The Theory of Pseudoknots

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Abstract

Classical knots in $\mathbb{R}^3$ can be represented by diagrams in the plane. These diagrams are formed by curves with a finite number of transverse crossings, where each crossing is decorated to indicate which strand of the knot passes over at that point. A *pseudodiagram* is a knot diagram that may be missing crossing information at some of its crossings. At these crossings, it is undetermined which strand passes over. Pseudodiagrams were introduced by Hanaki in [5] and further studied in [6]. Here, we introduce the notion of a *pseudoknot*, i.e. an equivalence class of pseudodiagrams under an appropriate choice of Reidemeister moves. In order to begin a classification of pseudoknots, we introduce the concept of a *weighted resolution set*, or *WeRe-set*, an invariant of pseudoknots. We compute the WeRe-set for several families of knots and extend many classical problems in knot theory to the theory of pseudoknots.

1 Introduction

Recently, Ryo Hanaki introduced the notion of a *pseudodiagram* of a knot, link or spatial graph [5]. A pseudodiagram of a knot is a knot diagram that may be missing some crossing information, as in Figure 1. In other words, at some crossings in a pseudodiagram it is unknown which strand passes over and which passes under. These undetermined crossings are called *precrossings*. Special classes of pseudodiagrams are knot diagrams and knot *shadows*, i.e. pseudodiagrams containing *only* precrossings. Pseudodiagrams were originally considered because of their potential to serve as useful models for biological objects related to DNA, but they are interesting objects in their own right.

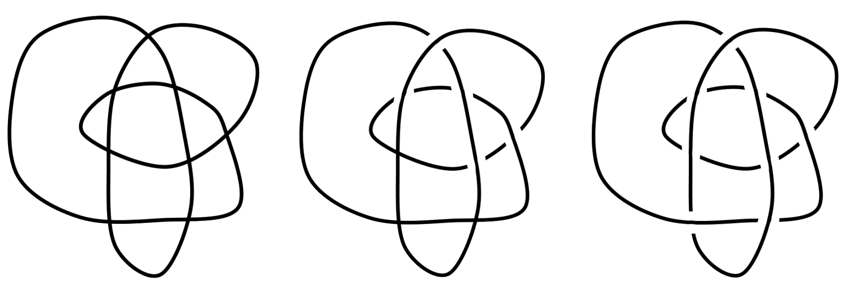

Figure 1: Examples of pseudodiagrams

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Of particular interest in pseudodiagram theory is the computation of the *trivializing number* and *knotting number* for pseudodiagrams. The trivializing number is the least number of precrossings that must be resolved into crossings in order to produce a pseudodiagram of the unknot. (That is, regardless of how the remaining precrossings are resolved, the unknot is always produced.) Similarly, the knotting number is the least number of precrossings that must be determined to produce a non-trivial knot. Much work has been done to analyze trivializing and knotting numbers [5], [6].

For the purposes of this paper, we are interested in studying the knot theory that arises from considering equivalence classes of pseudodiagrams under equivalence relations generated by a natural set of Reidemeister moves. We refer to these objects as *pseudoknots.* Our choice of the set of Reidemeister moves for pseudoknots, pictured in Figure 2, was inspired by the theory of singular knots. Singular knots are knots that contain a finite number of self-intersections. If we view precrossings as singular crossings, we recover all of the pseudoknot Reidemeister moves, with the notable exception of the pseudo-Reidemeister I (PRI) move.

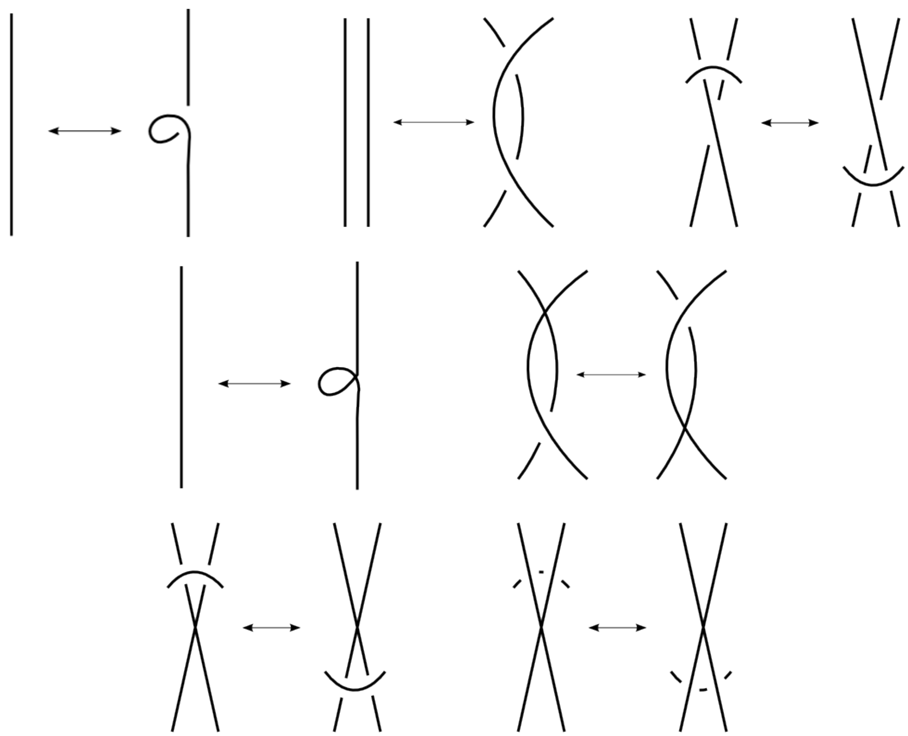

Figure 2: The pseudo-Reidemeister moves

We note that the PRI move ought to be included for the following reason. Consider a pseudodiagram P and the pseudodiagram P' that is related to P by a PRI move that introduces a new precrossing, c. Given a resolution of the precrossings in P', the canonical corresponding resolution of the precrossings in P produces the same knot type, regardless of how c was resolved in P'. So, in an important sense, PRI preserves the knot type of a pseudodiagram.

2 Weighted Resolution Sets

As with any knot theory, the primary question in pseudoknot theory asks how we might classify pseudoknots. That is, given any two distinct pseudoknots, how might we *prove* that they are distinct. One partial answer is to consider the set of all possible knots that can be produced by resolving all precrossings in a diagram of the pseudoknot. A more sophisticated answer is to consider the invariant we call the *weighted resolution set* of a pseudoknot.

Definition 1 *The* weighted resolution set *(or* WeRe-set*) of a pseudoknot P is the set of ordered pairs (K, p_K) where K is a resolution of P (i.e. a choice of crossing information for every precrossing) and p_K is the probability that K is obtained from P by a random choice of crossing information, assuming that positive and negative crossings are equally likely.*

To illustrate this definition, consider the pseudoknot T in Figure 3. There are $2^2 = 4$ ways to resolve the precrossings in the diagram. Three of the four resolutions result in the unknot, 0_1, while one resolution results in the (right-handed) trefoil, 3_1. Thus, the WeRe-set for this example is $\{(0_1, \frac{3}{4}), (3_1, \frac{1}{4})\}$. Note that if we resolve one of the precrossings of T to be a positive crossing, the WeRe-set of the resulting pseudoknot is $\{(0_1, \frac{1}{2}), (3_1, \frac{1}{2})\}$. This shows that the WeRe-set is indeed a more powerful invariant than the (unweighted) resolution set, as the added probabilities can distinguish these two pseudoknots.

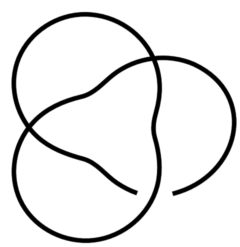

Figure 3: A pseudoknot, T, that produces the unknot and the trefoil.

Theorem 1 *The WeRe-set is an invariant of pseudoknots.*

Proof. It suffices to show that the WeRe-set of a pseudoknot is unchanged by the pseudo-Reidemeister moves. We will in fact show that all moves except the PRI move preserve the resolution multiset (i.e. the set of knots obtained by resolving precrossings in all possible ways, with multiplicity). Therefore, they preserve the WeRe-set.

First, all classical Reidemeister moves clearly preserve the resolution multiset since they involve no precrossings.

The two PRIII moves behave much like the classical moves. Regardless of which resolution is chosen for the single precrossing, a classical RIII move is possible on the resolution, so the knot type is unchanged. Thus the WeRe-set is also unchanged.

For PRII, we note that, both before and after the move, there are two possibilities for the precrossing resolution of the pseudoknot. In each case, there is precisely one precrossing resolution that yields alternating crossings while the other yields a local diagram that RII may be applied to. The alternating resolution that is obtained before the move is identical to the one obtained after the move. Thus, the knot types for this choice of resolution before and after the move are identical. The non-alternating choice of resolution also produces the same knot both before and after the PRII move because a simplifying RII move is possible in each case. It follows that the resolution multiset is unaffected by the PRII move, and thus so is the WeRe-set.

Finally, we consider the PRI move. We note that either resolution of the precrossing can be removed with a simplifying classical RI move, so this move does not change the knot type of any resolution. Since we have two choices for the precrossing in PRI, the multiplicity of every resolution is increased by a factor of two after the move that adds a precrossing is performed. Since doubling the multiplicity of each knot in the resolution multiset does not affect the ratios of the resolutions, the WeRe-set is unchanged. □

3 Pseudoknot Families

To help us understand the relationship between pseudoknots and their WeRe-sets, we compute the WeRe-sets for the shadows of various families of knots. Note that whenever we refer to such a shadow, we are considering the pseudoknot associated to a shadow of the standard projection of the knot.

3.1 Torus Pseudoknots

We begin by considering torus knots, with a focus on $(2, p)$-torus knots. A $(2, p)$-torus knot is particularly straightforward to analyze since it is the closure of a 2-braid with an odd number of crossings. It turns out that to determine the resolutions of a $(2, p)$-torus shadow, it suffices to consider the shadow of its corresponding 2-braid.

It is well known that braids can be represented by elements of a group called the *braid group* [2]. For example, a 2-braid can be represented by a word in the generators σ_1 and σ_1^{-1}, where σ_1 represents a negative crossing between the two strands and σ_1^{-1} represents a positive crossing. (Note that the fact that these generators are inverses follows from the RII move.) All resolutions of the shadow of a 2-braid, therefore, correspond to powers of σ_1. This leads us to the following result.

Lemma 1 *Suppose B is the shadow of a 2-braid with n crossings. Then there are $\binom{n}{k}$ ways to resolve the precrossings to obtain the braid σ_1^{n-2k}. Moreover, if B' is a pseudodiagram of a 2-braid with n precrossings and the classical crossings contribute a total of σ_1^l to the braid word for B', then there are $\binom{n}{k}$ ways to resolve B' to get a braid with reduced word σ_1^{l+n-2k}.*

Proof. If we choose $n - k$ of the n crossings to be negative (corresponding to σ_1) and k to be positive (corresponding to σ_1^{-1}), then the reduced braid word corresponding to the resulting braid is σ_1^{n-2k}. Furthermore, there are $\binom{n}{k}$ ways to choose the k positive crossings. The second statement follows as an immediate corollary. □

Theorem 2 *Every resolution of the shadow of a $(2, p)$-torus knot is a $(2, p - 2k)$-torus knot, with $0 \leq k \leq p$. Moreover, there are $\binom{p}{k}$ ways to obtain a $(2, p - 2k)$ torus knot from the $(2, p)$-torus shadow. In particular, since the unknot is its own mirror image (as it can be represented both as a $(2, 1)$- and $(2, -1)$-torus shadow), there are $2\binom{p}{\lfloor p/2 \rfloor}$ ways to get the unknot.*

Proof. A $(2, p)$-torus knot is the closure of the 2-braid with p crossings corresponding to the braid word σ_1^p. By the previous lemma, there are $\binom{p}{k}$ ways to get the braid σ_1^{p-2k} from the shadow of the 2-braid with p crossings, and the closures of these braids are precisely the $(2, p - 2k)$-torus knots. Clearly, no other knot types are obtainable from the standard projection, since every resolution is the closure of a 2-braid. □

Corollary 1 *The knotting number of the shadow of a $(2, p)$-torus knot is equal to $\frac{p+3}{2}$.*

Proof. Any crossing can be removed by a suitable resolution of an adjacent precrossing. So if we have chosen k crossings (say, positively) and have $p-k$ left to choose, the difference must be at least 3 to ensure that a non-trivial knot is produced. If we let $k = \frac{p+3}{2}$, then

$$k - (p-k) = \frac{p+3}{2} - \frac{p-3}{2} = 3.$$

This is the least number of crossing choices we can make to be sure we have a knot. $\square$

While the WeRe-sets of $(2,p)$-torus knot shadows are straightforward to describe, other kinds of torus knot shadows present more of a challenge. One interesting feature of the $(2,p)$-torus knots is that every resolution is itself a $(2,p)$-torus knot for some p. For the $(3,4)$-torus shadow, however, a similar result fails to hold.

Example 1 (The (3,4)-torus knot.) *The WeRe-set of the shadow of the (3,4)-torus knot is*

$$\{(0_1, \frac{88}{2^8}), (3_1, \frac{72}{2^8}), (4_1, \frac{4}{2^8}), (5_1, \frac{16}{2^8}), (5_2, \frac{32}{2^8}), (6_3, \frac{16}{2^8}), (8_{18}, \frac{2}{2^8}), (8_{19}, \frac{2}{2^8}), (8_{20}, \frac{16}{2^8}), (3_1\#3_1, \frac{8}{2^8})\}$$

and it contains knots which are not torus knots.

The $(3,7)$-torus knot is also of particular interest.

Example 2 (The (3,7)-torus knot.) *The probability of obtaining an unknot from a knot shadow is higher than the probability of obtaining any other knot type for shadows with up to 12 crossings. An exception to this rule is the shadow of the (3,7)-torus knot with the WeRe-set*

$$\{(0_1, \frac{2688}{2^{14}}), (3_1, \frac{2884}{2^{14}}), \ldots\}.$$

Figure 4: The shadows (a) of the $(3,4)$-torus knot; (b) of the $(3,7)$-torus knot.

3.2 Rational Pseudoknots

We turn our attention from pseudoknots related to torus knots to other interesting knot families. We consider the concrete example of shadows of twist knots (as in Figure 5) then discuss rational pseudoknots more generally. For more on rational and twist knots, see [1].

Theorem 3 *Every resolution of a twist knot shadow with n crossings is either the unknot or a twist knot with n or fewer crossings. The number of ways to obtain a twist knot with $n-2k$ crossings by resolving the shadow of a twist knot with n crossings is $2\binom{n-2}{k}$. The number of ways to obtain the unknot is $2^{n-1} + 2\binom{n-2}{\lfloor n-2/2 \rfloor}$.*

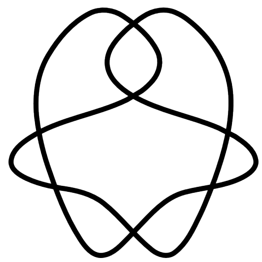

Figure 5: The shadow of a twist knot.

Proof. A twist knot is made up of a clasp and a twist, and each of these tangles can be viewed as braids. The clasp of a twist knot shadow can be resolved to alternate in either of two ways (corresponding to the braids σ_1^2 and σ_1^{-2}). It can also be resolved so that it is not alternating, in which case our resolution is the unknot.

By Lemma 1, there are $\binom{n}{k}$ ways for the twist tangle to be the braid σ_1^{n-2k}. If we have the braid σ_1^{n-2k} and the clasp is alternating, we will either get a twist knot with $n-2k$ crossings or one with $n-2k-1$ crossings, depending on which way the clasp alternates. Since we can get both the twist knot with positive crossings and the one with negative crossings, we multiply by 2 to get the total number of possible resolutions that will give us a particular twist knot. $\square$

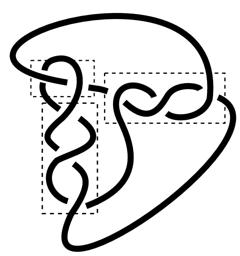

Figure 6: A rational knot.

Twist knots are a subfamily of the larger family of *rational knots* [4], [9]. We can generalize our strategy of computing WeRe-sets for twist knot shadows to strategies for computing rational knot shadow WeRe-sets. Recall that a rational knot can be viewed as the closure of a rational tangle, that is, a tangle made up of a sequence of horizontal and vertical twists. See Figure 6 for an example of a rational knot, and refer to [1] for a detailed discussion of rational tangle construction.

The sequence of twists $[a_1, a_2, ..., a_n]$ (meaning the first twist has a_1 crossings, the second has a_2 crossings, etc.) that creates a rational knot is key for classifying such knots. (Note that twist knots are associated to sequences of the form $[a_1, 2]$.) From this sequence, we can form the following continued fraction, which is instrumental in determining information about knot type.

$$a_n + \cfrac{1}{a_{n-1} + \frac{1}{a_{n-2} + \dots}}$$

If this is equal to p/q for some relatively prime $p, q \in \mathbb{Z}$ with p odd, then it is a knot. Moreover, a theorem by Schubert from [11] tells us exactly when two such knots will be equal.

Theorem 4 (Schubert, 1956) *Suppose that rational tangles with fractions p/q and p'/q' are given, where p and q are relatively prime as are p' and q'. (The numerators are always assumed to be*

positive, and the denominators may be negative.) If $K(p/q)$ and $K(p'/q')$ denote the corresponding rational knots obtained by taking numerator closures of these tangles, then $K(p/q)$ and $K(p'/q')$ are topologically equivalent if and only if
1. $p = p'$ and
2. either $q \equiv q' \mod p$ or $qq' \equiv 1 \mod p$.

Schubert's theorem, along with the following theorem, help us determine the WeRe-sets for shadows of rational knots.

Theorem 5 *Suppose we have a shadow of a rational knot with the sequence $[a_1, a_2, ..., a_n]$. Then the number of resolutions that yield the sequence $[a_1 - 2k_1, a_2 - 2k_2, ..., a_n - 2k_n]$ is*

$$\prod_{i=1}^{n} \binom{a_i}{k_i}.$$

Proof. We note that each element of the sequence $[a_1, a_2, ..., a_n]$ represents a tangle subdiagram that is a 2-braid. The total number of ways to get the sequence $[a_1 - 2k_1, a_2 - 2k_2, ..., a_n - 2k_n]$, then, is the product of the number of ways to get each element in the sequence, as determined in Lemma 1. □

4 Crossing Number of a Pseudoknot

Just as with classical knots, we have a natural notion of crossing number for pseudoknots,

Definition 2 *The* crossing number*, $cr(K)$, of a pseudoknot K is the minimum number of total crossings (both classical and precrossings) of any projection of that pseudoknot.*

Given this definition, one question we explored was the following.

Question 1 *Is the crossing number of a pseudoknot equal to the maximum crossing number of its resolutions? Equivalently, if it is possible to reduce any resolution of a pseudoknot, is it necessarily possible to reduce the pseudoknot?*

Using the WeRe-set invariant, we are able to answer this question negatively. Indeed, there are pseudoknots with crossing number strictly greater than the maximum of the crossing numbers of knots in the resolution set. Consider the pseudodiagram given in Conway notation by $3\ i\ -2$, where i denotes a single precrossing. The pseudoknot K given by this diagram has WeRe-set $\{(5_1, \frac{1}{2}), (5_2, \frac{1}{2})\}$, which is different from the WeRe-set of any pseudoknot with crossing number five. We can conclude that the minimum crossing number of K is six, while the maximum of the crossing numbers of knots in the WeRe-set is five.

There is a nice class of examples for which Question 1 can be answered positively. First, some terminology is needed.

Definition 3 *A crossing in a pseudodiagram is called* nugatory *if there exists a circle in the plane enclosing part of the knot which intersects the knot at the crossing and nowhere else.*

Definition 4 *A pseudodiagram is* potentially alternating *if it is has no nugatory crossings and if it is possible to resolve its precrossings so that the resulting knot diagram is alternating.*

We note that any shadow is potentially alternating, while our example above, $3\ i\ -2$, is not potentially alternating. Returning to our question, we find the following result.

Proposition 1 *Suppose a pseudoknot K has a potentially alternating diagram D_K. Then the crossing number of K is realized in D_K. Furthermore, the crossing number of K is equal to the maximum crossing number of the knots in K's resolution set.*

Proof. The result follows immediately from the fact that K has a reduced alternating knot K_a in its resolution set with $cr(K_a) = cr(D_K)$. □

Remark 1 *In our example* $3\ i\ -2$*, we saw that if crossing* i *is resolved one way, the knot* 5_1 *results. If the crossing is resolved the other way, we get the knot* 5_2*. An interesting question to consider is whether there is a similar example of an oriented pseudodiagram with a single non-nugatory precrossing such that both resolutions produce the* same *(oriented) knot rather than two different knots (so the WeRe-set has a single element). Such precrossings are called* cosmetic *or (non-nugatory)* degenerate *crossings. Nugatory crossings are also considered to be degenerate, but there are not yet any known oriented knots containing non-nugatory degenerate crossings (i.e. cosmetic crossings). This problem was first posed by X.-S. Lin and appears as Problem 1.58 in Rob Kirby's Problem List.*

5 Open Questions

We conclude with several interesting open questions about pseudoknots.

Question 2 *Is each pseudoknot uniquely determined by its WeRe-set?*

Question 3 *Do flypes involving nugatory precrossings preserve pseudoknot type?*

Question 4 *What is an appropriate definition of pseudobraids? In particular, when are two pseudobraids (or generalized pseudobraids) equivalent?*

Question 5 *In classical knot and braid theory, there are two Markov moves on braids that characterize when two braids have equivalent closures. If we generalize pseudoknots so that they can all be written as pseudobraids, can we come up with an algebraic criterion for when two pseudobraids represent the same pseudoknot?*

Acknowledgments

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