problem K7 is easy when C is two-dimensional, for then there are only countably many maximal segments in the boundary of C; thus the set $\operatorname{ex} C \sim \exp C$ is countable (each of its points being an endpoint of a maximal segment) and the proof is completed by use of Straszewicz's theorem quoted above. (In this case, the sets J_i are onepointed.) For higher dimensions, the problem is open, an affirmative solution being known only when C is smooth [2] and in certain other special cases.

[1] A convex body is said to be smooth provided it admits at each boundary point a unique supporting hyperplane.

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Comments by Grünbaum (K7)

Although this problem seems not to have been discussed in the literature in the terms presented here, it is related to a very large number of works that describe in detail the properties of various types of boundary points of convex bodies. For example, Asplund [Asp63] proves that the set of k-exposed points is dense in the set of k-extreme points, where the concepts are a natural generalization of the ones appearing in Straszewicz' theorem as the case $k = 0$. Bishop and Phelps [BP63] deal with related questions in infinite-dimensional Banach spaces. Choquet *et al.* [CCK66] discuss various aspects of problems at the confluence of K-6 and K-7; an open problem from [CCK66] is solved by Holický and Laczkovicz [HL04]. Mani-Levitska [ML93] and [KMZ07] are rich sources of additional references on topics of this kind.

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K8 Convex Hulls and Helly's Theorem

PROBLEM: For each pair of positive integers n and r, determine the smallest integer $k(= f(n,r))$ such that every set of k points in E^n can be divided into r pairwise disjoint sets whose convex hulls have a common point.

Consider the following three assertions:

(C) If a point p is in the convex hull of a set X in E^n, then p is in the convex hull of some at-most-$(n+1)$-pointed subset of X.

(R) Each set of $n+2$ points in E^n can be divided into two disjoint sets whose convex hulls have a common point.

(H) If F is a finite family of convex sets in E^n and each $n+1$ members of F have a common point, then there is a point common to all members of F.

The well-known results (C) and (H) are due respectively to Caratheodory 1907 and Helly 1923 (announced in 1919), while (R) was proved by Radon 1921, and used by him to establish Helly's theorem. Levi's axiomatic treatment [3] of Helly's theorem is based on (R). In recent years, it has been observed that any one of (C), (R), and (H) can be easily deduced from each of the other two, and each of the results has led to interesting generalizations and unsolved problems.

Now (R) implies that $f(n,2) \leq n+2$, and consideration of the vertices of a simplex shows that equality holds. R. Rado [4] has proved that always $f(n,r) \leq (r-2)2^n + n + 2$, with equality when $n = 1$ or $r = 2$. This is not the best possible result, for Prof. and Mrs. A. P. Robertson have proved that $f(2,3) = 7$; hence the problem above, posed by Rado in [4]. (Another interesting theorem in Rado's paper is simultaneously an extension of (C) and a refinement of his result on $f(n,r)$.)

Rado's proof of (R) is based on the following fact:

(S) For each set of $n+2$ points in E^n there is a hyperplane which contains n of the points and intersects the segment determined by the remaining two.

This leads to the following problem, indicated by Rado:

PROBLEM: For each pair of integers n and r with $0 < r < n$, determine the smallest integer $k(= g(n,r))$ such that for each set of k points in E^n there is an r-dimensional flat which contains $r+1$ of the points and intersects the convex hull of the remaining points.

From (S) it follows that $g(n, n-1) = n+2$, and from (R) that $g(3,1) = 5$; apparently $g(4,1)$ and $g(4,2)$ are still unknown.

There are other unsolved problems which involve convex hulls in a manner similar to that above. The following seem especially interesting:

PROBLEM: For each pair of integers $n \geq 2$ and $r \geq 3$, determine the smallest integer $k(= h(n,r))$ such that in each set of k-pointed subsets of E^n there is an r-pointed set X having each of its points in the boundary of its convex hull (equivalently, no point of X is interior to the convex hull of the remaining points of X).

PROBLEM: For each pair of integers $n \geq 2$ and $r \geq 3$, determine the smallest integer $k(= h'(n,r))$ such that in each set of k points in general position [1] in E^n there is an r-pointed set X having each of its points as an extreme point of its convex hull (equivalently, no point of X is in the convex hull of the remaining points of X).

It is easy to verify that $h \leq h'$, and it seems probable that $h = h'$. for the case $n = 2$, the last problem stated asks for the least integer k such that if X is a set of k points in the plane and no three points of X are collinear, then some r points of X form the vertices of a convex polygon. It is obvious that $h'(2,3) = 3$, easy to see that $h'(2,4) = 5$, and known that $h'(2,5) = 9$ [1]. In [2] there will appear a proof that $h'(2,n) \geq 2^{n-2}$. Szekeres conjectures that $h'(2,n) = 2^{n-2} + 1$.

[1] A subset X of E^n is in general position provided for $k = 1, \ldots, n-1$, no k-dimensional flat in E^n contains as many as $k+2$ points of X.

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Comments by Grünbaum (K8)

This problem, and the other problems discussed by Klee in the text, led to a very large literature. Already in 1960, Klee became aware of the paper [Bir59] by Birch, where he proves that

$$f(2, r) = 3r - 2$$

$$f(n, r) \leq rn(n + 1) - n^2 - n + 1$$

and conjectures that

$$f(n, r) = (n + 1)r - n;$$

this is in agreement with the result of the Robertsons mentioned above.

Birch's conjecture $f(n, r) = (n+1)r-n$ was affirmatively decided by Tverberg [Tve66]; a simpler proof appears in [Tve81]. Reay [Rea68] considered the generalization $f_k(n, r)$ of $f(n, r)$ in which the intersection of all the r sets is required to have dimension at least k. He conjectured that $f_k(n, r) = (n+1)r-n+k$, and proved that this holds if the starting set has a strong independence property. Reay and others proved various strengthening of this result, see, for example, [Rea79], [Rea82], [Rou90], [Rou09]. For additional references see the survey [Eck93]. Even simpler proofs are by Sarkaria [Sar92] and by Roudneff in [Rou01]. Moreover, there are now topological colorful versions of this theorem, see arXiv:0910.4987 and arXiv:0911.2692.

Also in 1960, Klee was aware that concerning Rado's problem, from the results of Gale [Gal56] or Caratheodory [Car11] it follows that $g(n, r) = \infty$ if $n \geq 2r + 2$; in particular, $g(4, 1) = \infty$.

In this context it is worth mentioning that a letter from Birch to Klee (in the Fall of 1961) contains the following paragraph:

> If $n \leq 2r + 1$, I assert that $g(n, r) = n + 2$. In fact, given points $P_1, \ldots, P_{n+2}$ there are real λ's such that $\sum_{i=1}^{n+2} \lambda_i P_i = 0$. At least $[(n + 3)/2] \geq n - r + 1$ of the λ's have the same sign, say $\lambda_1, \ldots, \lambda_{n-r+1}$ are all positive. Then the flat through $P_{n-r+2}, \ldots, P_{n+2}$ meets the convex hull of $P_1, \ldots, P_{n-r+1}$ in the point $\sum_{i=1}^{n-r+1} \lambda_i P_i = -\sum_{i=n-r+2}^{n+2} \lambda_i P_i$.

The last of the problems, concerning $h'(n, r)$, should be credited to Erdös and Szekeres [ES61]; in itself, this problem generated a huge literature. So did the related problem of "empty polygons" concerning $h''(n, r)$ which differs from $h'(n, r)$ by the requirement that the interior of the convex hull of the r-point set X contains no other of the k points in its interior.

Among the available results we should mention that

$$2^{r-2} + 1 \leq h'(2, r) \leq 2 + \binom{2r - 5}{r - 2}.$$

Erdös and Szekeres conjectured that the lower bound is tight. It is remarkable that none of the surveys mentioned below contain a reference to the paper of Kalbfleisch and Stanton [KS95], in which it is noticed that the construction establishing the tightness of the lower bound by Erdös and Szekeres is incomplete, and provided a complete construction. As it turns out, $h'(n, r) \leq h'(2, r)$ for $n \geq 3$. Many other results and problems on $h'(n, r)$ and $h''(n, r)$, with ample references, are contained in [BK01]; a survey with even more resources is [MS00].

Concerning $h''(n, r)$ it is worth mentioning that $h''(2, 4) = h'(2, 4) = 5$, but $h''(2, 5) = 10 > h'(2, 5) = 9$, and that $h''(2, r)$ does not exist for $r \geq 7$ (Horton [Hor83]). However, $h''(2, 6)$ is finite; estimates for its value can be found in Nicolás [Nic07] and Gerken [Ger08].

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K9 Expansive Homeomorphisms of the n-Cell

PROBLEM: Can an n-dimensional convex body admit an expansive homeomorphism?

Let C be an n-dimensional convex body. By a homeomorphism (of C onto C) is meant a continuous transformation T of C onto C, under which no two distinct points of C are carried onto the same point. Each point x of C gives rise to a sequence of points under successive iteration of the transformation T – $x, T_1x = Tx, T_2x = T(T_1x), \ldots, T_nx = T(T_{n-1}x), \ldots$. If T is a homeomorphism, then so are all the transformations T_n, and thus the points T_nx and T_ny must be distinct if x and y are distinct. On the other hand (by uniform continuity of T_n) there exists for each $\epsilon > 0$ a $\delta > 0$ such that the distance between T_nx and T_ny is less than ϵ whenever that between x and y is less than δ. In general, the number δ depends on n as well as on ϵ. For example, let C be the closed interval $[0, 1]$, $T0 = 0$, and $Tx = x^{1/2}$ for $0 < x \leq 1$. Then $T_n0 = 0$, while $T_nx = x^{1/2n}$ (for $x > 0$) and $\lim_{n\to\infty} T_nx = 1$. Thus no matter now close the point x may be to 0, there exists an n such that the distance between T_n0 and T_nx is at least $\frac{1}{2}$. If this sort of behavior is exhibited for all pairs of points, the transformation is said to be expansive. Specifically, the homeomorphism T of C onto C is expansive provided there is a positive number d such that whenever x and y are distinct points of C, then for some n the distance between T_nx and T_ny is at least d.

It is easily seen that if one convex body admits an expansive homeomorphism, then so do all others of the same dimension; thus attention may be restricted to spheres or simplexes. The following paragraph shows that a one-dimensional convex body cannot admit an expansive homeomorphism, but for $n > 2$ the problem is open. Expansive homeomorphisms were studied by Utz [3], who is responsible for the problem. An expansive homeomorphism of a (geometrically unpleasant) plane continuum was given by Williams [4]. For further information and references, see [1] and [2].

Now consider a homeomorphism h of an interval $[a, b]$ into itself. Clearly h must either interchange the endpoints a and b or leave them invariant; we suppose the latter, for the argument is similar in the other case. It is easy to establish the existence of an interval $[p, q]$ contained in $[a, b]$ such that $hp = p$, $hq = q$, and $hx \leq x$ for all x in $[p, q]$ or $hx \geq x$ for all x in $[p, q]$; we suppose the latter, for treatment of the other case is similar. It now happens that $h_ny \geq y$ for all y in $[p, q]$, whence (for such y) $|h_ny - h_nq| < d$ whenever $|y - q| < d$. Hence h is not expansive.

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Comments by Grünbaum (K9)

This problem was first posed by Utz [Utz50] in 1950. The first three examples of compact, connected metric spaces that admit an expansive homeomorphism were given by Williams [Wil55] in 1955; however, these are topologically complicated sets, and not convex bodies. However, in a later paper [Wil70] Williams established that there exists an expansive homeomorphism of the open unit disk onto itself. Many of the later results on the topic can be found (with references) in the paper [Mou08] by C. Mouron. It is not clear to me what is the situation concerning the original problem in dimensions ≥ 3. It is even less clear to me how to reconcile the apparently contradictory results of [Wil55] and of Coven and Keane [CK06] concerning expansive homeomorphisms of compact metric spaces. As a possible explanation of the contradiction, Mark Bun suggests that Williams' [Wil55] construction allows negative powers of the expansive homeomorphism, whereas Coven and Keane [CK06] restrict themselves to nonnegative powers. It is not completely clear which interpretation Klee used; if the former, then the problem for convex bodies appears to be open in dimensions $n \geq 2$.

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K10 Equichordal Points of Convex Bodies

PROBLEM: ⋆ Does there exist (for $n \geq 2$) an n-dimensional convex body which has two equichordal points?

If the set C is star-shaped from its interior point p (i.e. C contains every segment joining p to another point of C), p is called an equichordal point of C provided all chords of C which pass through p are of the same length. (For example, the center of a spherical region is an equichordal point.) Whether a plane convex body can possess two equichordal points was first asked by Fujiwara [5], and independently by Blaschke, Rothe, and Weitzenbock [1]. The problem has been considered by several authors, and some of their results can be extended to E^n. We discuss below only the two-dimensional case.

By the term ke-set, we shall mean a two-dimensional convex body which has at least k equichordal points. Fujiwara proved that there are no 3e-sets. Süss [12] showed that a 2e-set must be symmetrical relative to the line through its two equichordal points, and also relative to the midpoint of the segment joining them. Dirac [2] repeated some of Süss's results and proved that a 2e-set must be smooth (i.e., have a unique supporting line at each boundary point). He also obtained quantitative results on the chord $xpqy$ where p and q are the two equichordal points. Helfenstein [6] showed that the boundary curve cannot be six times differentiable at x and y. (But Wirsing [13] showed that the boundary curve must be analytic? Apparently Helfenstein is in error.) Dulmage [4] studied the tangent lines of 2e-sets. Linis [10] claimed to prove that the boundary curve of a 2e-set cannot be twice differentiable, but errors in his proof were discovered by Dirac [3].

Several authors have noted the existence of noncircular 1e-sets, and their connection with convex sets of constant width. For a point p and a line L in the plane, the foot of p on L is the intersection of L with the perpendicular through p. For a plane convex curve A, the pedal curve of A with respect to p is the curve whose points are the feet of p on the various supporting lines of A. If A is the boundary of a convex set of constant width, p is interior to A, and P is the pedal curve of A with respect to p, then P bounds a region which is star-shaped from p (though not necessarily convex) and which has p as an equichordal point. Conversely, if a curve P bounds a (star-shaped) region of which p is an equichordal point, and if P is the pedal curve with respect to p of a convex curve A, then A bounds a set of constant width. (Good references for this material are Fujiwara [5] and P.J. Kelly [9].)

It seems of interest to mention two classes of bodies analogous to those with equichordal points. Suppose the subset C of E^n is star-shaped from its interior point p. Then p is an equichordal point of C provided the sum $|x-p|+|y-p|$ has the same value for all chords x, y of C which pass through p. Analogously, we say that p is an equiproduct point of C provided the product $|x-p| \cdot |y-p|$ has the same value for all chords x, y as above, and an equireciprocal point provided the sum $|x-p|^{-1}+|y-p|^{-1}$ is constant.

Convex bodies with equiproduct points have been studied by Yanagihara [14, 15], Rosenbaum [7, 11] and J. B. Kelly [7]. Each point interior to a spherical body is an equiproduct point. The set of equiproduct points of a convex body is the intersection of a flat with the interior of the body. For $0 \leq k \leq n-2$, there are nonspherical n-dimensional convex bodies for which the set of equiproduct points is k-dimensional; however, if the set of equiproduct points is $(n-1)$-dimensional, then the body is spherical. If an n-dimensional convex body has two equiproduct points and is smooth, it must be spherical.

Now consider a surface S in E^n, enclosing a bounded region R which is star-shaped from its interior point p. It is clear that p is an equireciprocal point of R if and only if p is an equichordal point of the region which is enclosed by the image of S under inversion in a sphere centered at p. However, this simple relationship between equireciprocal points and equichordal points is not very useful in dealing with convex bodies, for convexity does not behave very simply under inversion. (For a discussion of inversive convexity, see P. J. Kelly and Straus [8].)

If the origin O is interior to a convex body C in E^n, the polar body C^O can be defined as the set of all points x such that $\sup_{y\in C}\langle x,y\rangle \le 1$ where $\langle\cdot,\cdot\rangle$ denotes the inner product. The set C^O is again a convex body having the origin in its interior, and $(C^O)^O = C$. It is easy to verify that C is of constant width if and only if O is an equireciprocal point of C^O. In the two-dimensional case, the pedal curve of the boundary of C is the image of the boundary of C^O under inversion in the unit circle. Thus the connection between constant width and equichordal points is by way of equireciprocal points. It appears that the following is open:

PROBLEM: Does there exist (for $n = 2$) an n-dimensional convex body which has two equireciprocal points?

The results of Süss [12] imply certain symmetry properties for a convex body with two equireciprocal points, and show that no plane convex body can have three equireciprocal points. The problem just stated can be transformed to one on convex bodies of constant width. For example, a negative solution to it is implied by an affirmative solution of the following:

PROBLEM: Suppose K is a convex cone in E^n, H a hyperplane whose intersection with K is an $(n-1)$-dimensional convex body of constant width, and H' is a hyperplane such that the orthogonal projection onto H of the set $H' \cap K$ is also of constant width. Must H be parallel to H'?

For further problems on convex bodies with equireciprocal points, see K11. Some other problems related to those above can be found at the end of Süss's paper [12]. Not all of them have been solved.

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Comments by Grünbaum (K10)

This question was very popular for a long time. Klee raised it again in 1969 in his first "Research Problems" column in the Monthly [Kle69], and repeated it in several later publications ([Kle71], [Kle79], [KW91]); it is also posed as Problem A1 in [CFG91], and in many other places. Each of the papers mentioned below has additional references to works relevant to the problem. The accepted wisdom is that the 1997 paper [Ryc97] of Rychlik finally disposed of the problem, by showing that no $2e$-curve (of a type that includes convex curves) can have two equichordal points.

However, the previous history has been made interesting by several partial results and even more so by various errors in claims about the problem. In the early articles about the problem it is established that if a $2e$-curve exists, it must have certain symmetry and smoothness properties, regardless of whether it is assumed to be convex or not; that much is not in dispute. An attempt by Linis [Lin57] to show that a $2e$-curve cannot exist has been discredited by Dirac's pointing out (see [Dir]) an error in the argumentation. But more unexpected has been the reaction to the work of Helfenstein [Hel56]. His rather infelicitously formulated claim is: "We shall show in this paper the non-existence of real and, in one special point, at least six times differentiable $2e$-curves." What his argumentation shows (assuming there are no errors, and nobody pointed out any specific errors in the paper) is that a contradiction is reached from the assumption of existence of a $2e$-curve together with the assumption that the curve assumed to exist is six times differentiable at a specific point. This would seem a reasonable result, and the non-existence would be established provided the differentiability assumption could be proved for all $2e$-curves assumed to exist. Hence one would think (and this was explicitly stated by Süss [Süs]) that the non-existence of $2e$-curves would follow from Helfenstein's result together with the following result of Wirsing [Wir58]: If a $2e$-curve exists, it must be analytic, that is, infinitely differentiable at all points. Indeed, to belabor the completely obvious, if a curve is infinitely differentiable at all points, then it is six times differentiable at a specific point – and the contradiction reached by Helfenstein proves the non-existence. It is mystery to me why Wirsing thought that Helfenstein's work must be in error since "it is contradicted by the present investigation" ("durch die vorliegende Untersuchung widerlegt wird"). But an even greater mystery is why Klee, in all his discussions of equichordality, accepted Wirsing's statement as valid, and Helfenstein's result as invalid. Wirsing committed a logical error, and Klee and others uncritically accepted it a valid. The reviews of Helfenstein's paper in the Math. Reviews and the Zentralblatt fail to claim any errors in it. Not all details of Helfenstein's proof are given in [Hel56] – but no error has been found either.

In the 1980's and 1990's several papers – combined or singly – proved the nonexistence of a Jordan curve with two equichordal points with respect to which it is star-shaped. The last of these is the paper by Rychlik [Ryc97] mentioned above, where the nonexistence is established after applying several advanced mathematical theories, on 72 pages. Some details on this, and on preceding results, are contained in the "Featured Review" [Vol] by Volkmer. It would be enjoyable and desirable to have available a more accessible presentation of this solution of the problem – but this has not happened in the more than a decade since the publication of [Ryc97].

With the approaching century-mark since the problem was first proposed, it would be even more enjoyable and desirable to have an independent confirmation – or rebuttal – of the results and claims of Helfenstein and Wirsing, and a clarification of the status of the problem and its solution.

Many additional references to relevant papers can be found in the papers by Klee listed below, as well as in [CFG91] and [Vol].

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K11 Antipodal Points of Convex Bodies

PROBLEM: Two points of a convex body will be called antipodal provided they lie in parallel supporting hyperplanes of the body. For various classes K of convex bodies, determine the smallest integer $k(= fK)$ such that every member of K admits a covering by k closed sets, none of which includes two antipodal points. Of special interest are the class C_n of all n-dimensional convex bodies, the class J_n of all members of C_n which are symmetric at the origin O, the class R_n of all rotund members of C_n, and the class W_n of all members of C which are of constant width.

A member of C_n is said to be rotund provided its boundary contains no line segment, and to be smooth provided it admits a unique supporting hyperplane at each boundary point. An interior point p of the convex body B is said to be an equireciprocal point of B provided the sum $|x-p|^{-1}+|y-p|^{-1}$ has the same value for all chords $[x, y]$ of B which pass through p. In addition to the classes of sets mentioned above, we shall refer later to the class S_n of all n-dimensional smooth convex bodies and the class Z_n of all those which have the origin O as an equireciprocal point. (Observe that if a convex body has an equireciprocal point, it must be smooth.)

Of the problems suggested above, determination of fW_n is the most interesting, due to its connection with the following problem of Borsuk [2]: Can every bounded subset of E^n be covered by $n+1$ subsets of smaller diameter? It is known that a set of diameter d lies in a convex body of constant width d, and that two points of such a body are antipodal if and only if their distance is d. Thus Borsuk's question amounts to asking whether $fW_n \leq n+1$. (From Borsuk's theorem [2] on the sphere, it follows [1] that $f\{C\} > n$ for every C in C_n.)

Hadwiger [7] proved that $f(W_n \cap S_n) = n+1$, and the same argument shows that $fS_n = n+1$. If a body C in E^n is rotund and symmetric about the origin O, then two points x and y of the boundary of C are antipodal if and only if $x = -y$; from this it follows easily that $f\{C\} = n+1$. Now consider, more generally, an n-dimensional convex body C in which each boundary point x admits a unique antipodal point Λx. Then Λ maps the boundary of C onto itself, and is a continuous involution without fixed-point; from results of Yang [11] it follows that $f\{C\} = n+1$.

By a theorem of Macbeath and Mahler [9] (or by earlier results of Behrend) there follows the existence in E^n of two concentric spherical bodies U (of diameter u) and V such that every convex body in E^n is affinely equivalent to some convex body C with $U \subset C \subset V$. Then clearly two antipodal points of C must be at distance at least u, while by compactness V can be covered by a finite number k_n of closed sets of diameter less than u. Thus $f\{C\} \leq k_n$, and since f is affine-invariant we conclude that

$$n+1 = fS_n = f(J_n \cap R_n) \leq fW_n \begin{array}{c} \leq fJ_n \leq \\ \leq fR_n \leq \end{array} fC_n < \infty.$$

It is further clear that $fC_n = fP_n$, where P_n consists of all polyhedral members of C_n; and similarly, $fJ_n = f(J_n \cap P_n)$. The n-dimensional cube has 2^n vertices, each two of which are antipodal. This shows that $f(J_n \cap P_n) \geq 2^n$, and suggests the following additional

PROBLEM: For various classes K of convex bodies, determine the largest integer $k(= gK)$ such that some member of K contains k points which are pairwise antipodal.

It is clear that always $2 \leq gK \leq fK$, and can be verified that $gS_n = 2$, $gW_n = n+1$, $gJ_n = g(J_n \cap P_n)$, and $gC_n = gP_n$.

A theorem of Levi [8] implies that for a two-dimensional convex body C, $f\{C\} = 4$ if and only if C is a parallelogram. Thus when C is not a parallelogram, $f\{C\} = 3$ and $g\{C\} = 2$ or 3. Borsuk's problem has been solved in E^3 by Eggleston [4] and Grünbaum [6]; from their result it follows that $fW_3 = 4$.

(We digress slightly –.) In contrast to the notion of antipodal vertices of a convex polyhedron P, we may consider the notion of adjacent vertices, these being vertices x and y such that the segment $[x, y]$ is the intersection with P of some supporting hyperplane. For $n \leq 3$, an n-dimensional polyhedron has at most $n+1$ pairwise adjacent vertices. But, in striking contrast to the fact that gC_n is finite for all n, a theorem of Gale [5] asserts that for all $n \geq 4$, there exist n-dimensional convex polyhedra having an arbitrarily large number of pairwise adjacent vertices. See also Caratheodory [3]. An interesting problem on adjacent vertices appears at the end of Gale's paper.

Now in order to justify a different formulation for the problem on f, let us establish the following fact: for a convex body C and a point p interior to C, fC is equal to the smallest integer j for which there exist j convex bodies $B_1, \ldots, B_j$, each having p in its interior, none including two antipodal points, and such that C is the convex hull of their union. By definition, the set C can be covered by $f\{C\}$ closed sets, $X_1, \ldots, X_{f\{C\}}$, none of which includes two antipodal points. Let U be a closed neighborhood of p which is interior to C, and for each i let X'_i be the convex hull of the set $X_i \cup U$. It is easy to see that no set X'_i includes two antipodal points, and consequently $j \leq f\{C\}$. Conversely, suppose the sets B_i are as described above, and for each a let B'_a be the set of points of the form $\sum_{i=1}^{j} t_i b_i$ where $t_a \geq i/j$, $\sum_{i=1}^{j} t_i = 1$, and always $b_i \in B_i$ and $t_i \geq 0$. It can be verified that C is covered by the sets B'_i, and that each such set is closed and fails to include two antipodal points. Thus $f\{C\} \leq j$ and the proof is complete.

Now for an interior point p of a convex body C, let us denote by $h(p, C)$ the smallest integer k for which there exist k convex bodies $D_1, \ldots, D_k$ whose intersection is C, and such that for each D_i and each line L through p, the segment $D_i \cap L$ is longer than the segment $C \cap L$. (The value of $h(p, C)$ may vary with p; for example, if C is a parallelogram, then $h(p, C) = 4$ when p is the center of C but otherwise $h(p, C) = 3$.) Recall that for a convex body C which has O as an interior point, the polar body C^O is the set of all x such that $\sup_{y \in C} \langle x, y \rangle \leq 1$; C^O is again a convex body having O as an interior point, and $(C^O)^O = C$. From well-known properties of the operation ^{O}, in conjunction with the result of the preceding paragraph, the following may be deduced: if C is a convex body having O as an interior point, then $f\{C\} = h(O, C^O)$ and $h(O, C) = f\{C^O\}$. Now for a class K of convex bodies, let K' be the class of all members of K which have O as an interior point, and let hK be the largest of the integers $h(O, C)$ for C in K'. Then we have $fC_n = hC'_n, fJ_n = hJ_n, fR_n = hS'_n, n+1 = fS_n = hR'_n$, and $fW_n = hZ_n$. Thus, in particular, Borsuk's problem is equivalent to the following:

PROBLEM: Suppose C is a convex body in E^n, having O as an equireciprocal point – say $|x|^{-1} + |y|^{-1} = r$ for each chord $[x, y]$ of C which passes through O. Do there exist convex bodies $D_1, \ldots, D_{n+1}$ whose intersection is C, such that for each i and each chord $[u, v]$ of D_i which passes through O, $|u|^{-1} + |v|^{-1} < r$?

Finally, let us consider one more problem which is related to those above. Suppose C is a smooth convex body in E^n which has O as an interior point, and let B denote the boundary of C. For each y in B, let n_y denote the unit outward normal to B at p. For each nonzero $x \in E^n$, let H_x denote the set of all y in B for which $\langle x, y \rangle > 0$ and N_x the set of all y in B for which $\langle x, n_y \rangle > 0$. Clearly B can be covered by $n+1$ sets of the form H_x, and by $n+1$ of the form N_x. Now let wC denote the smallest integer k such that B can be covered by k sets of the form $H_x \cap N_x$. With the aid of a theorem of McMinn [10], it can be proved that $h(O, C) \leq wC$ when $n = 3$ (and it seems probable that this holds also for higher dimensions), so the relationship of the function w to the previous problems is evident. In particular, we note the following

PROBLEM: For each n, determine the largest possible value of wC for n-dimensional smooth convex bodies having O as an interior point. Perform the same task for n-dimensional smooth convex bodies which are symmetric about O, and for those which have O as an equireciprocal point.

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Additional remarks.

The literature on Borsuk's problem and its relatives is fairly extensive. Two papers of Heppes should have been mentioned – Acta Math. Hung. **7** (1956) 159-162 and Mat. Lapok **7** (1956) 108-111. For a discussion in Minkowski planes see Grünbaum, Bull. Res. Council Israel **7F** (1957) 25-30. Further unsolved problems appear at the end of that paper.

Comments by Grünbaum (K11)

This is actually a collection of problems; concerning several of them there are many developments to report.

Concerning gK, besides the facts mentioned by Klee, it was established in [DG62] that $gC_n = 2^n$ for all n. For $n = 3$ Bisztriczky and Böröczky [BB05], Bezdek, Bisztriczky and Böröczky [BBB05], and Schürmann and Swanepoel [SS06] investigate refinements of this result. Martini and Soltan [MS05] give a detailed survey of antipodality properties, with many references and open problems. Complementing results of Croft [Cro81], it was proved in [Grü63b] that a strictly antipodal set in E^3 can have at most 5 points.

Borsuk's problem in Minkowski planes has been studied in [Grü57]; the special case in which the unit disk is a regular hexagon is considered by Xu, Yuan and Ding [XYD04].

[Grü63a] is a detailed survey of results related to the Borsuk problem available till the early 1960's. While the following decades brought several advances on particular questions related to the Borsuk problem, the main question remained unsolved for all $n \geq 4$. However, a major change in the results and attitudes toward the whole field came in 1993, with the unexpected result by Kahn and Kalai [KK93] that the number of sets of smaller diameter needed to cover a given set grows exponentially with the dimension n. The lowest dimension in [KK93] for which the number of covering sets exceeds by more than 1 the dimension is $n = 1825$. This has been lowered by several people, but is still quite high. See Raigorodskii's papers [Rai04], [Rai08] for details. But it is remarkable that the problem is still unsolved in the case $4 \leq n \leq 322$. For this bound see Hinrichs [Hin02].

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K12 Squaring the Rectangle

PROBLEM: What numbers can be attained as the orders of perfect squarings of squares? of simple squarings? of simple perfect squarings? In particular, what is the smallest possible order for a perfect squaring of a square?

PROBLEM: What numbers can be attained as the ratio of the two dimensions of a simply perfectly squared rectangle? In particular, can the number 2 be so attained?

A dissection of a rectangle into a finite number $n(>1)$ of non-overlapping squares is called a squaring of the rectangle, of order n, and the side-lengths of the n squares are called the elements of the squaring. A squaring of order n is called perfect provided it has n different elements and simple provided no union of k of its members is a rectangle for $1 < k < n$. The squaring of rectangles has been studied by many authors, commencing with Dehn [4], who proved that every squared rectangle has commensurable sides and elements. An interesting systematic treatment is that of Brooks, Smith, Stone and Tutte [2], who correlate the squaring of rectangles with some aspects of electrical network theory. See also the expository paper by Tutte [9]. They prove a strong converse of Dehn's theorem – every rectangle with commensurable sides admits infinitely many different perfect squarings. This was proved independently by R. Sprague [7], who earlier had given the first example of a perfectly squared square [6]. His squaring was of order 55, but Willcocks [10] later produced a squaring of order 24. (The lowest order known for a perfect squaring of a square.) The squaring of Willcocks was not simple, but simple perfect squarings were discussed in [2], by Bouwkamp in [1], and in a second paper by Brooks, Smith, Stone, and Tutte [3].

The second problem stated above is mentioned in [2], where it is remarked that perhaps a simple perfect squaring is admitted by every rectangle whose sides are commensurable. In [8] it is remarked that such a squaring definitely is admitted when the ratio of the sides is 15/17 (as well as when the ratio is unity).

For additions to the list of references below, consult the bibliographies of [2], [10], and [8]. See also [5].

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Comments by Grünbaum (K12)

Several parts of the first problem enjoyed great popularity over a long time. An abundant source of references is Federico [Fed79] in 1979. In particular, answers to the question of the minimal order of a perfect squared square kept improving over time. The first such squared square had order 55; it was found by Sprague [Spr39] in 1939. Brooks *et al.* [BSST40] in 1940 describe a squared square of order 26. Willcocks [Wil48], [Wil51] found a squared square of order 24; although [Wil48] was published in 1948, the existence of this squared square was not generally known till 1951 [Wil51]. This was long considered to be the smallest possible order. However, Duijvestijn [Dui78] found in 1978 a squared square of order 21; this is known to be the smallest order possible (and Duijvestijn's square the unique squared square of this order). For additional information about known squared squares of orders 21 to 24 see Duijvestijn [Dui93]. The same paper also contains solutions of the second problem: There exist simply perfect squared rectangles with ratio of sides 2 : 1. The smallest is of order 22, and there are eight of order 23. A very detailed account of the history of the quest for perfect squared squares and rectangles is given by Federico [Fed79] in 1979. It was written before the discovery by Duijvestijn of the squared square of order 21, but managed to mention it in a note added in proof. Another interesting account, in particular of the early history of these problems, is on pages 127 – 134 of [Mau81]. A first-person account of the history is in Tutte [Tut98] published in 1998.

The flood of papers dealing with squared squares and rectangles has abated somewhat in recent years, but a related topic deserves mention here. Is it possible to find a tiling of the whole plane using one square each of size n, for $n = 1, 2, 3, \dots$? The affirmative answer was provided by Henle and Henle [HH08] in 2008, where a discussion of the history of related questions is presented, and several open problems are listed.

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K13 Sets of Visibility

PROBLEM: Suppose X is a compact subset of E^n such that each point of X lies in a unique minimal convex set of visibility. Must X include a point of visibility?

We say that a point of X is visible (in X) from the point p of X provided X contains the entire segment $[x, p]$. The point p is a point of visibility of X provided each point of X is visible from p. (Alternatively, we say that X is starshaped from p.) A set of visibility is a subset V of X such that each point of X is visible from some point of V. If we denote by $\mathcal{U}_x$ the family of all convex sets of visibility which contain the point x, the above problem asks whether X must be starshaped from some point if for each point x of X there is a unique member of $\mathcal{U}_x$ which contains no other member of $\mathcal{U}_x$. The problem is mentioned in [7] by Valentine, who established an affirmative answer for $n = 2$ and obtained some related results in the n-dimensional case. It can be verified that if each point of X lies in a unique minimal convex set of visibility, and X includes two or more points of visibility, then every point of X is a point of visibility – that is, X is convex. Thus an affirmative solution of the problem leads to the conclusion that X is convex or starshaped from exactly one point. Obviously every convex set satisfies the condition of the problem; however, if X is the union of a circular disc with two segments from its center which protrude beyond the boundary, then X is starshaped from exactly one point but does not satisfy the condition of the problem. (This example is due to Valentine [7].)

In considering this and other problems which deal with starshapedness, it may be useful to recall the following interesting theorem of Krasnosselsky [4]: Suppose a compact subset X of E^n is such that for each set Y of at most $n + 1$ points in X, there is a point of X from which all points of Y are visible. Then X is starshaped. Krasnosselsky's theorem is based on the following well-known theorem of Helly: If a family $\mathfrak{F}$ of compact convex sets in E^n is such that each $n + 1$ members of $\mathfrak{F}$ have a common point, then there is a point common to all members of $\mathfrak{F}$. Even when the number $n + 1$ is replaced by a smaller number, interesting properties of the family $\mathfrak{F}$ are implied by the given intersection condition. (See Horn and Valentine [2], Horn [1], and Klee [3].) It would be of interest to find corresponding extensions of Krasnosselsky's theorem. One such extension is the theorem of Horn and Valentine [2] which asserts that if X is a compact simply connected subset of E^2, and for each two points of X there is a point from which both are visible, then X is a union of convex sets every two of which have a common point. This result fails in the absence of simple connectedness. It would be of interest to study the situation when X is not simply connected, and to consider the n-dimensional case.

In a similar spirit is the theorem of Valentine [8] which asserts that if X is a connected closed subset of E^2 which contains, for each three of its points x, y, and z, at least one of the segments $[x, y], [y, z]$, and $[x, z]$, then X is the union of three convex sets having nonempty intersection. It would be of interest to find a similar condition for sets which are the union of two convex sets, and to extend the consideration to n dimensions.

In connection with sets which are the union of finitely many convex sets, an example due to Motzkin [5] may be found instructive.

In the paper [6], Straus and Valentine prove that if X is a connected closed subset of E^n and each point of X lies in a unique maximal $(n - 1)$-dimensional convex subset of X, then X is convex. An unsolved problem is mentioned at the end of their paper.

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Comments by Grünbaum (K13)

The title of Problem K-13 is somewhat misleading, since the contents cover a much broader array of topics. But the text of K-13 puts in stark perspective how things have changed over the last half-century. The feeling that the reader will not be familiar with the theorems of Helly and Krasnosselsky (now spelled Krasnosel'skii) appears strange today, with a literature on these topics that would fill a library of books. One way of seeing the difference between the attitudes in the 1960's and more recent times is to compare the survey [DGK63] published in 1963 or the book [HDK64] published in 1964 with the collections [GS91] edited by Gritzmann and Sturmfels in 1991, and [Pac91] edited by Pach in 1993, the surveys [BMS97] by Boltyanski, Martini and Soltan published in 1997, [BMP05] by Brass, Moser and Pach published in 2005, [Kar08] by Karasëv published in 2008, and the collections of problems [CFG91] by Croft, Falconer and Guy and [KW91] by Klee and Wagon, both in 1991.

In staying close to the formulation in Valentine's paper, Klee makes the understanding of the problem needlessly complicated. Moreover, it seems to lead him to claim that the example mentioned shows that the converse of the problem is invalid. It may be that the converse is invalid, but the example does not work. The example Klee produces (taken from Valentine's paper where it was used for another purpose) is the set X that consists of a circular disk together with two segments from the center O that protrude beyond the disk. The segments are not collinear, since then X would be starshaped from each point of a segment, contradicting Klee's assumption. But if the segments are not collinear, then the unique minimal set of visibility for X that includes a given point x is the segment Ox, except if $x = O$, when it is just O. Hence this is not a counterexample to the converse.

For a more recent proof of Horn's result, and an account of its history, see Webster [Web96].

Valentine's result that if for any triplet of points in a set X at least one of the segments determined by them is contained in X, then X is the union of at most three convex sets has been generalized to all dimensions and to k-tuples of points, but with no precise number of convex sets involved. See [MV99] for details about the results and the history of the problem.

For a discussion of extensions and relatives of Krasnosel'skii's theorem and a long list of references, see Section E3 of [CFG91].

The example of Motzkin, and the paper by Santaló mentioned by Klee, deal with the existence or non-existence of Helly numbers for sets that are unions of pairs of convex sets. This has also led to considerable literature, the most general result being that of N. Amenta [Ame96], confirming and old conjecture of Grünbaum and Motzkin [GM61]: If F is a family of sets in Euclidean d-space, such that every subfamily is the disjoint union of at most k closed sets, then F has Helly number at most $k(d + 1)$. Other relevant generalizations are given by Swanepoel [Swa02] and N. Halman [Hal08], and many others.

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K14 Unit Distances

PROBLEM: For each integer $n > 1$, determine the smallest integer $k(= D_n)$ such that the n-dimensional Euclidean space E^n can be covered by k sets, none of which includes two points at unit distance.

Clearly $D_1 = 2$, for we may take as one set the union of all half-open intervals $[2j, 2j+1)$ and as the other the union of all half-open intervals $[2j-1, 2j)$ (j an integer). And it is evident that $D_n \geq n+1$. Now let d be a number greater than 1 and let L_n be the cubic lattice of side-length d in E^n – i.e., the set of all points of E^n whose coordinates are all integral multiples of d. Let S_n be the set of all points of E^n which lie at distance less than $(d-1)/2$ from some point of L_n . Then clearly the fundamental cube $\{x \in E^n : 0 \leq x^i \leq d\}$ can be covered by finitely many translates of S_n, and in fact these translations cover the entire space E^n. It follows that D_n is finite, and the coverings described suggest modification of the problem by placing additional restrictions on the sets of the covering – either on the individual sets (for example, topological conditions such as openness or closedness) or on their inter-relationship (for example, translation-equivalence or congruence).

For $n = 2$, the above problem was suggested to us by John Isbell, who attributes it to Edward Nelson. The following reasoning, due to Isbell, shows that $4 \leq D_2 \leq 8$. Consider a covering of E^2 in which each of the eight sets is a union of squares of unit diagonal whose vertices are taken from square lattices of side-length $1/\sqrt{2}$. The squares are arranged in the following pattern:

$$\begin{array}{cccc} A & B & C & D \\ E & F & G & H \\ C & D & A & B \\ G & H & E & F \end{array}$$

If each open square is augmented by half its boundary in the appropriate way, the resulting sets A, B, C, D, E, F, G, and H satisfy the condition of the problem. Consequently $D_2 \leq 8$.

Now suppose E^2 is covered by three sets A, B, and C, none including two points at unit distance. For each point a of A, let X_a and Y_a denote the circles centered at a with radii 1 and $\sqrt{3}$ respectively. Then of course X_a misses A and hence $X_a \subset B \cup C$. For each point p of Y_a there is a unit equilateral triple pqr such that q and r both lie in X_a, and consequently $p \in A$. But then $Y_p \subset A$, and since Y_p intersects X_a we have a contradiction which shows $D_2 \geq 4$.

We include two additional remarks which may conceivably be useful in connection with the above problem, and are interesting in any case:

1) In some ways, the significance of "unit distance" with respect to E^2, E^3, etc., is quite different from that with respect to E^1. For example, there are many transformations of E^1 onto itself which preserve some distances and not others. (Move all the integer points one unit to the right, the others one unit to the left – only the integral distances are preserved.) But a theorem of Beckman and Quarles [2] asserts that if $n \geq 2$ and T is a transformation of E^n into itself which preserves unit distances ($||Tx - Ty|| = 1$ whenever $||x - y|| = 1$), then in fact T is an isometry (always $||Tx - Ty|| = ||x - y||$). (Of course it suffices to assume that T preserves any fixed positive distance.) A theorem of Mazur and Ulam [1, p. 166] then implies that T is in fact an affine isometry.

2) Covering problems such as the one above can be interpreted as graph-coloring problems. We may imagine an abstract graph whose vertices are the points of E^n, and in which two vertices are adjacent (that is, are the "endpoints" of a common "edge") if and only if they lie at unit distance in E^n. The problem then is to color the vertices of this graph, using the smallest possible number of colors, in such a way that no two adjacent vertices have the same color. It has been proved by Erdös and de Bruijn [3] that if a graph

requires at least k colors for such a coloring of its vertices, then the same is true of some finite subgraph. Thus for the problem above, it suffices to consider finite subsets of E^n.

There are many other covering problems which involve distances and are similar in spirit to the one above. For example, a result of Erdös and Kakutani [4] (based on the continuum hypothesis) implies that the line can be covered by countably many sets $S_1, S_2, \ldots$, such that no S_i includes two different point-pairs with the same distance. (If w, x, y, and z are in S_i and $||w-x|| - ||y-z|| = 0$, then $w = y$ and $x = z$ or $w = z$ and $x = y$.) It is not known whether E^n admits such a covering for $n \geq 2$.

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Comments by Grünbaum (K14)

This was one of Klee's favorite problems, which he brought up at many occasions; for example, it was Problem (A) in [Kle79] and Problem 8 in [KW91]. It is also the topic of Section G10 of [CFG91]. In each of these a number of related problems are raised as well. A very prolific writer on these topics is A. M. Raigorodskii; see for example [RK08], [RS08].

The history of the problem is quite interesting and convoluted. It is investigated in detail by Soifer [Soi09] in 2009. This book dispels various misattributions of the main problems and of the results concerning it. It also contains much original material, and presents many older results that seem to have been forgotten, that do not appear in other recent accounts. Soifer gives an authoritative and detailed description of the many directions in which the problem has been generalized, specialized, and modified. Anybody interested in Problem K-14 or in any of its numerous ramifications should consult this volume.

With all of this, the original problem is still open even for $n = 2$, where it is only known that $4 \leq k \leq 7$.

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K15 Sections of Concentric Convex Bodies

PROBLEM: Suppose K' and K are n-dimensional convex bodies with a common center of symmetry p. Suppose further that for each hyperplane H through p, the $(n-1)$-dimensional area of the section $K' \cap H$ is less than or equal to that of the section $K \cap H$. Must the n-dimensional volume of K' be less than or equal to that of K?

The problem is due to Busemann and Petty [2], and is unsolved even for $n = 3$. The background and some applications of the problem are discussed in [2], where it is pointed out that an affirmative solution is known when K' is an ellipsoid but not when K is a sphere, and that even the latter case is of considerable interest.

In [1], Busemann shows by examples that the answer to the above problem is negative if K' and K are assumed merely to be starshaped from their common center of symmetry p, or if both are assumed to be convex but the condition that p should be their common center is dropped. (Similar examples have been found by L. Danzer (unpublished).)

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Comments by Grünbaum (K15)

This problem has attracted much attention and received an almost complete solution. The first result was by Larman and Rogers [LR75] in 1975; they provided a negative answer for all dimensions $n \geq 12$. In 1994 Gardner [Gar94] gave an affirmative answer in case $n = 3$, and Zhang [Zha94] seemed to complete the solution for the intermediate n by giving a negative answer for all $n \geq 4$. However, it turned out (see [Zha99]) that Zhang's proof is valid only for $n \geq 5$, and that for $n = 4$ the problem has an affirmative solution. It is remarkable that according to the Math. Reviews Citation Index, [Zha99] is among the references in 52 papers; also, [Sch08] has a bibliography with 68 entries. Obviously, there is still considerable activity on related questions (see, for example, [Rub09]). Some of this centers on a problem of Shephard [She64] that is somewhat dual to K-15:

If K' and K are centrally symmetric convex bodies in E^n such that the projection of K' on any $(n-1)$-dimensional hyperplane has $(n-1)$-content that is smaller than or equal to that of the projection of K, is the n-dimensional volume of K' necessarily smaller than or equal to the volume of K?

As shown by Petty [Pet67] and Schneider [Sch67], the answer is affirmative for $n = 2$ and negative for $n \geq 3$. For additional results and references see Koldobsky *et al.* [KRZ04] and Schuster [Sch08].

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K16 Characterization of Ellipsoids

PROBLEM: Suppose $2 \le k < n$, and C is an n-dimensional centered convex body of which all central k-sections are affinely equivalent. Must C be an ellipsoid?

An n-dimensional ellipsoid is a subset of E^n consisting of all points $x = (x^1, \ldots, x^n)$ for which $\sum_1^n (x^i)^2 a_i^2 \le 1$ (for an appropriately chosen coordinate system and numbers $a_i > 0$); equivalently, it is a nonsingular affine image of the solid sphere in E^n. The ellipsoids play an important role in the general theory of convex bodies, and many characterizations have been given. (For some references, see pp. 142-143 of [2]). For $k = 2$ and $n = 3$, the above problem was solved affirmatively by Süss [6], who raised the further question as to whether projective equivalence is sufficient. (See also Auerbach [1].) Specifically, if C is a 3-dimensional centered convex body of which all central plane sections are projectively equivalent, must C be an ellipsoid? Süss's problem appears still to be open. (It was called to our attention by W. Fenchel.)

The problem above is mentioned by Dvoretzky [5], who obtains an affirmative solution when E^n is replaced by an arbitrary infinite-dimensional Banach space and the term "ellipsoid" is suitably interpreted. His result is based on the fact that centered convex bodies of high dimensionality must have nearly spherical sections. (See [5] for details and a precise formulation.)

Among other unsolved problems concerning characterizations of ellipsoids, two due to Busemann and Petty [4] are especially interesting for their significance in the theory of Minkowskian geometries. (See [4] for details.)

PROBLEM: Let C be an n-dimensional centered convex body, and for each central $(n-1)$-section S of C let V_S be the maximum volume attained by the cones which are based on S and have vertex in the boundary of C. If V_S has the same value for all S, must C be an ellipsoid?

(The answer is negative for $n = 2$ [3], and the problem is open for $n \ge 3$)

PROBLEM: Suppose C is an n-dimensional centered convex body, and consider all pairs (S, p) such that p is a boundary point of C and S, among the central $(n-1)$-sections of C, is one which maximizes the volume of the cone from p over S. For each such p and S, let $R_{p,S}$ denote the quotient A_1/A_2, where A_2 is the $(n-1)$-dimensional area of S and A_1 is the $(n-1)$-dimensional area of the image of C under the affine projection which maps C into the hyperplane containing S and p into the center of C. Then if $R_{p,S}$ has the same value for all p and S as described, must C be an ellipsoid?

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Comments by Grünbaum (K16)

The result of Süss [Süs26] was proved by Auerbach *et al.* [AMU35] as well. However, the mysterious reference to Auerbach [Aue40] is to the 1940 paper that claims to solve Süss's projective question. However, Fubini [Fub] expressed a reservation about the validity of this claim, and this seems to be the reason for Fenchel's opinion that the problem is still open. On the other hand, Corollary 3.3.5 in Thompson's book [Tho96] presents a solution of Süss's problem for all $n \geq 3$, utilizing the Löwner ellipsoid. Countless references to characterizations of ellipsoids, or inner-product spaces among Minkowski spaces, can be found in Chapter 3.4 of [Tho96], in Amir's book [Ami86], and in Section 3 of the survey Heil and Martini [HM93]. The quest for characterizations of ellipsoids continues unabated, with too many and too diverse approaches to give details here.

The paragraph dealing with Dvoretzky's work is somewhat misleading, since Dvoretzky's papers are not relevant to the problem K-16.

References

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K17 Inscribed Squares

PROBLEM: Must a Jordan curve contain the vertices of a square?

A Jordan curve is a plane set which is a continuous biunique image of the circle. That is, a subset J of E^2 is a Jordan curve provided there exist continuous real functions x and y on $[0,1]$ such that: i) J is the set of all points $(x(t), y(t))$ for $0 \leq t \leq 1$; ii) if $0 \leq s < t \leq 1$, then $(x(s), y(s)) = (x(t), y(t))$ if and only if $s = 0$ and $t = 1$.

The above problem has an interesting history in that several authors, even though restricting their attention to special classes of Jordan curves, have published unconvincing "solutions" (always affirmative). The most recent publications on the problem appear to be those of Šnirelman [5], Christensen [2], and Ogilvy [3]. Šnirelman (whose paper is an expanded version of one published in 1929) supplies a valid proof that if a Jordan curve has continuous curvature, then it admits an inscribed square and also admits a "complete system of inscribed rhombi" (see the paper for the definition). Christensen proved that a convex Jordan curve admits an inscribed square, and discusses the shortcomings of some earlier "proofs". Ogilvy claims to solve the general problem, but his approach can at best be described as a heuristic.

(Further remarks by Grünbaum: It is an old result that every closed convex curve contains the vertices of an affine-regular hexagon. Is this true also of all Jordan curves? Must a centrally symmetric Jordan curve contain the vertices of an affine-regular octagon?)

It is natural to wonder about higher-dimensional analogues of the problem. Bielecki [1] has settled one aspect by means of an example of a 3-dimensional convex body in which no rectangular parallelopiped can be inscribed. On the other hand, Pucci [4] proves that in every 3-dimensional convex body there is an inscribed regular octahedron.

There are many other interesting problems involving "inscribed" squares and other figures. We shall mention one which was proposed by S. Kakutani: Let S be a square plane region of unit side, B its boundary. Does there exist a positive number $d < 1$ such that

i) for each continuous function f on S, the graph of f contains the vertices of a square of side d?

ii) for each continuous function f on S which vanishes everywhere on B, the graph of f contains the vertices of a square of side d which is parallel to the plane of S?

(Intuitively: Is there a chair which is small enough to be "useful" on every "large" mountain?)

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Comments by Grünbaum (K17)

PROBLEM: Must a Jordan curve contain the vertices of a square?

This is a very popular problem, dear to Klee; it is specifically mentioned in [Kle79] and in [KW91]. The problem has a rich history, studded with misunderstandings and errors. It seems to have originated with O. Toeplitz, with a publication in 1911 which I have not been able to see.

As far as I can tell, the original problem is still open, although it has been answered in the affirmative for many special types of curves. For example, it is known that the answer is affirmative for convex curves (Emch [Emc13], 1913), for sufficiently smooth curves (for example, Shnirel'man [Shn44], 1944; Stromquist [Str89], 1989), and for polygonal curves (for example, Pak [Pak], 2008).

More details about the history of the problem can be found in [KW91] and in the recent surveys [Pak] and [Kar08]. These papers give many references, and specify the shortcomings and errors contained in many papers that contain supposed solutions.

A considerable number of generalizations and variants of the problem have been proposed and studied. For instance, Stromquist [Str89] shows that sufficiently smooth curves in E^3 contain the vertices of a skew regular quadrangle. The existence of regular skew $(2n)$-gons with $n \geq 3$ under similar conditions seems open. However, at least one recent paper [Wu04] has a misleading title, since it calls "regular" polygons that are merely equilateral (have edges of same length).

I have not been able to get the paper by Pucci listed in the problem. This is unfortunate, since

- The Math. Reviews did not get the paper.
- The Zentralblatt review by G. Aumann states that every bounded, open, convex set in E^3 admits a *regular* inscribed octahedron – something that is rather hard to believe.
- I seem to remember hearing (very, very long ago) that Pucci's proof is invalid.

If anybody has a copy of the Pucci paper, or knowledge about the situation – I would be very grateful for any communication.

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K18 Some Inequalities Connected with Simplexes

PROBLEM: Suppose p is a point of an n-dimensional simplex S, and for $0 \le i \le n-1$, let D_i denote the sum of the distances from p to the various i dimensional faces of S. What relations among the numbers D_i must persist for all p and S as described? In particular, if $n = 3$ (so that S is a tetrahedron) is it true that

i) $D_0 > \sqrt{8} D_2$ (D. K. Kazarinoff) ?

ii) $D_0 > D_1$ (A. Shields) ?

iii) $D_1 > 2D_2$ (N. D. Kazarinoff and A. Shields) ?

The first inequality is an analogue of the Erdös-Mordell inequality for triangles which asserts that $D_0 \ge 2D_1$. (See pp. 12-14, 28 of [1] and a simpler proof in [2].) D. K. Kazarinoff stated the validity of the inequality (i) for all tetrahedra, but did not publish his proof (see pp. 120-121 of [1]). His methods were applied in [3] to establish validity of (i) when the tetrahedron in question has a trihedral "right angle", and also when it contains its own circumcenter (and the multiplier $\sqrt{8}$ is shown to be the best possible for this latter class of tetrahedra.) The inequalities (ii) and (iii) were conjectured in a letter from Shields and N. D. Kazarinoff, who observe that the inequality $D_0 = \frac{2}{3} D_1$ is easily established for tetrahedra.

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Comments by Grünbaum (K18)

The Erdös-Mordell inequality is a very popular topic, that has had all sorts of proofs, generalization, and modifications. For example, the ratio of distances D_1/D_0 has been generalized to convex n-gons in the plane with the result that $D_1/D_0 \leq \cos(\pi/n)$; see Lenhard [Len61]. For an extension to n-gons in 3-space see Pech [Pec94]. Some results for higher-dimensional simplices can be found in [Spi71].

The result attributed to Shields and Kazarinoff should probably be $D_0 \geq \frac{2}{3}D_1$.

Many other related inequalities can be found in Mitrinović *et al.* [MPV89]

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K19 Angular Diameter and Width of Plane Convex Bodies

PROBLEM: Study the class of plane convex bodies C in terms of the θ-diameter $D_\theta(C)$ and the θ-width $W_\theta(C)$. In particular, determine the greatest lower bound of the quotient $D_{\pi/2}/D_0$.

When C is a plane convex body and $0 \leq \theta < \pi$, the θ-diameter $D_\theta(C)$ (resp. the θ-width $W_\theta(C)$) is defined as the length of the longest (resp. the shortest) chord of C which joins two rays of support enclosing an angle of θ. Thus D_0 and W_0 are the ordinary diameter and width, and C is of constant width if and only if $D_0(C) = W_0(C)$. It would be interesting to study the sets of constant θ-width – those for which $D_\theta = W_\theta$. In [1,3], J. W. Green has established some relationships among the functions $D_{\pi/2}$, $W_{\pi/2}$, D_0, and W_0, showing that

$$\frac{D_{\pi/2}}{D_0} > \frac{\sqrt{7}-1}{3}, \qquad \frac{D_{\pi/2}}{W_0} \geq \frac{1}{\sqrt{2}}, \qquad \frac{W_{\pi/2}}{D_0} \leq \frac{1}{\sqrt{2}}, \qquad \text{and } \frac{W_{\pi/2}}{W_0} < 1,$$

with equality in either case only for the circular disc. He shows further that each of the last three bounds is the best possible, but that the first is not. He conjectures in [3] that $D_{\pi/2}/D_0 \geq 1/2$, but has informed us recently that a counterexample has been found by R. H. Sorgenfrey. It remains, then, to determine the greatest lower bound of the quotient $D_{\pi/2}/D_0$.

Let us mention here another problem, related to those above only in that it too was proposed by Green [2]:

PROBLEM: Suppose F is a family of subsets of E^n which includes each intersection of a subfamily of its members, each translate of one of its members, and each point-reflection of one of its members. Then if F includes some nonempty bounded open set, must F include some nondegenerate convex sets?

References

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Comments by Grünbaum (K19)

The only paper I was able to find that appears to be (rather remotely) related to Problem K-19 is Santaló's [San55]. It is not completely clear what is established; in the review by H. Busemann (MR0082692) an equation is said to be an inequality.

On the other hand, the problem from Green [Gre52] mentioned at the end has had an affirmative solution in the 2-dimensional case, see Eggleston [Egg71]. The higher-dimensional case seems still to be open; I conjecture that the answer is negative for all dimensions ≥ 3.

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K21 A Diophantine Problem on Triangles

PROBLEM: Does there exist a triangle which has integral sides, medians, and area?

While this problem is phrased in geometric language, it belongs more properly to Number Theory. We include it as an especially attractive sample of a large class of Diophantine geometric problems. For a discussion of many such problems, see Chapters 4 and 5 of [1].

A triangle with rational sides and area has been called a Heron triangle, and a summary of their properties known in 1920 can be found in Chapter 5 of [1]. In 1905, H. Schubert claimed to prove that a Heron triangle can have at most one rational median, but his proof was incorrect. (The above problem asks whether all medians of Heron triangles can be rational.) H. G. Eggleston [2] also gave a faulty solution to the problem, but later proved [3] that no isosceles Heron triangle can have all rational medians. (This problem was suggested to us by N. D. Kazarinoff and Allen Shields.)

References

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Comments by Grünbaum (K21)

The problem is repeated in [KW91], as Problem 14.2. On page 205 of [KW91] is given (as an exercise) an example of a Heron triangle with sides 52, 102 and 146, which is said to have to medians of integer lengths. The example is credited to R. H. Buchholz and R. L. Rathbun in 1987.

The problem appears to be still unsolved, but an affirmative solution was found to the modified question in which only two of the medians are required to be integral; see the very interesting papers [BR97], [BR98] by Buchholz and Rathbun. It seems highly probable that the answer to the original problem is negative.

It may be noted that the problem of determining triangles with all sides and all medians of rational lengths goes back to Euler [Eul11]. For a detailed exposition, and new results, see [Buc02].

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K22 Polyhedral Sections of Convex Bodies

PROBLEM: For each pair of integers n and r with $2 \leq n \leq r$, determine the smallest possible dimension $A^f(n,r)$ (resp. $S^f(n,r)$) of a convex polyhedron P such that every n-dimensional convex polyhedron having $r+1$ maximal faces is affinely equivalent (resp. similar) to some proper section of P.

(By proper section of P we mean the intersection of P with a flat which intersects the relative interior of P.) C. Davis [1] has observed that if a convex polyhedron has $r+1$ maximal faces, then it is affinely equivalent to a proper section of an r-simplex. Consequently $A^f(n,r) \leq r$. And Klee [2] has proved that $\xi^{a,f}(n,r) \geq n(r+1)/(n+1)$, where $\xi^{a,f}(n,r)$ (resp. $\xi^{s,f}(n,r)$) is the smallest possible dimension of a convex body Q (not necessarily polyhedral) such that every n-dimensional convex polyhedron having $r+1$ maximal faces is affinely equivalent (resp. similar) to some proper section of Q. Thus

$$n(r+1)/(n+1) \leq \xi^{a,f}(n,r) = A^f(n,r) \leq r.$$

For $r < 2n+1$, this implies that $\xi^{a,f}(n,r) = A^f(n,r) = r$, but we do not know whether $A^f(2,5)$ is equal to r or to 5, or whether $A^f(2,5) = \xi^{a,f}(2,5)$.

From results of Naumann [4] and Klee [2] it follows that

$$n(r+2)/(n+1) \leq \xi^{s,f}(n,r) \leq S^f(n,r) \leq 2^n(n+1)(r+1)$$

but the upper bound especially may be subject to much improvement. Melzak shows [3] that $\xi^{s,f}(2,2) = 3$ and Klee [2] that $S^f(2,2) > 3$, but the exact values of $\xi^{s,f}(2,3)$ and $S^f(2,2)$ are unknown.

Rather than considering the functions A^f, S^f, $\xi^{a,f}$, and $\xi^{s,f}$ defined above, it is perhaps more natural to consider functions A^v, S^v, $\xi^{a,v}$, $\xi^{s,v}$, whose definitions are obtained from those above by replacing "$r+1$ maximal faces" with "$r+1$ vertices". For these new functions we have (from [2]) the same lower bounds as those above, but are currently able to obtain upper bounds only by observing that if a polyhedron has $r+1$ vertices then it certainly has fewer than 2^{r+1} maximal faces (or by other observations almost this crude). It would be of interest to improve these crude upper bounds. Of course we have $A^v(2,r) = A^f(2,r)$, etc., since a two-dimensional polyhedron has the same number of maximal faces as vertices. However, the values of $A^v(3,4)$ and $S^v(3,3)$ are unknown.

For further related theorems, problems, and references, see [2].

References

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Comments by Grünbaum (K22)

There seems to be a typo in the line above the paragraph that starts with "From results of Naumann ...". Instead of "to r or 5" should be "to r or 4", which are the bounds in the displayed formula.

Klee's statement that he proved $S^f(2,2) > 3$ in [Kle60] is unjustified; there is an error in the proof, as he pointed out in his review (MR 0170266) of Shephard's paper [She64]. Shephard proved that Melzak's conjecture in [Mel59], that $S^f(2,2) > 3$ is correct. In less formal expression, this means that no 3-dimensional bounded convex polyhedron admits as sections similar images of all triangles. As Shephard mentions at the end of his paper, any 3-sided infinite prism does have sections that are similar to arbitrary triangles.

Concerning the affine version of the problem, it was first proposed in 1935 by S. Mazur, as Problem 41 of the famous "Scottish Book" [Mau81]. Klee commented in his remarks following Problem 41 on page 112 in [Mau81]:

> By very simple reasoning, Grünbaum [Grü58] showed that no 3-dimensional Banach space is isometrically universal for all 2-dimensional Banach spaces. Bessaga [Bes58], with more complicated reasoning, obtained the result with "3" replaced by "finite". Further refinements were contributed by Melzak [Mel58], Klee [Kle60], and Lindenstrauss [Lin66].

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K23 Asymptotes of Convex Sets

PROBLEM: For each n-dimensional closed convex subset C of E^n, let αC denote the set of all integers j between 1 and $n-1$ such that C admits a j-asymptote. Determine what subsets of $\{1,\ldots,n-1\}$ can be realized as the set αC for some C in E^n.

A j-asymptote of C is a j-dimensional flat F which lies in $E^n \setminus C$ but includes points arbitrarily close to C. It is clear that C admits a j-asymptote if and only if the orthogonal projection of C on some $(n-j)$-dimensional linear subspace fails to be closed. Asymptotes are studied in [2], where it is proved that if the closed convex set $C(\subset E^n)$ admits no boundary ray, then αC is empty or $\alpha C = \{1,\ldots,n-1\}$. A similar situation occurs for a closed convex cone K in E^n. Of course K admits no $(n-1)$-asymptote, but it is known for $2 \leq k \leq n-1$ that all of K's projections are closed (and, in fact, K is polyhedral) if and only if all its k-projections are closed [1]; consequently αK is empty or $\alpha K = \{1,\ldots,n-2\}$.

Now consider the following three closed convex subsets of E^3:

$$\begin{aligned}
U &= \{(x,y,z) : z \geq 0,\ z^2 \geq x^2 + y^2\} \\
V &= \{(x,y,z) : x \geq 0,\ xy \geq 1\} \\
W &= \{(x,y,z) : x \geq 0,\ xy \geq 1,\ z \geq (x+y)^2\}.
\end{aligned}$$

It is easy to verify that $\alpha U = \{1\}$, $\alpha V = \{1,2\}$, and $\alpha W = \{2\}$. Thus for $n \leq 3$ it is true that every subset of $\{1,\ldots,n-1\}$ can be realized as the set αC for appropriately constructed n-dimensional closed convex sets C. We do not know whether this is true for large values of n. In particular, can the sets $\{1\}$ and $\{1,3\}$ be realized as αC for $C \subset E^4$? (It is easy to verify that certain sets of consecutive integers can always be realized in E^n – the sets $\{j, j+1, \ldots, n-2\}$ for $1 \leq j \leq n-2$ and $\{j, j+1, \ldots, n-1\}$ for $1 \leq j \leq n-1$. For $n = 4$, these cover all the possibilities except the two mentioned.)

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Comments by Grünbaum (K23)

In a later paper [Kle67], Klee gave a complete answer to the problem: Every subset of $\{1, 2, \ldots, n-1\}$ is the set αC for some convex body C in Euclidean n-space. For certain extensions see, for example, [Goo86], [AC94], [ET08].

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