# Eigenvalues of the Response Matrix and the Neumann-to-Dirichlet Map

Jerry Li

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**Abstract**

Here we explore various spectral properties of the Dirichlet-to-Neumann map resulting from a Kirchhoff matrix. Using previous bounds of the eigenvalues of the Schur complement of a matrix in terms of those of the matrix, we find some nice bounds for the eigenvalues of the response matrix in terms of those of the Kirchhoff matrix. We also find an interiorization property of the Neumann-to-Dirichlet map to derive a matrix given the eigenvalues and eigenvectors of the Kirchhoff matrix whose eigenvalues contain the reciprocals of those of the response matrix. Finally, we give directions for future research.

# 1 Introduction

Here we provide a brief overview of notation and past work. In §2 we describe previous work which gives nice interlacing properties of the eigenvalues of the response and the Kirchhoff matrix and its submatrices. In §3 we describe the Neumann-to-Dirichlet map and a transformation between Neumann-to-Dirichlet maps with interiorized vertices. In §4 we use the results from the previous sections to obtain bounds and expressions for the trace of the Neumann-to-Dirichlet map in terms of thoe for the Kirchhoff matrix and various submatrices. In §5 we use the transformation derived in §3 to get an "expression" for the eigenvalues of the response matrix. In §6 we use a Green's function derived by Ian Zemke to get an alternate expression for the eigenvectors and eigenvalues of the response matrix, and in §7 we describe further directions for research.

*Notation*

This paper follows the notation in [2], except we will always work on connected graphs. We let $M_n$ denote the space of $n \times n$ matrices, and $\{e_j\}_{j=1}^n$ the standard basis vectors for $\mathbb{R}^n$. Let $G = (V, \partial V, E)$ be a graph, where $V$ is a set of vertices, $E \subset V \times V$ is the set of edges, and $\partial V \subset V$ is a set of boundary vertices. We write $\text{int}V = V - \partial V$. If $a, b \in V$ we write $ab \in E$ to be the edge (if it exists) between $a$ and $b$. Let $\gamma : E \to \mathbb{R}_{>0}$ be a conductivity function. We will sometimes consider $\gamma$ to be function on $V \times V$ with $\gamma(ab) = 0$ if $ab \notin E$. Then we call $\Gamma = (G, \gamma)$ a network. We will write $K \in M_{|V|}$ to be the standard Kirchhoff matrix for the network $\Gamma$ where we order the entries so that the boundary vertices come before in the column order, so that

$$K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \tag{1}$$

where $A$ is a $|\partial V| \times |\partial V|$ sized submatrix representing the boundary to boundary conductivities, $B$ is a $|\partial V| \times |\text{int}V|$ sized matrix denoting the boundary to interior edges, and $C$ is $|\text{int}V| \times |\text{int}V|$ sized and denotes the interior to interior conductivities. We will let $\Lambda$ be the response matrix for this graph. [2] shows that

$$\Lambda = A - BC^{-1}B^T$$

where this is the block decomposition as above and $C^{-1}$ exists because [2] shows that it is positive definite. We note that both $K$ and $\Lambda$ and later the Neumann-to-Dirichlet map implies an ordering of the boundary vertices, and we will refer to this order as the order induced by the corresponding matrix.

In working with eigenvectors, it is often useful to use two somewhat contradictory notations for these said eigenvectors. If $\phi$ is an eigenvector for $K$ (resp. $\Lambda$) we will often write $\phi_i$ to be the $i$th entry in this eigenvector (or if we are talking about the $j$th entry in the $i$th eigenvector we will write the eigenvector to be $\phi_i$ and its entry $\phi_{ij}$. The meaning of the statement should be clear from context) but also we will sometimes consider $\phi$ as a function from $V$ (resp. $\partial V$) to the real line, and so if $x \in V$ (resp. $x \in \partial V$) we will often write $\phi(x)$ to denote the value of the eigenvector at vertex $x$. Finally, as the all ones vector comes up quite often, we will write $\mathbf{1}$ to denote it, with $\mathbf{1}_n$ meaning the all ones vector of length $n$, and $\mathbf{1}_{m,n}$ meaning the $m \times n$ matrix of all ones. We will also let $\mathbf{0}$ denote the vector of all zeroes, and $\mathbf{0}_n$ and $\mathbf{0}_{m,n}$ in analogy to $\mathbf{1}_n$ and $\mathbf{1}_{m,n}$.

*Elementary Results*

We first briefly give a fairly basic interpretation of the eigenvectors of $\Lambda$. We know that $\Lambda$ is defined to be the linear map from boundary voltages to boundary currents, so that if $\phi$ is a boundary voltage, then if $v = [\phi, \psi]^T$ where $|\psi| = |\text{int}V|$ is the $\gamma$-harmonic voltage function on the interior of the network it induces, then $\Lambda\phi$ is the boundary current made by this $\gamma$-harmonic function. Hence an eigenvector of $\Lambda$ is a set of boundary voltages which produce boundary currents proportional to the original voltages, and the proportion is the eigenvalue. Of course, this is a somewhat dissatisfying interpretation and more physically meaningful one would definitely be useful.

The following elementary result is very useful:

**Proposition 1.1.** *For any response matrix (or Kirchhoff matrix interpreted as a response matrix) $\Lambda$, we have that $\lambda = 0$ is an eigenvalue of $\Lambda$ with multiplicity zero, and the eigenvectors of $0$ are the vectors in the span of $\mathbf{1}$.*

*Proof.* [2] shows that the kernel of $\Lambda$ is just the span of $\mathbf{1}$, and so $\mathbf{1}$ is an eigenvector of $\Lambda$ with eigenvalue $\lambda = 0$; furthermore as the kernel of $\Lambda$ is simply the span of $\mathbf{1}$, it is the only eigenvector with eigenvalue $\lambda = 0$. □

**Corollary 1.2.** *If $H \in M_n$ is a response matrix (or a Kirchhoff matrix interpreted as a response matrix), then if $\phi$ is a non-constant eigenvector of $H$, then $\phi$ has row sum zero.*

*Proof.* $\phi$ is non-constant and hence by Proposition 1.1 does not share an eigenvalue with $\mathbf{1}$; hence they are orthogonal, and therefore $\phi$ has row sum zero. □

We note that in general eigenvectors of the Kirchhoff matrix are not $\gamma$-harmonic as defined in [2]; see [5] for some motivation for why this shouldn't keep anyone up at night. In general, when talking about the eigenvalues of $K$ or $\Lambda$, we will let $\lambda_1 = 0$ correspond to $\phi_1 = \mathbf{1}$. Note also that because Kirchhoff and response matrices (and later on Neumann-to-Dirichlet maps) are normal matrices, all their eigenvalues are non-negative.

*A Calculation*

Here we provide a simple way of representing the values of a Kirchhoff or response matrix in terms of its eigenvectors and eigenvalues. Although the proof of this is very straightforward, it leads to some useful results. Let $H \in M_n$ be symmetric real-valued (in general this can be Hermitian). By the spectral theorem, we know that $H = UDU^T$ where $U$ is an unitary matrix where the columns are the orthonormal eigenvectors of $H$ and $D$ is a diagonal matrix where the $i$th diagonal entry is the eigenvalue corresponding to the eigenvector in the $i$th column of $U$. Let $\{\phi_i\}_{i=1}^n$ be the eigenvectors with $\{\lambda_i\}_{i=1}^n$ being their corresponding eigenvalues. It is then not difficult to see that by explicitly doing the matrix calculation that

$$H_{ij} = (UDU^T)_{ij} = \sum_{k=1}^{n} \lambda_k \phi_{ki} \phi_{kj},$$

or in our function notation,

$$H(x, y) = \sum_{k=1}^{n} \lambda_k \phi_k(x) \phi_k(y).$$

Hence, by Proposition 1.1, we have the following:

**Proposition 1.3.** *If $H \in M_n$ is a Kirchhoff or response matrix with eigenvectors $\mathbf{1}, \phi_2, \dots, \phi_n$ and corresponding eigenvalues $0, \lambda_2, \dots, \lambda_n$, then*

$$H_{ij} = \sum_{k=2}^{n} \lambda_k \phi_{ki} \phi_{kj}$$

*or in our alternative notation,*

$$H(x, y) = \sum_{k=2}^{n} \lambda_k \phi_k(x) \phi_k(y).$$

This provides us with this intuition about how the values in the Kirchhoff matrix are affected by its eigenvalues. With notation as above, we can vary the eigenvalues while keeping the eigenvectors unchanged, and thus consider $H_{ij}$ as a function of the eigenvalues for each $i, j$.

## 2 Prior Work

Work has been done when considering general Schur complements of Hermitian matrices. If $H \in M_n$ is a Hermitian matrix, let $\lambda_i^{\downarrow}(H)$ be the $i$th largest eigenvalue of $H$, and $\lambda_i^{\uparrow}(H)$ the $i$th smallest eigenvalue of $H$.

**Theorem 2.1.** *Let $H$ be a Hermitian positive semi-definite matrix, written as*

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}$$

*where $H_{11} \in M_r$ is non-singular. Then*

$$\lambda_i^{\downarrow}(H) \geq \lambda_i^{\downarrow}(H/H_{11}) \geq \lambda_{i+k}^{\downarrow}(H). \tag{2}$$

By a little bit of rewriting, and the fact that $C$ is positive definite ([2]), we get this following result:

**Corollary 2.2.** *If $K \in M_n$ is a Kirchhoff matrix and $\Lambda \in M_{n-k}$ its response matrix with $k$ interior nodes, then for $i \leq n-k$*

$$\lambda_i^{\downarrow}(K) \geq \lambda_i^{\downarrow}(\Lambda) \geq \lambda_{i+k}^{\downarrow}(K).$$

This is a result similar to Cauchy's classical result about the eigenvalues of submatrices of Hermitian matrices, and is somewhat striking. This is made more so by the following theorem.

**Definition 2.1.** Let $K \in M_n$ and $A \in M_r$ where $n \geq r$. We say $A$ is *embeddable* in $K$ if there exists a unitary matrix $U$ so that $A$ is the $r \times r$ submatrix at the top left of $UKU^*$.

**Theorem 2.3.** Fan-Pall, [4] *Let $K \in M_n$ and $A \in M_r$ where $n \geq r$. Then $A$ is embeddable in $K$ if and only if*

$$\lambda_i^{\downarrow}(K) \geq \lambda_i^{\downarrow}(A) \geq \lambda_{i+n-r}^{\downarrow}(K)$$

*for all $1 \leq i \leq r$.*

Corollary 2.1 and Theorem 2.3 imply that the Schur complement of a matrix is embeddable in the original matrix; however, even in the special case of Kirchhoff matrices and response matrices the unitary matrix involved seems to sometimes be very complicated.

We can also use a special case of Weyl's inequalities to get a somewhat sharper bound:

**Theorem 2.4.** Weyl, [3] *Let $A, B \in M_n$ be Hermitian. For each $1 \leq k \leq n$ we have*

$$\lambda_k^{\uparrow}(A) + \lambda_1^{\uparrow}(B) \leq \lambda_k^{\uparrow}(A+B) \leq \lambda_k^{\uparrow}(A) + \lambda_n^{\uparrow}(B).$$

**Corollary 2.5.** *If $K \in M_n$ is a Kirchhoff matrix block decomposed as in Equation 1, and $\Lambda \in M_{n-k}$ is the response matrix resulting from taking the Schur complement of $C \in M_k$, then for all $1 \leq j \leq n-k$ $\lambda_j^{\uparrow}(\Lambda) \leq \lambda_j^{\uparrow}(A)$.*

*Proof.* By Theorem 2.4 $\lambda_k^{\uparrow}(\Lambda) \leq \lambda_k^{\uparrow}(A) + \lambda_n^{\uparrow}(-BC^{-1}B^T)$; however note that for all $x$ we have that

$$x^T(BC^{-1}B^T)x = (B^Tx)^TC^{-1}(B^Tx) \geq 0$$

as by [2] $C^{-1}$ is positive definite; therefore $BC^{-1}B^T$ is positive semi-definite, and so all of its eigenvalues are non-negative; hence $-BC^{-1}B^T$ is negative semi-definite, and so all of its eigenvalues are non-positive, from which we get the desired bound. □

Corollary 2.5 provides a generalization of Theorem 2.10 in [6]; it would also be interesting to attempt to find an interlacing between the eigenvalues for the Neumann-to-Dirichlet map and those of $A$ in the block decomposition of $K$; however the author was unable to do so.

## 3 The Neumann-to-Dirichlet Map

Let $\Lambda \in M_n$ be a response matrix (or a Kirchhoff matrix interpreted as a response matrix) for some connected network $\Gamma$. This takes boundary voltages to boundary currents; hence it is natural to consider the map which takes boundary currents to boundary voltages, which we call a Neumann-to-Dirichlet map. This map is not unique (see the below discussion about the Neumann problem); however, there is only one map which sends boundary currents to boundary voltages with row-sum zero. [1] found an explicit form for this map; on connected graphs this Neumann-to-Dirichlet map (which from this point on we will call the Neumann-to-Dirichlet map) $H$ (eta) can be written as

$$H = (\Lambda^2 + \mathbf{1}_n)^{-1}\Lambda, \tag{3}$$

where [1] shows that the expression makes sense as $\Lambda^2+\mathbf{1}_n$ is invertible. Whenever we have a Kirchhoff matrix $\Lambda$ (resp. response matrix), we will always call the matrix given by Equation 3 its Neumann-to-Dirichlet map or its associated Neumann-to-Dirichlet map. [1] also notes that

$$H = (\Lambda^2 + \alpha\mathbf{1}_n)^{-1}\Lambda$$

for all $\alpha > 0$ scalars, hence we do not have to worry about whatever constant they may have had. [1] also shows that $H$ is a symmetric, semi-definite matrix with row-sum zero and positive diagonal entry. It is, however, not in general a Kirchhoff matrix (i.e. some of the non-diagonal entries may be positive). We explore several basic properties of $H$ below.

First we consider its obvious connection to the Neumann problem. Given a network $\Gamma = (G, \gamma)$ where $G = (V, \partial V, E)$, given a vector of currents $\xi$ where $|\xi| = |\partial V|$, the Neumann problem consists of finding a voltage vector $\phi = [v^T, \delta^T]^T$ on the entire graph (i.e. $|\phi| = |V|$) so that $\phi$ is $\gamma$-harmonic and $K\phi|_{\partial V} = \xi$. [1] shows that this solution is unique up to summing up to a constant, and so if we force $\phi$ to have boundary sum zero, then this solution is unique, and [1] shows that $H\xi = v$, where $H$ is the Neumann-to-Dirichlet map associated with the response matrix.

The most important spectral property of the Neumann-to-Dirichlet map for our purposes is the following:

**Proposition 3.1.** *Let $\Lambda \in M_n$ be a response matrix and $H$ be as in Equation 3. A vector $\phi$ is an eigenvector of $\Lambda$ if and only if it is an eigenvector for $H$, and if $\lambda$ is its eigenvalue for $\Lambda$, then if $\lambda = 0$, its eigenvalue for $H$ is also 0, but if $\lambda \neq 0$, then $1/\lambda$ is its eigenvalue for $H$, and vice-versa.*

*Proof.* Suppose $\phi$ is an eigenvector of $\Lambda$ with eigenvalue 0. Then obviously by construction $H\phi = 0$, so $\phi$ is also an eigenvector of $H$ with eigenvalue 0. Suppose then that its eigenvalue is $\lambda \neq 0$. Then we note that as by Corollary 1.2

$$(\Lambda^2 + \mathbf{1}_n)\phi = \Lambda^2\phi = \lambda^2\phi,$$

we get that

$$\frac{1}{\lambda^2}\phi = (\Lambda^2 + \mathbf{1}_n)^{-1}\phi.$$

Hence we get that as $\Lambda\phi = \lambda\phi$,

$$(\Lambda^2 + \mathbf{1}_n)^{-1}\Lambda\phi = \frac{1}{\lambda}\phi,$$

so $\phi$ is an eigenvector of $H$ with eigenvalue $1/\lambda$, as claimed. To show the other direction it suffices to note that by the above we can find $n$ linearly independent eigenvectors with the eigenvalues as in the claim; hence these are the only eigenvalues for $H$; hence we are done. $\square$

This makes sense; if we put some voltages which happen to generate a current which is proportional to the voltages, then as the voltages must already have row-sum zero, if we put the current into the Neumann-to-Dirichlet map, we get the original voltages but with the reciprocal of the proportion. Incidentally, this results in this curious, but for our purposes useless, property of the Neumann-to-Dirichlet map:

**Corollary 3.2.** *If $\Lambda$ is a response matrix, then for all $k \geq 2$,*

$$H = (\Lambda^k + \mathbf{1}_n)^{-1}\Lambda^{k-1}.$$

*Proof.* An identical argument to that used in the proof of Proposition 3.1 shows that the eigenvectors and eigenvalues of $H' = (\Lambda^n + \mathbf{1}_n)^{-1}\Lambda^{n-1}$ are identical to that of $H$; it is not difficult to see that $H'$ is still symmetrical and hence can be diagonalized and therefore because the eigenspaces are the same we get that $H' = H$. □

Note that this spectral condition guarantees that that $H$ is the Moore-Penrose generalized inverse of $\Lambda$ (see [3]). It also may be worthwhile to consider other powers of $H$ that result from taking different combinations of powers $\Lambda$ in the above expressions, but we do not in this paper. Instead, let us trudge onwards.

### *3.1 Interiorizing Vertices in the Neumann-to-Dirichlet Map*

If $G' = (V, \partial V, E)$ is a graph with boundary, and $v \in \partial V$, we let the interiorized graph be the graph $G = (V, \partial V - \{v\}, E)$; that is, we simply demote $v$ to be an interior node without changing any edges. If $\Gamma' = (G', \gamma)$ is an electrical network and if $v \in \partial V$ we let $\Gamma$ be the network with the same conductivities but with $G$ being the interiorized graph. This notation may seem backwards but it makes the notation in the proof in the following discussion more consistent. The notation simply means that objects on the non-interiorized graph are notated with a prime and those on the interiorized graph do not; hence $G'$ for instance denotes the original graph.

Let $n > 1$ be a fixed integer. Consider the following linear map:

$$P_n = \begin{pmatrix} 1 & 0 & \dots & 1/(n-1) \\ 0 & 1 & \dots & 1/(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 1/(n-1) \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

The following is a proof of this claim:

**Theorem 3.3.** *Let $P_n$ be as above, let $H' \in M_n$ be a Neumann-to-Dirichlet matrix for an electrical network $\Gamma' = (G', \gamma)$ where $G' = (V, \partial V, E)$ and $v_n$ be the nth boundary vertex in the order implied by $H'$. Then the $n-1 \times n-1$ sized submatrix involving the first $n-1$ rows and columns of the following matrix*

$$\tilde{H} = P_n H' P_n^T$$

*is the Neumann-to-Dirichlet matrix $H$ of the modified electrical network $\Gamma = (G, \gamma)$ where the $n$th boundary vertex, call it $v_n$, has been interiorized (i.e. $G = (V, \partial V - \{v_n\}, E)$).*

The intuition behind this is rather straightforward. It is not difficult to see that $H$ already sends currents over the first $n-1$ vertices (and zero on the $n$th vertex) to valid voltages which solve the Neumann-to-Dirichlet problem. The only issue is that the row sums of these resulting voltages are not correct and are off by some constant factor of the voltage of the last vertex; this is what the transformation $P_n H' P_n^T$ rectifies. It is quite possible to think of the $P_n$ as the matrix with the last row omitted, in which case it is not necessary to take submatrices of the product; however, for our purposes it is more convenient to think of the $P_n$s as square. However, before we embark fully on the proof of this theorem, we require a few lemmas about the behavior of $P_n$.

**Lemma 3.4.** $P_n^T[\mathbf{1}_{n-1}, 0]^T = \mathbf{1}$, *where $\mathbf{1}_{n-1}$ is the corresponding all ones vector of length $n-1$.*

*Proof.* This is obvious by inspection. □

**Lemma 3.5.** *For any $M \in M_n$, the nth row and column of $P_n M P_n^T$ are zeroes.*

*Proof.* This is again obvious by inspection. $\square$

**Lemma 3.6.** *If $v$ is a length $n$ vector with last entry zero and row sum zero, then $P_n^T v = v$.*

*Proof.* Write $v = [v_1, v_2, \ldots, v_{n-1}, 0]^T$. By assumption $\sum_{i=1}^{n} v_i = \sum_{i=1}^{n-1} v_i = 0$. But then

$$P_n^T v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n - 1 \\ \frac{1}{n-1}\sum_{i=1}^{n-1} v_i \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n - 1 \\ 0 \end{pmatrix} = v.$$

$\square$

**Lemma 3.7.** *Suppose $v = [v_1, v_2, \ldots, v_n]^T$ has row-sum zero. Then $v' = P_n v$ also has row-sum zero, the first $n-1$ entries of $v'$ differ by $v$ by the constant amount of $v_n/(n-1)$, and the nth entry of $v'$ is zero.*

*Proof.* We calculate.

$$v' = \begin{pmatrix} 1 & 0 & \ldots & 1/(n-1) \\ 0 & 1 & \ldots & 1/(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & 1/(n-1) \\ 0 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 + v_n/n-1 \\ v_2 + v_n/n-1 \\ \vdots \\ v_{n-1} + v_n/n-1 \\ 0 \end{pmatrix}$$

so

$$\sum_{i=1}^{n} v_i' = \sum_{i=1}^{n-1} v_i' = v_n + \sum_{i=1}^{n-1} v_i = 0$$

as $v$ had row-sum zero by assumption. $\square$

*Proof. of Theorem 3.3* We let the submatrix involving the first $n-1$ rows and columns of $P_n H' P_n^T$ be $A$. Then by Lemma 3.5 we can write

$$P_n H' P_n^T = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix}.$$

where $\mathbf{0}$ is the $n-1$ sized column vector of all zeroes. Consider first of all what $A$ does to the constant vector $\mathbf{1}$ of length $n-1$. Let $u$ be the column vector of length $n$ created by appending a 0 to $\mathbf{1}$. By Lemma 3.4, we have that $P_n^T u = \mathbf{1}_n$. However, $H'\mathbf{1} = 0$, so $P_n H' P_n^T u = 0$, which implies that $A\mathbf{1} = 0$, and as $H\mathbf{1} = 0$, these two expressions agree in this special case, and indeed, for any $v \in \text{span}\{\mathbf{1}\}$, $A\mathbf{1} = H\mathbf{1} = 0$.

Now let $\psi = [\psi_1, \psi_2, \ldots, \psi_{n-1}]^T$ be a vector of length $n-1$ with row sum zero. We wish to show that $H\psi = A\psi$. By the uniqueness of the Neumann problem, there is exactly one set of voltages $\phi = [v^T, \delta^T]$ on $G$ where $|v| = n-1$ so that $v$ has row sum zero and $K\phi|_{\partial V - v_n} = \psi$. By definition, $H\psi = v$. Furthermore, we know that $K\phi$ at $v_n$ is zero, because the current at every interior node of a $\gamma$-harmonic function is zero. Hence if we consider $\psi' = [\psi^T, 0]^T$, we know that by the uniqueness of the Neumann problem on $\Gamma'$ there exists exactly on set of voltages $\phi' = [v'^T, \delta'^T]$ on $G'$ where $|v'| = n$ so that $v'$ has boundary row sum zero and $Kv'|_{\partial V} = \psi'$, and moreover since $\phi$ also solves this Neumann problem, every entry in $\phi'$ and $\phi$ differ by some constant amount. Let $v' = [v_1', v_2', \dots, v_n']^T$. But then, consider this following calculation:

$$P_n H' P_n^T \psi' = P_n H' \left( \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 1/(n-1) & \dots & 1/(n-1) & 1/(n-1) & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{n-1} \\ 0 \end{pmatrix} \right)$$

$$= P_n H' \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{n-1} \\ \frac{1}{n-1}\sum_{i=1}^{n-1} \psi_i \end{pmatrix} = P_n H' \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{n-1} \\ 0 \end{pmatrix} = P_n v',$$

but by Lemma 3.7 if we let $v''$ be the first $n-1$ entries of $P_n v'$, these differ only by a constant from the first $n-1$ entries in $v'$, which differ only by a constant from those in $v$, but $v''$ has row-sum zero (as the last entry in $P_n v'$ is zero and $P_n v'$ has row-sum zero) and so does $v$, so $v'' = v$. Then we get that

$$P_n H' P_n^T \psi' = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix} \begin{pmatrix} \psi \\ 0 \end{pmatrix} = \begin{pmatrix} A\psi \\ 0 \end{pmatrix},$$

so $A\psi = v = H\psi$ for all $\psi$ with row-sum zero.

But now we are done, as it is well known that $\mathbb{R}^{n-1} = \operatorname{span}\{\mathbf{1}_{n-1}\} \oplus (\operatorname{span}\{\mathbf{1}_{n-1}\})^\perp$. Hence for all $v \in \mathbb{R}^{n-1}$ we can write $v = c\mathbf{1} + u$ where $u$ has row sum zero, and therefore $Av = Au = Hu = Hv$. $\square$

*Repeated Interiorization*

We may continue the process shown above as many times as we wish (well, less than $n$ times).

**Corollary 3.8.** *Let $H'' \in M_n$ be a Neumann-to-Dirichlet map for a network $\Gamma$. If $v_{n-1}, v_n \in \partial V$ are the last two boundary vertices in the ordering implied by $H''$, then if $H'$ is the Neumann-to-Dirichlet map for the network with $v_n$ interiorized, and $H$ is the Neumann-to-Dirichlet map for the network with both $v_n$ and $v_{n-1}$ interiorized, then $H$ is equal to the $(n-2) \times (n-2)$ dimensional submatrix of*

$$P_{n-1}' P_n H'' P_n^T P_{n-1'}^T$$

*where $P_{n-1}'$ is the $n \times n$ dimensional matrix with the $(n-1) \times (n-1)$ entries are those for $P_{n-1}$ and the last row and column and 0.*

*Proof.* This follows from exactly the considerations as above. $\square$

In general, if $H_K \in M_n$ is some Neumann-to-Dirichlet map and the last $k < n$ boundary vertices in the order implied by $H_K$ are interiorized then if $H$ is the Neumann-to-Dirichlet map for the resulting graph, we have that $H$ is the $(n-k) \times (n-k)$ sized submatrix of the product

$$\tilde{H} = (P_{n-k}^* P_{n-k+1}^* \dots P_n) H_K (P_{n-k}^* P_{n-k+1}^* \dots P_n)^T \tag{4}$$

where the $P_m^*$ for $m < n$ are interpreted to be the $n$-dimensional matrix with the top $m \times m$ submatrix being the original $P_m$ and the remaining entries all zeroes.

It will be useful later on to have a better form for the product of the $P_n$s:

**Lemma 3.9.** *In general, for $0 \le k < n-1$ we have that*

$$(P_{n-k}^* P_{n-k+1}^* \dots P_n) = \begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix}.$$

*Proof.* We leave this simple inductive calculation to the reader. $\square$

For the rest of the paper, we will denote the above product as $P_{n,k}$. It is not hard to show by analogy to Lemma 3.6 $P_{n,k}^T x = x$ for all $x$ with row-sum zero and whose last $k$ entries are zero. It is also possible to think of the $P_{n,k}$ as non-square matrices by omitting the last $k$ rows, but again it is more useful for our purposes to think of them as square matrices.

We briefly note here that this form of the Neumann-to-Dirichlet map associated with the response matrix and the Fan-Pall Theorem ([4]) gives us an easy proof of Corollary 2.2 because we note that if we let

$$P_{n,k}^{\dagger} = \begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & I_k \end{pmatrix}$$

then $H$ is the top-left hand corner of $P_{n,k}^{\dagger} H_K P_{n,k}^{\dagger}$ and $P_{n,k}^{\dagger}$ is obviously unitary. By Theorem 2.3 and manipulating a few indices and reciprocals then it is easy to get Corollary 2.2.

## 4 Bounds and Expressions for the Trace of the Neumann-to-Dirichlet Map and Response Matrix

A concept now useful is the concept of the trace of a matrix:

**Definition 4.1.** The *trace* of a matrix $M \in M_n$, denoted $\mathrm{tr}M$, is defined to be the sum of the diagonal entries in $M$; that is, $\mathrm{tr}M = \sum_{i=1}^n M_{ii}$.

It is not difficult to see that the trace operator is commutative and linear. But that implies that the trace is similarity invariant, and so by Schur's unitary decomposition theorem ([3],) we get that if $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of $M$, then $\mathrm{tr}M = \sum_{i=1}^n \lambda_i$. There are some straightforward bounds, given previous estimates:

**Proposition 4.1.** *If $K$ is a Kirchhoff matrix with associated Neumann-to-Dirichlet map $H_K$, and $\Lambda$ is a response matrix arising from taking the Schur complement of a principle submatrix of $K$ with $K$, with associated Neumann-to-Dirichlet map $H$, then*

$$trK \geq trA \geq tr\Lambda \tag{5}$$

*and*

$$trH_K \geq trH. \tag{6}$$

*where $A$ is as in Equation 1.*

*Proof.* The first bound follows directly from Corollary 2.5 and the fact that $BC^{-1}B^T$ is positive semi-definite as in the proof of Corollary 2.5, and the second bound follows immediately from Proposition 3.1 and Theorem 2.1. $\square$

With notation as in Equation 4, by analogy with Lemma 3.5 we see that the last $k$ rows and columns of $A$ are zero; hence $\mathrm{tr}H = \mathrm{tr}\tilde{H}$. But

$$\begin{aligned}
\mathrm{tr}(A) &= \mathrm{tr}\left(P_{n,k} H_K P_{n,k}^T\right) \\
&= \mathrm{tr}\left(\begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix} H_K \begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix}^T\right) \\
&= \mathrm{tr}\left[H_K\left(\begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix}^T \begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix}\right)\right].
\end{aligned}$$

Another elementary calculation shows that

$$\begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix}^T \begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix} = \begin{pmatrix} I_{n-k,n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \frac{1}{n-k}\mathbf{1}_{k,n-k} & \frac{1}{n-k}\mathbf{1}_{k,k} \end{pmatrix}.$$

Write

$$H_K = \begin{pmatrix} A_H & B_H \\ B_H^T & C_H \end{pmatrix}$$

where $A_H$ is $(n-k)\times(n-k)$ dimensional. Then

$$\begin{aligned} H_K \begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix}^T \begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix} &= H_K \begin{pmatrix} I_{n-k,n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \frac{1}{n-k}\mathbf{1}_{k,n-k} & \frac{1}{n-k}\mathbf{1}_{k,k} \end{pmatrix} \\ &= \begin{pmatrix} A_H & B_H \\ B_H^T & C_H \end{pmatrix} \begin{pmatrix} I_{n-k,n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \frac{1}{n-k}\mathbf{1}_{k,n-k} & \frac{1}{n-k}\mathbf{1}_{k,k} \end{pmatrix} \\ &= \begin{pmatrix} A_H + B_H\frac{1}{n-k}\mathbf{1}_{n-k,k} & \mathbf{0}_{n-k,k} \\ B_H^T + C_H\frac{1}{n-k}\mathbf{1}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix}. \end{aligned}$$

as $H_K$ has row-sum zero. Hence:

**Proposition 4.2.** *Let $H_K$ be decomposed as above. Then $trH = tr(A_H + B_H\frac{1}{n-k}\mathbf{1}_{n-k,k})$.*

However, we cannot find any way to bound tr$H$ in terms of only tr$A_H$ because the non-diagonal entries of $H_K$ are not necessarily non-positive.

## 5 "Expressions" for the Eigenvalues of the Response Matrix

*An "Expression" for the Eigenvalues of the Interiorized Matrix*

Here we give an actual expression for the eigenvalues of the interiorized matrix in terms of the eigenvalues and eigenvectors of the original matrix, although the expression is pretty horrible and probably useless. We use notation as above. Assume $v$ is an eigenvector of $H$ with eigenvalue $\lambda \neq 0$ (which corresponds to the reciprocal of an eigenvalue for the response matrix with the last vertex interiorized). We make it a vector of length $n$ by appending a 0 to it. Let $\{\phi_1 = \mathbf{1}, \phi_2, \ldots, \phi_n\}$ be orthogonal eigenvectors of the original matrix, with corresponding eigenvalues $\{0, \lambda_2, \ldots, \lambda_n\}$ (we note that the last $n-1$ of these correspond to the reciprocals of the eigenvalues of the Kirchhoff matrix). Then $v = \sum_{i=2}^n a_i\phi_i$, as by Corollary 1.2 it has row sum zero and so the coefficient of $\mathbf{1}$ must be zero. We also note that as the last row and column of $P_nH'P_n^T$ is zero, we have that $v$ (with the added zero) is an eigenvector of $P_nH'P_n^T$. Finally, we also let $\{b_j\}_{j=1}^n$ be the unique scalars so that $e_n = \sum_{i=1}^n b_j\phi_j$. By Lemma 3.6, as the $n$th entry in $v$ is zero, $P_n^Tv = v$, so

$$\lambda v = P_nH'P_n^Tv = P_nH'v = P_n\left(\sum_{i=2}^n a_i\lambda_i\nu_i\right).$$

If $\phi_i = [\phi_{i1}, \phi_{i2}, \ldots, \phi_{in}]^T$, let $\tilde{\phi}_i$ be the length $n-1$ vector $[\phi_{i1}, \phi_{i2}, \ldots, \phi_{i(n-2)}, \phi_{i(n-1)}, 0]$; that is, $\tilde{\phi}$ is just the result of replacing the last entry in $\phi$ with zero. We note that

$$P_n\phi_i = \tilde{\phi}_i + \frac{\phi_{in}}{n-1}\tilde{\mathbf{1}};$$

hence

$$\lambda v = \sum_{i=2}^n a_i\lambda_i\left(\tilde{\phi}_i + \frac{\phi_{in}}{n-1}\tilde{\mathbf{1}}\right).$$

However, $\tilde{\phi_i} = \phi_i - \phi_{in} e_n = \phi_i - (\phi_{in} \sum_{j=1}^n b_j \phi_j)$, so plugging this expression into the top expression, we get that as $\phi_{1n} = 1$

$$\lambda v = \sum_{i=2}^n a_i \lambda_i \left[ \phi_i - \left( \phi_{in} \sum_{j=1}^n b_j \phi_j \right) + \frac{\phi_{in}}{n-1} \left( \mathbf{1} - \left( \sum_{j=1}^n b_j \phi_j \right) \right) \right].$$

Rearranging, we get that

$$\lambda v = \mathbf{1} \left( -\frac{nb_1}{n-1} \sum_{i=2}^n a_i \lambda_i \phi_{in} + \frac{1}{n-1} \sum_{i-2}^n a_i \lambda_i \phi_{in} \right) + \sum_{i=2}^n \phi_i \left( a_i \lambda_i - \frac{nb_j}{n-1} \sum_{i=2}^n a_i \lambda_i \phi_{in} \right).$$

As $v$ has row sum zero, this gives us that either $\sum_{i=2}^n a_i \lambda_i \phi_{in} = 0$ or $b_1 = \frac{1}{n}$; a curious result, but in any case the point is that the coefficient of $\mathbf{1}$ must be zero (because $v$ has row sum zero), so

$$\lambda v = \lambda \sum_{i=2} a_i \phi_i = \sum_{i=2}^n \phi_i \left( a_i \lambda_i - \frac{nb_j}{n-1} \sum_{i=2}^n a_i \lambda_i \phi_{in} \right),$$

so by orthogonality

$$\lambda a_i = a_i \lambda_i - \frac{nb_j}{n-1} \sum_{i=2}^n a_i \lambda_i \phi_{in}$$

for all $2 \le i \le n$. Hence, if we let $\Omega$ be the $(n-1) \times (n-1)$ dimensional matrix so that

$$\Omega_{ij} = \begin{cases} -\frac{nb_{(j+1)}}{n-1} \lambda_{(i+1)} \phi_{(i+1)n} & \text{if } i \neq j; \\ \lambda_{(i+1)} - \frac{nb_{(j+1)}}{n-1} \lambda_{(i+1)} \phi_{(i+1)n} & \text{if } i = j. \end{cases} \tag{7}$$

(we have to shift everything by one because previously we were starting with indexing at 2) for $1 \le i, j \le n-1$, then every non-zero eigenvalue of $H$ is an eigenvalue of $\Omega$, and if $v$ is a non-constant eigenvector for $H$, letting $v'$ be the length $n$ vector whose first $n-1$ entries are those in $v$ and whose last entry is zero, then if $v' = \sum_{i=2}^n a_i \phi_i$, then $[a_2, \ldots, a_n]^T$ is an eigenvector for $\Omega$. It is easy to see that these arguments are reversible; hence if $[a_2, \ldots, a_n]^T$ is an eigenvector for $\Omega$ so that $v = \sum_{i=2}^n a_i \phi_i$ has last entry zero (it is also quite trivial that the dimension of the eigenvectors of $\Omega$ which do not satisfy this property is exactly 1, as the other $n-2$ dimensions are taken up by the coefficients of eigenvectors of $H$) with eigenvalue $\lambda$, then $\lambda$ is an eigenvalue of $H$ and $\tilde{v}$ is its corresponding eigenvector.

*An "Expression" for the Eigenvalues of a General Neumann-to-Dirichlet Map*

The same ideas as the ones used in the previous section can actually be used to get a matrix in terms of the eigenvalues and eigenvectors of the Kirchhoff matrix whose eigenvalues are the eigenvalues are exactly those for the Neumann-to-Dirichlet map. As can be expected, this matrix is even worse than the one given above, but nonetheless it gives an almost-closed form expression for the eigenvalues and eigenvectors of the response matrix in terms of those for the Kirchhoff matrix. Let $H_K$ be the Neumann-to-Dirichlet map arising from the Kirchhoff matrix and $H$ the Neumann-to-Dirichlet map after $k$ vertices have been interiorized. By the calculations done in 2.1, we get that the non-zero eigenvalues of $H$ are precisely the nonzero eigenvalues of

$$\begin{pmatrix} I_{n-k} & \frac{1}{n-k} \mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix} H_K \begin{pmatrix} I_{n-k} & \frac{1}{n-k} \mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix}^T.$$

Again, let $\{\mathbf{1}, \phi_2, \ldots, \phi_n\}$ be an orthogonal set of eigenvectors for $H_K$, with corresponding eigenvalues $\{\lambda_1 = 0, \lambda_2, \ldots, \lambda_n\}$. Let $v$ be any non-constant eigenvector of $H$ with eigenvalue $\lambda$ and extend it to be of length

$n$ by appending $k$ zeroes onto it, and relabel this extended eigenvector to be $v$. Write $v = \sum_{i=2}^{n} a_i\phi_i$, where again the coefficient of the constant term must be zero because by Corollary 1.2 $v$ has row sum zero. We note that because its last $k$ entries are zero and it has row sum zero,

$$\begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix}^T v = v$$

just as in the case with one interiorized vertex. Hence as $\lambda v = Hv$, (these are the $n-k$ sized $v$s without zeroes appended) we have that

$$\begin{aligned} \lambda v &= \begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix} H_K \begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix} v \\ &= \begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix} H_K v \\ &= \begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix} \left( \sum_{i=2}^{n} a_i\lambda_i\phi_i \right). \end{aligned}$$

As before, if $w$ is a vector, let $\tilde{w}$ be the vector with the last $k$ entries zero but with the first $n-k$ entries matching those in $w$. Furthermore, let $e_j = \sum_{l=1}^{n} b_l^j \phi_l$ be its unique representation in terms of the eigenvectors of $H_K$. Then we note that for all $\phi_i$, we have that

$$\begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix} \phi_i = \tilde{\phi_i} + \frac{1}{n-k}\left( \sum_{j=n-k+1}^{n} \phi_{ij} \right) \tilde{\mathbf{1}},$$

but

$$\tilde{\phi_i} = \phi_i - \sum_{j=n-k+1}^{n} \phi_{ij} e_j = \phi_i - \sum_{j=n-k+1}^{n} \phi_{ij} \left( \sum_{l=1}^{n} b_j^l \phi_l \right) = \phi_i - \sum_{l=1}^{n} \phi_l \left( \sum_{j=n-k+1}^{n} b_j^l \phi_{ij} \right).$$

Let $\omega_{il} = \sum_{j=n-k+1}^{n} b_j^l \phi_{ij}$ and $\sigma_i = \sum_{j=n-k+1}^{n} \phi_{ij}$. Then

$$\begin{aligned} \begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix} \phi_i &= \tilde{\phi_i} + \frac{1}{n-k}\sigma_i \tilde{\mathbf{1}} \\ &= \phi_i - \sum_{l=1}^{n} \omega_{il}\phi_l + \frac{1}{n-k}\sigma_i \left( \phi_1 - \sum_{l=1}^{n} \omega_{1l}\phi_l \right), \end{aligned}$$

so

$$\begin{pmatrix} I_{n-k} & \frac{1}{n-k}\mathbf{1}_{n-k,k} \\ \mathbf{0}_{k,n-k} & \mathbf{0}_{k,k} \end{pmatrix} \sum_{i=2}^{n} a_i\lambda_i\phi_i = \sum_{i=2}^{n} a_i\lambda_i \left[ \phi_i - \sum_{l=1}^{n} \omega_{il}\phi_l + \frac{1}{n-k}\sigma_i \left( \phi_1 - \sum_{l=1}^{n} \omega_{1l}\phi_l \right) \right].$$

Rearrange the above so that it is in a standard basis representation for $v$ in terms of the eigenvectors of $H_K$. As the above expression is equal to $\lambda v$ the coefficient for $\phi_1 = \mathbf{1}$ must be zero, and for $i \geq 2$ the coefficient of $\phi_i$ is

$$a_i\lambda_i - \sum_{p=2}^{n} a_p\lambda_p w_{pi} - \frac{\sigma_i w_{1i}}{n-k} \sum_{p=2}^{n} a_p\lambda_p$$

so for all $i$,

$$\lambda a_i = a_i\lambda_i - \sum_{p=2}^{n} a_p\lambda_p w_{pi} - \frac{\sigma_i w_{1i}}{n-k}\sum_{p=2}^{n} a_p\lambda_p.$$

Hence if we consider the $(n-1)\times(n-1)$ dimensional matrix $\Omega$ given by

$$\Omega_{ij} = \begin{cases} -\lambda_{(i+1)}\left(\omega_{(i+1)(j+1)} + \frac{\sigma_{(j+1)}w_{1(j+1)}}{n-k}\right) & \text{if } i \neq j; \\ \lambda_{(i+1)} - \lambda_{(i+1)}\left(\omega_{(i+1)(j+1)} + \frac{\sigma_{(j+1)}w_{1(j+1)}}{n-k}\right) & \text{if } i = j. \end{cases} \tag{8}$$

for $1 \leq i,j \leq n-1$ then the non-zero eigenvalues of $\Omega$ are eigenvalues of $H$, and if $[a_2,\ldots,a_n]$ is an eigenvector of $\Omega$ with the last $k$ entries of $v = \sum_{i=2}^{n} a_i\phi_i$ zero (which must occur for a set of eigenvectors of $\Omega$ of dimension $n-k$), then the first $n-k$ entries of $v$ form an eigenvector of $H$ and the last $k$ entries are zero. Hence:

**Theorem 5.1.** *Let $K$ be a Kirchhoff matrix with eigenvectors $\{\phi_1 = \mathbf{1}, \phi_2, \ldots, \phi_n\}$ and corresponding eigenvalues $\{\nu_1 = 0, \nu_2, \ldots, \nu_n\}$. Let $\lambda_i = 1/\nu_i$ for all $2 \leq i \leq n$, and let $\Omega$ be as in Equation 8. Then every eigenvalue of the response matrix for the network with the last $k$ vertices in the order induced by the Kirchhoff matrix being interior vertices is a nonzero eigenvalue of $\Omega$, and if $\lambda$ is an eigenvalue of $\Omega$ with corresponding $[a_2,\ldots,a_n]$ with the last $k$ entries of $\sum_{i=2}^{n} a_i\phi_i$ zero, then $1/\lambda$ is an eigenvalue of the response matrix.*

It would be nice to see how the eigenvalues of $\Omega$ which are not eigenvalues of $H$ relate to anything; however, because of the complicated nature of the matrix, even for just one interiorizing step, the author cannot find a good relation.

## 6 A Green's Function Representation

Let $n = |V|$ and $\{\phi_1, \phi_2, \ldots, \phi_n\}$ the orthonormal eigenvectors of $K$ so that if $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ are the corresponding eigenvalues, $\lambda_1 = 0$. We primarily use a Borg-Levinson-like result due to Ian Zemke (the source paper contains a typo although the proof is unchanged; the sum should be from $i = 2$ to $n$, not $i = 1$ to $n$):

**Lemma 6.1.** [7] *Suppose $f : \partial V \to \mathbb{R}$ be so that $\sum_{y\in\partial V} f(y) = 0$. Then if we define $u : V \to \mathbb{R}$ by*

$$u(x) = \sum_{y\in\partial V} f(y)G(x,y)$$

*where*

$$G(x,y) = \phi_1(x)\phi_1(y) + \sum_{i=2}^{n} \frac{\phi_i(x)\phi_i(y)}{\lambda_i}$$

*then $u$ is $\gamma$-harmonic and $Ku|_{\partial G} = f$.*

Suppose $\phi_\Lambda$ is an eigenvector of $\Lambda$ with eigenvalue $\lambda_\Lambda \neq 0$. Then if we let $\psi$ be so that $v = [\phi_\Lambda, \psi]^T$ is $\gamma$-harmonic, we have that $Kv = [\lambda_\Lambda\phi_\Lambda, 0]^T$.

In the notation of Lemma 6.1, let $f = \lambda_\Lambda\phi_\Lambda$. By Corollary 1.2, this has row-sum zero, and so $u$ is a

$\gamma$-harmonic function with current $f$ on the boundary. Furthermore, we see that

$$\begin{aligned}
u(x) = \sum_{y\in\partial V} \lambda_\Lambda \phi_\Lambda(y) G(x,y) &= \lambda_\Lambda \sum_{y\in\partial V} \phi_\Lambda(y) G(x,y) \\
&= \lambda_\Lambda \sum_{y\in\partial V} \phi_\Lambda(y) \left( \frac{1}{n} + \sum_{i=2}^{n} \frac{\phi_i(x)\phi_i(y)}{\lambda_i} \right) \\
&= \lambda_\Lambda \left( \frac{1}{n} \sum_{y\in\partial V} \phi_\Lambda(y) + \sum_{y\in\partial V} \phi_\Lambda(y) \left( \sum_{i=2}^{n} \frac{\phi_i(x)\phi_i(y)}{\lambda_i} \right) \right) \\
&= \lambda_\Lambda \sum_{y\in\partial V} \phi_\Lambda(y) \left( \sum_{i=2}^{n} \frac{\phi_i(x)\phi_i(y)}{\lambda_i} \right) \\
&= \lambda_\Lambda \sum_{i=2}^{n} \frac{\phi_i(x)}{\lambda_i} \left( \sum_{y\in\partial V} \phi_\Lambda(y)\phi_i(y) \right).
\end{aligned}$$

*Remark* 6.1. Another way to get formula (4) above is by the calculation made in section 1.4. Indeed, we note that the coefficient of each $\phi_\Lambda(y)$ is merely the corresponding entry in the Neumann-to-Dirichlet map when we interpret the Kirchhoff matrix as a response matrix, as by Proposition 3.1 the eigenvectors are preserved and the eigenvalues are inverted (except the zero eigenvalue, which remains unchanged).

By the uniqueness of the solution to the Neumann problem, we know that $u$ is the unique $\gamma$-harmonic function which satisfies $Ku|_{\partial V} = \lambda\phi$ up to a constant. As we know that $\phi_\Lambda$ is an eigenvector for $\Lambda$ with eigenvalue $\lambda_\Lambda$ so $v$ also solves the Neumann problem, we have the following:

**Proposition 6.2.** *If $u$ is as above we have that $u = v + c\mathbf{1}$, for some scalar $c \in \mathbb{R}$. In particular, we have that if $x, y \in V$ then $u(x) - u(y) = v(x) - v(y)$.*

It is quite difficult, it seems, to get a good estimate for the $c$ in Proposition 6.2. However, we do note that by the fourth line above we get that

$$\sum_{x\in V} u(x) = \lambda_\Lambda \sum_{y\in\partial V} \phi_\Lambda(y) \left( \sum_{i=2}^{n} \frac{\phi_i(y)}{\lambda_i} \left( \sum_{x\in V} \phi_i(x) \right) \right) = 0,$$

so $u$ has row sum zero; therefore in general it is not the case that $c = 0$. However, that implies that if we let

$$c = \sum_{x\in \mathrm{int} V} u(x) = \lambda_\Lambda \sum_{y\in\partial V} \phi_\Lambda(y) \left( \sum_{i=2}^{n} \frac{\phi_i(y)}{\lambda_i} \left( \sum_{x\in \mathrm{int} V} \phi_i(x) \right) \right)$$

then $\tilde{u} = u + (c/|\partial V|)\mathbf{1}$ does in fact have boundary row sum zero. Hence:

**Proposition 6.3.** *$\tilde{u} = v$ is the unique solution of the Neumann problem with $\sum_{x\in\partial V} v = 0$. Hence $\tilde{u}|_{\partial V} = \phi_\Lambda$.*

*A Resulting Bound*

Let $a \in \partial V$ and $\{b_j\}_{j=1}^{k} \subset V$ with $ab_j \in E$ and $ab_j$ being the only connections for $a$, with conductance $\gamma_j$. With notation as above, we now assume that $\phi_\Lambda$ is normalized, and we note that as $\phi_\Lambda$ is an eigenvector for $\Lambda$, we have that

$$\sum_{j=1}^{k} \gamma_j(\phi_\Lambda(a) - \psi(b_j)) = \sum_{j=1}^{k} \gamma_j(u(a) - u(b_j)) = \lambda_\Lambda \phi_\Lambda(a).$$

Using the expression derived above for $u$, we get that the above implies that

$$\sum_{j=1} \gamma_j \left[ \lambda_\Lambda \sum_{i=2}^{n} \frac{\phi_i(a) - \phi_i(b_j)}{\lambda_i} \left( \sum_{y \in \partial V} \phi_\Lambda(y) \phi_i(y) \right) \right] = \lambda_\Lambda \phi_\Lambda(a)$$

so if we cancel the $\lambda_\Lambda$s and let $\gamma = \sum_j \gamma_j$, we get that

$$\phi_\Lambda(a) = \sum_{j=1}^{k} \gamma_j \left[ \sum_{i=2}^{n} \frac{\phi(a) - \phi_i(b_j)}{\lambda_i} \left( \sum_{y \in \partial V} \phi_\Lambda(y) \phi_i(y) \right) \right]$$

so

$$\begin{aligned} |\phi_\Lambda(a)| &= \lambda_\Lambda \left| \sum_{j=1}^{k} \gamma_j \left[ \sum_{i=2}^{n} \frac{\phi_i(a) - \phi_i(b_j)}{\lambda_i} \left( \sum_{y \in \partial V} \phi_\Lambda(y) \phi_i(y) \right) \right] \right| & (9) \\ &\leq \sum_{j=1}^{k} \gamma_j \sum_{i=2}^{n} \left| \frac{(\phi_i(a) - \phi_i(b_j))}{\lambda_i} \left( \sum_{y \in \partial V} \phi_\Lambda(y) \phi_i(y) \right) \right| & (10) \\ &\leq \lambda_\Lambda \sum_{j=1}^{k} \gamma_j \sum_{i=2}^{n} \frac{2}{\lambda_i} |\phi_\Lambda||\phi_i| & (11) \\ &\leq \sum_{j=1}^{k} \gamma_j \sum_{i=2}^{n} \frac{2}{\lambda_i} = 2\gamma \sum_{i=2}^{n} \frac{1}{\lambda_i} & (12) \end{aligned}$$

by normality and the fact that conductances and the eigenvalues are positive. I'm not sure how useful this bound is, however.

## 7 Work in Progress

Here we present several ideas the author had which did not result in anything; however, the author thinks that these ideas probably warrant future research.

### *7.1 Characterizing the Eigenvalues of Kirchhoff Matrices*

Here we consider this problem: Let $\{\phi_2, \phi_3, \ldots, \phi_n\}$ be $n-1$ orthonormal eigenvectors with all row-sums zero. What $\{\lambda_2, \ldots, \lambda_n\}$ would, after diagonalizing with respect to $\mathbf{1}$ and the above collection of eigenvectors, create a valid Kirchhoff matrix?

**Proposition 7.1.** *Let $H$ be as in Proposition 1.3. For each $i, j$, there exists a $k \geq 2$ so that $\phi_{ki}\phi_{kj} \neq 0$. Hence $H_{ij}$ changes linearly with $\lambda_k$ (if we keep every other $\lambda_j$ constant) with slope $\phi_{ki}\phi_{kj}$.*

*Proof.* Suppose $\phi_{ki}\phi_{kj} = 0$ for all $k \geq 2$. Let $V_i = \{\phi_k : \phi_{ki} = 0\}$ and define $V_j = \{\phi_k : \phi_{ki} \neq 0, \phi_{kj} \neq 0\}$. For each $\phi \in V_i, V_j$ make the $n-2$-length vector $v_k$ created by omitting both $\phi_{ki}$ and $\phi_{kj}$, and redefine $V_i$ and $V_j$ to be their projections onto this space. By their constructions, every vector in $V_i$ is orthogonal to every vector in $V_j$; hence as their spans are independent vector spaces on an $(n-2)$ sized vector space, if every vector in $V_i$ is linearly independent of every other vector in $V_i$, and similarly for $V_j$, then this is a $n-1$ dimensional spanning set for a $n-2$ dimensional vector space, which is impossible. Suppose then without loss of generality $v_1 \in V_i$ and $v_1 = \sum_{r=1}^{k} a_v v$ for some scalars $a_v$, where $k = |V_i|$. But as for any $v \in V_i$ we have that if $\phi$ is the corresponding eigenvector it came from as $\phi_i = 0$ by assumption and $\phi$ is orthogonal to

**1**, it must be the case that $\phi_j = -\sum_{p \neq i,j} \phi_p = -\sum_{p=1}^{n-2} v_p$. Plugging this into the linear dependence we see that as

$$\sum_{p=1}^{n-2} v_{1p} = \sum_{r=1}^{k} a_v \sum_{p=1}^{n-2} v_{rp},$$

so therefore as $\phi_i = 0$ for all $\phi \in V_i$, we have that $\phi_{1p} = \sum_{r=1}^{k} a_v \phi_{rp}$ for every $p$; hence the $\phi$ were originally linearly independent, which is impossible as the original eigenvectors were orthogonal and hence linearly independent. The same argument for $V_j$ shows that $V_j$ is also linearly independent. This is a contradiction; hence we are done. □

**Corollary 7.2.** *Let $H$ be as in Proposition 1.3. If $H_{ij} = 0$ for some $i, j$, there is some $\lambda_k$ so that increasing $\lambda_k$ forces $H_{ij} < 0$, and similarly there is a $\lambda_k$ so that increasing $\lambda_k$ forces $H_{ij} > 0$.*

*Proof.* By Proposition 7.1, there is a $k \geq 2$ so that $\phi_{ki}\phi_{kj} \neq 0$. If $\phi_{ki}\phi_{kj} < 0$, we are done, as then increasing $\lambda_k$ decreases $H_{ij}$ by Proposition 1.3. If $\phi_{ki}\phi_{kj} > 0$, as $\lambda_k > 0$ for all $k$, this implies there must be some other $k' \geq 2$ so that $\phi k' i \phi_{k'j} < 0$. Hence we are done. The symmetric argument shows the second claim. □

Hence if $K$ is a Kirchhoff matrix for some network $\Gamma = (G, \gamma)$, every edge in the complementary graph of $G$ imposes a bound on how we can vary some of the eigenvalues of matrices with the same eigenvectors as $K$ while keeping them valid Kirchhoff matrices.

A related idea to the above considerations is considering the type of eigenvectors are possible given a set graph topology. If $ij$ is in the complementary graph, for instance, then

$$H_{ij} = \sum_{k=2}^{n} \lambda_k \phi_{ki} \phi_{kj} = 0;$$

therefore if $\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_k\beta_k$ are all edges in the complementary graph, we can construct the matrix $P$ with entries $P_{ij} = \phi_i(\alpha_j)\phi_i(\beta_j)$ (we are switching into the functional notation), then the values in the null-space of $P$ correspond to lists of eigenvalues which preserve the graph topology given the eigenvectors. In particular, if $P$ is non-singular, then the only eigenvalues are the zero eigenvalues, which means that the graph induced is the graph with no edges, which does not preserve the graph topology; hence $P$ cannot be non-singular; otherwise the vectors used to construct the matrix cannot be eigenvectors of a graph with the desired graph topology. However, the author cannot find any good results about the nature of this matrix $P$ even in the special case when it is square, and in the general case, when it is not square (i.e. when there are not exactly $n$ edges in the complementary graph) then the situation is even worse.

### *7.2 Bounds on the Dirichlet Eigenvalues on Special Graphs*

A more detailed discussion of the ramifications of Dirichlet eigenvalues is given in [5]. Here we will just consider the problem of finding eigenvalues of $C$ when $K$ is block decomposed as in Equation 1. Any eigenvalue of $C$ we will call a *Dirichlet eigenvalue* and any associated eigenvector a *Dirichlet eigenvector*. As $C$ is positive definite ([2]), symmetric and real-valued, we have that all the eigenvalues of $C$ are nonzero. For some time the author attempted to find good expressions for these eigenvalues in terms of anything; however, the only positive result he found was the following.

**Proposition 7.3.** *Let $G$ be a graph with no interior nodes which have only edges to other interior nodes. Then if we let $B$ be as in Equation 1, then if $b_i = \max_j\{-B_{ij}\}$ is the maximum for the absolute values of the values in the $i$th column of $B$, then if $\lambda$ is a Dirichlet eigenvalue, we must have $\lambda \geq \min_i\{b_i\}$.*

*Proof.* Suppose to the contrary we had $\lambda < \min_i\{b_i\}$. For the $i$th column of $B$ let $\sigma(i)$ be a value so that $B_{i\sigma(i)} > \lambda$. As $C_{ii}$ is the sum of the absolute values of the $C_{ij}$ for $j \neq i$ plus the sum of the absolute values of the $B_{ij}$, we have that $C_{ii} - \lambda + \sum_{j \neq i} C_{ij} = \xi_i > 0$ still because $\lambda < \min_i\{b_i\}$ and thus is less than the sum of any column of $B$. But then note that if we construct the diagonal matrix $D$ whose entry in the $(i,i)$th place is $\xi_i$, we get that $C^\dagger = C - \lambda I - D$ is symmetric, has positive diagonal entries, row sum zero, and non-diagonal entries negative; in short, $C^\dagger$ is a Kirchhoff matrix, and $C - \lambda I = C^\dagger + D$. But then for all $u \in \mathbb{R}^\ltimes$

$$u^T(C - \lambda I)u = u^T C^\dagger u + u^T D u;$$

but $u^T D u \geq 0$ for all $u$ and $u^T C^\dagger u \geq 0$; hence this matrix is positive definite, and so in particular $\det(C - \lambda I) > 0$; so $\lambda$ cannot be an eigenvalue of $C$. □

We note that the above proof definitely fails to have any meaning if any interior node has only edges to other interior nodes; while the proof still works, it merely states that every eigenvalue of $C$ is nonnegative, which we already know.

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