THE DIRICHLET BOUNDARY VALUE PROBLEM ON A FINITE CIRCULAR NETWORK USING EQUILIBRIUM MEASURE

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Abstract. If the equilibrium measure is obtained for a finite network we may resolve the solution to the Dirichlet problem as well as other important kernels. We seek the equilibrium measure to the Dirichlet boundary value problem on a finite circular network. A solution for the principal equilibrium on specific circular network is given and comments on the difficulty for generalizing the equilibrium measure.

Contents

1. Introduction

First we need some preliminaries, the formal problem statement as well as the description of the finite circular network. A more in depth coverage of the preliminaries and proofs for statements (where needed) can be found at [1] and [2].

The main focus of this paper will be on resolving the equilibrium measure, because with the equilibrium measure we can solve the Dirichlet problem as well as resolve the Poisson kernel and the Neumann to Dirichlet map. The equilibrium measure is composed of two parts, what I call the principle and symmetric. It turns out that generalizing the principle equilibrium measure is straight forward, but the symmetric equilibrium measure is very complicated to write down (or so it seems). A portion of this paper will be dedicated to exposing what has been found as well as the difficulties that arise with attempting to generalize the symmetric equilibrium measure.

2. Preliminaries

We will only consider networks $\Gamma(V, E)$ that are simply connected and finite, with V as the set of verticies and E as the set of edges. The set $C(V)$ will be the vector space of real valued functions on V. The main difference operator we will be using is the Laplacian on Γ which is given by:

$$\mathcal{L}(u)(x) = \sum_{x \sim y} c(x, y)(u(x) - u(y)) \tag{1}$$

[1] A function $G : \bar{F} \times F \to \mathbb{R}$ is the Green's function if and only if $\forall y \in F$, $G_y : \bar{F} \times F \to \mathbb{R}$ is the unique solution of

$$\begin{cases} \mathcal{L}G_y = \epsilon_y, & \text{on } F \\ G_y = 0, & \text{on } \delta(F) \end{cases} \tag{2}$$

where $\epsilon_y(x)$ is the discrete Dirac delta function ($\epsilon_y(y) = 1$ and null otherwise). Here a distinction is made, but required, whenever the phrase "equilibrium measure" appears this is meant to include both the principle and symmetric. Otherwise principle or symmetric is implied.

The principle equilibrium measure is a function v^F such that:

$$\begin{cases} \mathcal{L}v^F = 1, & \text{on } F \\ v^F = 0, & \text{on } \delta(F) \end{cases} \tag{3}$$

The symmetric equilibrium measure is a function v_y^F such that:

$$\begin{cases} \mathcal{L}v_y^F = 1, & \text{on } F \setminus \{y\} \\ v_y^F = 0, & \text{on } \delta(F) \cup \{y\} \end{cases} \tag{4}$$

We state the fundamental relationship between the Green's function and the equilibrium measure:

$$G = \frac{v^F - v_y^F}{|v^F - v_y^F|} v^F(y) \tag{5}$$

where we define $|v^F - v_y^F| \equiv \sum_{z \in F} (v^F - V_v^F)(z)$.

There are some other assumptions we make, even though the Schroedinger operator is used in [2] we let the potential be null, and we assume unit conductances on the network.

3. Problem statement

The main objective is to solve the Dirichlet boundary value problem, in order to solve this problem we seek the Green's function. A solution to the Dirichlet problem is a function u that satisfies the following properties:

$$\begin{cases} \mathcal{L}(u)(x) = f, & \text{on } F \\ u = 0, & \text{on } \delta(F) \end{cases}$$

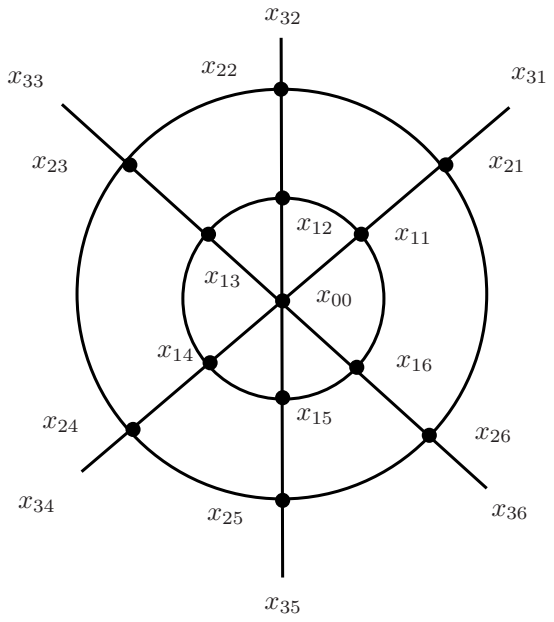
where $f \in C(F)$ and $g \in \delta(F)$. It is also stated in [2] that:

$$u(x) = \sum G(x, y) f(y)$$

Thus to find the function u we use the Green's function, but to find the Green's function we need the equilibrium measure, as seen in (5). Refer to [2] for information about the Poisson kernel. So now our main focus is resolving the equilibrium measure. We did not find the general expression for the symmetric equilibrium measure in general, but we will continue with the goal to obtain the equilibrium measure.

4. The Circular Planar Network m circles n rays

Figure 1. The network Γ



Here we give a description of the finite network we will be working on. Figure 1 above will be described. It has m circles and n rays, we will consider the case

in which $m = 2$ and $n = 3$. We define the set F and it's boundary as $F = \{x_{00}, x_{11}, x_{12}, \ldots, x_{26}\}$ and $\delta(F) = \{x_{31}, x_{32}, \ldots, x_{36}\}$, respectively. The notation x_{ij} is read "The vertex j at circle i", as a convention x_{00} is the vertex located at the center of the circle.

5. Resolving principle equilibrium measure

In this section we will obtain the principle equilibrium measure using two different techniques. The first technique will be the process of expanding the Laplacian at every vertex in F, getting a system of equations and then solving that system. And the second technique is from Encinas, Carmona, and Bendito [3], using information about the symmetry of the graph. With the second technique we still end up with a system of equations but the number of equations is reduced considerably.

5.1. **Technique 1.** Referring to equation (3), we want a function v^F that satisfies those two properties. In order to find this function we expand the Laplacian about every vertex in F and use the information on the boundary where necessary. We start at the center vertex x_{00}. Let $v^F = v$. Then we have by (1):

$$\begin{aligned}\mathcal{L}(v)(x_{00}) &= (v_{00} - v_{11}) + (v_{00} - v_{12}) + \ldots + (v_{00} - v_{16}) \\ &= 6v_{00} - (v_{11} + \ldots + v_{16}) \\ &= 1\end{aligned} \tag{1a}$$

This yields one equation in seven unknowns. We need more equations to find the principle equilibrium measure. The other equations will be given by expanding the Laplacian about the rest of the verticies in F. Expanding the Laplacian about x_{11} we have:

$$\begin{aligned}\mathcal{L}(v)(x_{11}) &= (v_{11} - v_{00}) + (v_{11} - v_{12}) + (v_{11} - v_{16}) + (v_{11} - v_{21}) \\ &= 4v_{11} - v_{00} - v_{12} - v_{16} - v_{21} = 1\end{aligned} \tag{1b}$$

Expanding the Laplacian about nodes $\{x_{12}, \ldots, x_{16}\}$ we have:

$$\mathcal{L}(v)(x_{12}) = 4v_{12} - v_{00} - v_{11} - v_{13} - v_{22} = 1 \tag{1c}$$

$$\mathcal{L}(v)(x_{13}) = 4v_{13} - v_{00} - v_{12} - v_{14} - v_{23} = 1 \tag{1d}$$

$$\mathcal{L}(v)(x_{14}) = 4v_{14} - v_{00} - v_{13} - v_{15} - v_{24} = 1 \tag{1e}$$

$$\mathcal{L}(v)(x_{15}) = 4v_{15} - v_{00} - v_{14} - v_{16} - v_{25} = 1 \tag{1f}$$

$$\mathcal{L}(v)(x_{16}) = 4v_{16} - v_{00} - v_{11} - v_{15} - v_{26} = 1 \tag{1g}$$

Then finally the Laplacian about nodes $\{x_{21}, \ldots, x_{26}\}$ we have:

$$\mathcal{L}(v)(x_{21}) = 4v_{21} - v_{11} - v_{22} - v_{26} - v_{31} = 1 \tag{1h}$$

$$\mathcal{L}(v)(x_{22}) = 4v_{22} - v_{12} - v_{21} - v_{23} - v_{32} = 1 \tag{1i}$$

$$\mathcal{L}(v)(x_{23}) = 4v_{23} - v_{13} - v_{22} - v_{24} - v_{33} = 1 \tag{1j}$$

$$\mathcal{L}(v)(x_{24}) = 4v_{24} - v_{14} - v_{23} - v_{25} - v_{34} = 1 \tag{1k}$$

$$\mathcal{L}(v)(x_{25}) = 4v_{25} - v_{15} - v_{24} - v_{26} - v_{35} = 1 \tag{1l}$$

$$\mathcal{L}(v)(x_{26}) = 4v_{26} - v_{16} - v_{21} - v_{25} - v_{36} = 1 \tag{1m}$$

Notice that for equations $(1h)$ through $(1m)$ we can apply the boundary data to eliminate six variables $(v_{31}, \ldots, v_{36})$. We now put our system into matrix-vector notation $\mathbf{A}v = \mathbf{1}$:

$$\begin{bmatrix} 6 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} v_{00} \\ v_{11} \\ v_{12} \\ \vdots \\ v_{24} \\ v_{25} \\ v_{26} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Looking at the matrix $\mathbf{A}$ we see it has a lot of structure, the entries of the main diagonal are the the degree of the vertex corresponding to the row of the entry (row 1 corresponds to x_{00}, row 2 corresponds to x_{11} and so on). The matrix $\mathbf{A}$ has weak diagonal dominance (which can be easily verified). Also $\mathbf{A}$ is symmetric (self-adjoint but all entries of $\mathbf{A}$ are real),is invertible, and is positive definite. In section 6 we give MATLAB code to verify invertibility, positive definiteness, as well as symmetry. Below we give the solution to $v = \mathbf{A}^{-1}\mathbf{1}$.

$$v = \begin{bmatrix} 3.5000 \\ 3.3333 \\ 3.3333 \\ 3.3333 \\ 3.3333 \\ 3.3333 \\ 3.3333 \\ 2.1667 \\ 2.1667 \\ 2.1667 \\ 2.1667 \\ 2.1667 \\ 2.1667 \end{bmatrix}$$

The components of the solution vector v represent the coefficients of the equilibrium measure.

Theorem 1: *The principle equilibrium measure v^F on the network Γ is given by the following expression*:

$$V^F = \frac{7}{2}\epsilon_{00}(x) + \frac{10}{3}\Big(\epsilon_{11}(x) + \cdots + \epsilon_{16}(x)\Big) + \frac{13}{6}\Big(\epsilon_{21}(x) + \cdots + \epsilon_{26}(x)\Big)$$

Proof: *We need to verify the principle equilibrium measure satisfies the following two properties: $\mathcal{L}(V^F) = 1$ on F and $v^F = 0$ on $\delta(F)$. First we check property one:*

Notice if $x = x_{00} \in F$ then $V^F(x_{00}) = 7/2$. Referring to equation (1a) we have $\mathcal{L}(v^F)(x_{00}) = 6v_{00} - (v_{11} + \ldots + v_{16}) = 6 * 7/2 - 6 * 10/3 = 1$. Doing the same procedure for the remaining vertices in F we see that property one is satisfied.

Property two is straight forward, if $x \in \delta(F)$ we have $v^F = 0$, since none of the discrete Dirac Delta functions are centered about any boundary nodes. □

5.2. **Technique 2.** This technique exploits the symmetry of the network to reduce the number of unknowns, as well as the number of equations. The way that technique 2 accomplishes this is by realizing that the principle equilibrium measure must have equal value on every vertex that lies on the same circle. For instance $v_{11} = v_{12} = \ldots = v_{16}$, this is confirmed by looking at the solution vector v in subsection 5.1.

Let $\alpha_1 = v_{00}$, $\alpha_2 = v_{11} = v_{12} = \ldots = v_{16}$, and $\alpha_3 = v_{21} = v_{22} = \ldots = v_{26}$. Rewriting equations (1a) through (1m) with this insight we obtain the following:

$$\begin{aligned} 6\alpha_1 - 6\alpha_2 &= 1 \\ 2\alpha_2 - \alpha_1 - \alpha_3 &= 1 \\ 2\alpha_3 - \alpha_2 &= 1 \end{aligned}$$

This system has three equations and three unknowns which is a drastic improvement from technique 1 which has 13 equations and 13 unknowns. One might also notice that this is a telescoping series, which can help make the system of equations more compact. We may generalize the coefficients for the principal equilibrium measure very easily.

For the network Γ with m circles and n rays the coefficients of the principle equilibrium measure may be written as:

$$\begin{aligned} 2n(\alpha_1 - \alpha_2) &= 1 \\ 2\alpha_2 - \alpha_1 - \alpha_3 &= 1 \\ &\vdots \\ 2\alpha_{m-1} - \alpha_{m-2} - \alpha_m &= 1 \\ 2\alpha_m - \alpha_{m-1} &= 1 \end{aligned}$$

Theorem 2: *The principle equilibrium measure v^F on the network Γ with m circles and n rays is given by the following expression*:

$$v^F = \alpha_1\epsilon_{00}(x) + \alpha_2\Big(\epsilon_{11}(x) + \cdots + \epsilon_{1,2n}(x)\Big) + \cdots + \alpha_m\Big(\epsilon_{m1}(x) + \cdots + \epsilon_{m,2n}(x)\Big)$$

Proof: *The same argument is used as in Theorem 1.* □

6. The coefficients of the principal equilibrium measure in general

In section 5.2 we did not solve for the coefficients of the principal equilibrium measure, in this section we give analytic expressions for the coefficients. This derivation is from Zhen Wei [4].

$$\begin{aligned} n(\alpha_1 - \alpha_2) &= 1 \\ &\vdots \\ -(\alpha_1 - \alpha_2) + (\alpha_2 - \alpha_3) &= 1 \\ -(\alpha_{m-2} - \alpha_{m-1}) + (\alpha_{m-1} - \alpha_{m-3}) &= 1 \end{aligned}$$

so let

$$\begin{aligned} \Phi_1 &= \alpha_1 - \alpha_2 = 1/n \quad \text{then} \quad \Phi_1 = 1/n \\ \Phi_2 &= 1 + \Phi_1 = 1 + 1/n \\ \Phi_3 &= 1 + \Phi_2 = 1 + 1 + 1/n \\ &\vdots \\ \Phi_{m-1} &= 1 + \Phi_{m-2} \\ &= 1 + 1 + \Phi_{m-3} = 1 \times (m-2) + \Phi_1 \\ &= m - 2 + 1/n \end{aligned}$$

so

$$\begin{aligned} \alpha_m &= (m-2) \times 1 + 1/n + 1 = (m-1) + 1/n \\ \alpha_{m-1} &= (m-2) \times 1 + 1/n + \alpha_m = (2m-3) + 2/n \\ \alpha_{m-2} &= (m-3) \times 1 + 1/n + \alpha_{m-1} = (3m-6) + 3/n \\ \alpha_{m-4} &= (m-4) \times 1 + 1/n + \alpha_{m-2} = (4m-10) + 4/n \\ &\vdots \\ \alpha_1 &= [m - (m-1+1)] \times 1 + 1/n + \alpha_2 \\ &= 1/n + (m-1)/n + m \times (m-1)/2 \\ &= m/n + (m^2 + m)/2 \end{aligned}$$

□

7. The symmetric equilibrium measure

The symmetric equilibrium measure is where we experienced difficulty with attempting to generalize. We have good faith that a generalization does exist, but will not be very elegant. In this section we will give exposition about the case when $m = 2$ and $n = 3$. In this special case we have 13 different symmetric equilibriums to consider. To think about this crudely, this implies that we have 13 different sets of 12 by 12 matrices to invert (assuming they are invertible). If we want to generalize this, then in the general case we would have to invert $2mn+1$ different sets of $2mn$ by $2mn$ matrices (assuming they are invertible). This does seem do-able but might rely on computational techniques to solve, and at the end of the day the final

expression for the symmetric equilibrium measure will be rather complicated. The expression for the coefficients of the symmetric equilibrium measure will depend on what node you have chosen for y.

7.1. **Resolving the symmetric equilibrium measure.** As in section 5.1 we let $m = 2$ and $n = 3$. For our example we will let $y = x_{00}$. Essentially we are removing x_{00} from our interior data and making it part of our boundary data. We expand the Laplacian at all the interior nodes $(F \setminus \{x_{00}\})$:

$$\begin{aligned}
3v_{11} - v_{12} - v_{16} - v_{21} &= 1 \\
3v_{12} - v_{11} - v_{13} - v_{22} &= 1 \\
3v_{13} - v_{12} - v_{14} - v_{23} &= 1 \\
3v_{14} - v_{13} - v_{15} - v_{24} &= 1 \\
3v_{15} - v_{14} - v_{16} - v_{25} &= 1 \\
3v_{16} - v_{15} - v_{11} - v_{26} &= 1 \\
4v_{21} - v_{11} - v_{22} - v_{26} &= 1 \\
4v_{22} - v_{12} - v_{21} - v_{23} &= 1 \\
4v_{23} - v_{13} - v_{22} - v_{24} &= 1 \\
4v_{24} - v_{14} - v_{23} - v_{25} &= 1 \\
4v_{25} - v_{15} - v_{24} - v_{26} &= 1 \\
4v_{26} - v_{16} - v_{25} - v_{21} &= 1
\end{aligned}$$

putting this in vector matric notation we have:

$$\begin{bmatrix}
3 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 4
\end{bmatrix}
\begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{24} \\ v_{25} \\ v_{26} \end{bmatrix}
=
\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

And the solution vector for this system is given by:

$$v = \begin{bmatrix} 3.0000 \\ 3.0000 \\ 3.0000 \\ 3.0000 \\ 3.0000 \\ 3.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \end{bmatrix}$$

Here we end up with the result that all the nodes that lie on the same circle end up having the same coefficients. This case is very different from the other cases that must be done.

Next we consider $y = x_{11}$, to see the contrast between the symmetry. As done earlier in this section, we expand the Laplacian at all the interior nodes $(F \setminus \{x_{11}\})$:

$$\begin{aligned}
5v_{00} - v_{12} - v_{13} - -v_{14} - v_{15} - v_{16} &= 1 \\
3v_{12} - v_{00} - v_{13} - v_{22} &= 1 \\
4v_{13} - v_{00} - v_{12} - v_{14} - v_{23} &= 1 \\
4v_{14} - v_{00} - v_{13} - v_{15} - v_{24} &= 1 \\
4v_{15} - v_{00} - v_{14} - v_{16} - v_{25} &= 1 \\
3v_{16} - v_{00} - v_{15} - v_{26} &= 1 \\
3v_{21} - v_{22} - v_{26} &= 1 \\
4v_{22} - v_{12} - v_{21} - v_{23} &= 1 \\
4v_{23} - v_{13} - v_{22} - v_{24} &= 1 \\
4v_{24} - v_{14} - v_{23} - v_{25} &= 1 \\
4v_{25} - v_{15} - v_{24} - v_{26} &= 1 \\
4v_{26} - v_{16} - v_{25} - v_{21} &= 1
\end{aligned}$$

In matrix vector notation we have:

$$\begin{bmatrix} 5 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} v_{00} \\ v_{12} \\ \vdots \\ v_{24} \\ v_{25} \\ v_{26} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and}$$

the solution vector is given by:

$$v = \begin{bmatrix} 3.4661 \\ 3.2484 \\ 3.2747 \\ 3.2841 \\ 3.2747 \\ 3.2484 \\ 1.6697 \\ 2.0045 \\ 2.1001 \\ 2.1211 \\ 2.1001 \\ 2.0045 \end{bmatrix}$$

We note here that $v_{12} = v_{16}$, $v_{13} = v_{15}$, $v_{22} = v_{26}$, and $v_{23} = v_{25}$. All the other values are unique. A common theme appears to be present in all of the symmetric equilibriums - there is some sort of symmetry among the coefficients (thus the name "symmetric equilibrium"). The issue becomes now that we are required to repeat this entire process for the remaining vertices in $F \setminus \{x_{00}, x_{11}\}$. To help reduce the work done it seems that there are three cases you can exploit.

• First case is when $y = x_{00}$, this case is unique from all the other cases that will be considered because it reduces the degree of all the nodes on the first circle by one. This was the first example we considered, and we saw that the symmetry was that all the nodes that lie on the same circle end up have the same coefficients.

• Second case is if we let y be a node on the first circle, this is the second example we considered, and we saw that there was a line that had unique equilibrium coefficients along it at every node (this line always contains the node that was removed, $y = x_{11}$). This line divide the graph into two parts and nodes on corresponding sides of the graph were the same ($v_{12} = v_{16}$, $v_{13} = v_{15}$, $v_{22} = v_{26}$, and $v_{23} = v_{25}$).

• The last case is in which we let y be a node on the mth circle. In this situation we should expect another type of symmetry that is different from the first two cases.

If we let y be a node on a circle between one and m, they should also have similar

structure, but when we change nodes, we change entries around in the matrix, and this pattern seems hard to track. In summary of this section, an expression for the symmetric equilibrium should exist, it just will have a very complicated form (or so it seems).

8. Computational verification section 5.1

Another useful idea not pursued is the use of block structure. Many of the matrices associated with this problem seem to exhibit very nice form.

```
%%%%====%%%%==== For the system presented in section 5.1 ====%%%%====%%%%
%%Constructing main diagonal
for i=2:13
    D(i)= 4;
end

D(1)=6;

mainDiagonal = diag(D);
%%
%=%Constructing off diagonals
for i=1:12
    offdiagonal(i)=-1;
end

Oh= zeros(13,13);
Lowerd = diag(offdiagonal,-1);
Upperd = diag(offdiagonal,1);

DiagonalMatrix = Oh + mainDiagonal+Upperd+Lowerd;
%=%
for i=3:7
    DiagonalMatrix(i,1)=-1;
end

for i=3:7
    DiagonalMatrix(1,i)=-1;
end

DiagonalMatrix(7,2)=-1;

for i=9:13
    DiagonalMatrix(i,(i-10)+3)=-1;
end

DiagonalMatrix(13,8)=-1;

DiagonalMatrix(2,7)=-1;

for i=2:6
```

```
    DiagonalMatrix(i,(i-2)+9)=-1;
end

DiagonalMatrix(8,13)=-1;

A=DiagonalMatrix

b=ones(13,1);
%Crude but the backslash method works
x=A\b

%Notice that all the det's are positive
for i=1:10
     det( A(1:i, 1:i) )
   end
%Check that A'-A=0
A'-A

%END OF CODE

%%%%====%%%%==== For the system presented in section 5.1 ====%%%%====%%%%
```

9. Computational verification section 7.1

```
%%%%====%%%%==== For the system presented in section 7.1, y=x_{00} ====%%%%====%%%%

D=zeros(12,1);

for i=1:6
    D(i)= 3;
end
for i=7:12
    D(i)= 4;
end

mainDiagonal = diag(D);
for i=1:11
    offdiagonal(i)=-1;
end

Oh= zeros(12,12);
Lowerd = diag(offdiagonal,-1);
Upperd = diag(offdiagonal,1);

DiagonalMatrix = Oh + mainDiagonal+Upperd+Lowerd;


for i=7:12
    DiagonalMatrix(i,i-6)=-1
```

```
end

DiagonalMatrix(12,7)=-1

for i=1:6
    DiagonalMatrix(i,i+6)=-1
end

DiagonalMatrix(1,6)=-1
DiagonalMatrix(7,12)=-1
DiagonalMatrix(7,6)=0

A=DiagonalMatrix;

b=ones(12,1);
x=A\b

%END OF CODE

%%%%====%%%%==== For the system presented in section 7.1, y=x_{00} ====%%%%====%%%%



%%%%====%%%%==== For the system presented in section 7.1, y=x_{11} ====%%%%====%%%%
D=zeros(12,1);
D(1)=5;
D(2)=3;
for i=3:5
    D(i)= 4;
end
 D(6)= 3;
 D(7)= 4;
 D(8)= 3;
for i=9:12
    D(i)=4;
end
mainDiagonal = diag(D);
for i=1:11
    offdiagonal(i)=-1;
end




Oh= zeros(12,12);
Lowerd = diag(offdiagonal,-1);
Upperd = diag(offdiagonal,1);
```

```
DiagonalMatrix = Oh + mainDiagonal+Upperd+Lowerd;

for i=3:6
   DiagonalMatrix(i,1)=-1
end


for i=2:6
   DiagonalMatrix(1,i)=-1
end


%for i=9:12
 %   DiagonalMatrix(i,(i-9)+2)=-1
%end

DiagonalMatrix(12,7)=-1;
DiagonalMatrix(7,12)=-1;
DiagonalMatrix(2,8)=-1

for i=3:6
    DiagonalMatrix(i,(i-3)+9)=-1
end
DiagonalMatrix(7,7)=3
DiagonalMatrix(2,9)=0
DiagonalMatrix(7,6)=0
DiagonalMatrix(8,8)=4
DiagonalMatrix(8,8)=4
DiagonalMatrix(6,7)=0

for i=8:12
    DiagonalMatrix(i,(i-8)+2)=-1
end

A=DiagonalMatrix;

b=ones(12,1);
x=A\b

%END OF CODE

%%%%====%%%%==== For the system presented in section 7.1, y=x_{11} ====%%%%====%%%%
```

Appendix A. REFERENCES

References

[1] A. Carmona and A.M. Encinas: *An Introduction to discrete vector calculus on finite networks.* Grupo VARIDIS REU 2010, UW, (2010).
[2] A. Carmona and A.M. Encinas: *An Introduction to boundary value problems on finite networks.* Grupo VARIDIS REU 2010, UW, (2010).
[3] E. Bendito, A. Carmona, A. M. Encinas: *Solving boundary value problems on networks using equilibrium measures.* J. Funct. Anal. 171 (2000), 166-175.
[4] Ginny Zhen Wei, Virginia, Oxford, zw4mu@virginia.edu.