The Construction of 2^n to 1 Graphs

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Abstract. Independent cycles are created due to the presence of special quadrilaterals called independent connectors in the R-Multigraph. It will be shown that an independent cycle generates 2 sets of positive conductivities. It is demonstrated as to how to create a 2^n to 1 graph by forcing its R-Multigraph to have n independent cycles.

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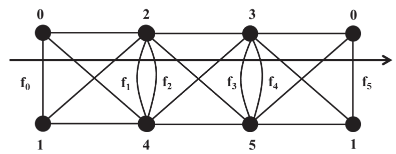
1 Preliminaries

Definition 1. Given a R-Multigraph, let $f_0(x), f_1(x), ..., f_i(x), ...f_n(x)$ satisfy the response matrix condition for $\lambda_{j,k}$. So $\lambda_{j,k} = f_0(x) + f_1(x) + ... + f_i(x) + ... + f_n(x)$. A *cycle* is a path starting at $f_0(x) = x$ and going through a connected sequence with at least 1 quadrilateral to $f_i(x)$ (where $i \neq 0$) only using the response matrix condition and quadrilateral rules starting with $f_0(x) = x$. No other unknown edge in the R-Multigraph can be determined uniquely by knowing $f_0(x) = x$ and the response matrix. See [1] for another definition.

Definition 2. The *length* of a cycle is the number of quadrilaterals in its path.

Example 1. Here is an example of a cycle beginning at $f_0(x)$ and ending at $f_5(x)$.



A cycle of length 3

The cycle starts at $f_0(x)$ and goes through 3 quadrilaterals to $f_5(x)$ using the response matrix condition and the quadrilateral rules generated by $f_0(x) = x$.

$f_0(x) = x$

$f_1(x) = \frac{\lambda_{0,2}\lambda_{1,4}}{f_0(x)}$

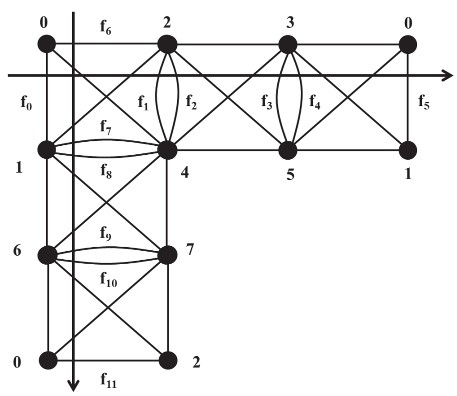
$f_2(x) = \lambda_{2,4} - f_1(x)$

$f_3(x) = \frac{\lambda_{2,3}\lambda_{4,5}}{f_2(x)}$

$f_4(x) = \lambda_{3,5} - f_3(x)$

$f_5(x) = \frac{\lambda_{0,3}\lambda_{1,5}}{f_4(x)}$

Example 2. Here is an example of two cycles both of length 3.



Two cycles both of length 3

Cycle One

$f_0(x_1) = x_1$

$f_1(x_1) = \frac{\lambda_{1,2}\lambda_{0,4}}{f_0(x_1)}$

$f_2(x_1) = \lambda_{2,4} - f_1(x_1)$

$f_3(x_1) = \frac{\lambda_{2,5}\lambda_{3,4}}{f_2(x_1)}$

$f_4(x_1) = \lambda_{3,5} - f_3(x_1)$

$f_5(x_1) = \frac{\lambda_{1,3}\lambda_{0,5}}{f_4(x_1)}$

Cycle Two

$f_6(x_2) = x_2$

$f_7(x_2) = \frac{\lambda_{1,2}\lambda_{0,4}}{f_6(x_2)}$

$f_8(x_2) = \lambda_{1,4} - f_7(x_2)$

$f_9(x_2) = \frac{\lambda_{4,6}\lambda_{1,7}}{f_8(x_2)}$

$f_{10}(x_2) = \lambda_{6,7} - f_9(x_2)$

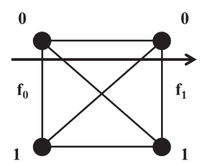
$f_{11}(x_2) = \frac{\lambda_{2,6}\lambda_{0,7}}{f_{10}(x_2)}$

By the quadrilateral rule, it is also true that $f_1(x_1) = \frac{f_6(x_1)f_7(x_1)}{f_0(x_1)}$. However, this equation cannot be used in cycle 1 since $f_1(x_1)$ cannot be determined uniquely by knowing $f_0(x_1) = x_1$ and the response matrix.

Lemma 1. A cycle is always of length at least 3.

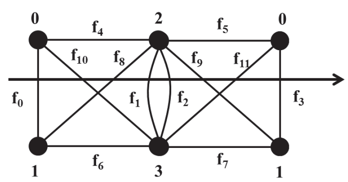
Proof. Suppose it is possible to have cycles of length less than 3. By definition, a cycle must have length at least 1 quadrilateral. This leaves us to consider cycles of length 1 and 2.

A cycle of length 1 has 1 quadrilateral in its path. However, since $f_0(x)$ and $f_1(x)$ satistfy the response matrix condition, this contradicts with the very shape of the quadrilateral.



Since the vertices 0 and 1 appear twice in the same figure, this is not a quadrilateral at all.

A cycle of length 2 has 2 quadrilaterals in its path.



All edges are unknown.

However, we cannot uniquely determine the $f_j(x)$'s by just knowing the response matrix and that $f_0(x) = x$. This contradicts the definition of cycle. □

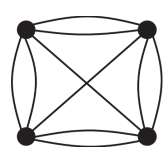
Definition 3. Two cycles are *independent* if their paths share no f_j's.

In example 2, cycle 1 and cycle 2 are independent. Although some λ's are shared, there are no f_j's in common.

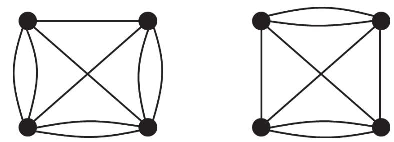
Independent cycles in R-Multigraphs are created by the presence of special quadrilaterals called independent connectors.

Definition 4. An *independent connector* is a quadrilateral involved in the construction of at most 2 independent cycles. An independent connector that is involved in the construction of 2 independent cycles is called a *2-connector* whereas one that is involved in the construction of 1 independent cycle is called a *1-connector*.

Example 3. Examples of 2-connectors and 1-connectors are shown.

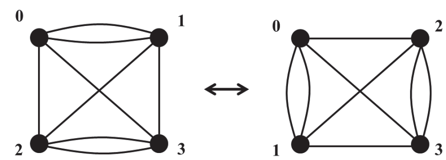


A 2-connector

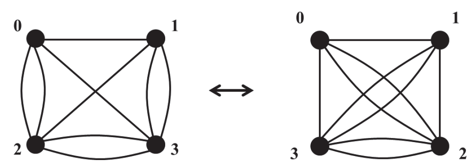


1-connectors

Note that other independent connectors can be formed from these examples by a simple relabeling of vertices. For example,



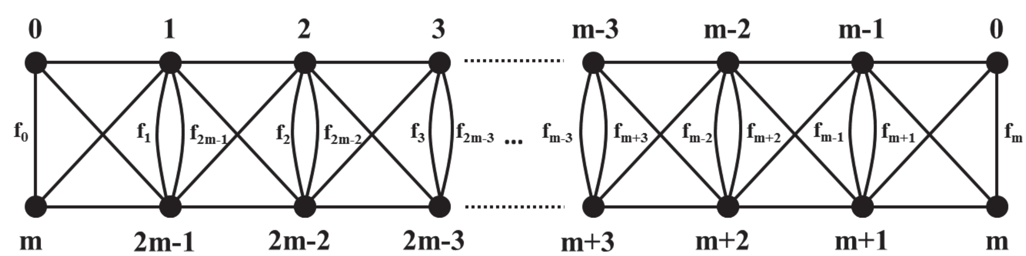
These 1-connectors are the same after a relabeling of vertices.



These 1-connectors are the same after a relabeling of vertices.

Theorem 1. Suppose a graph's R-Multigraph is a single cycle. Then the graph is 2 to 1.

Proof. Suppose that the cycle is of length $m \geq 3$. We label the cycle in the following fashion:



The cycle is of length m.

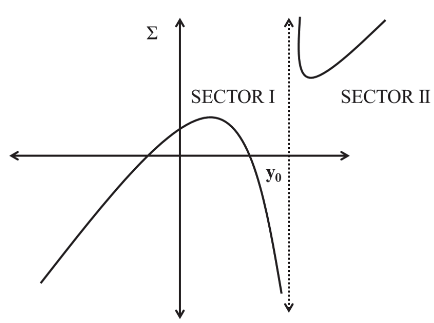
Assume that $f_0(x) = x$ and obtain the following system of equations for the cycle using the response matrix condition and the quadrilateral rules.

Sign of Derivative	Equation
+	$f_0(x) = x$
−	$f_1(x) = \frac{\lambda_{0,2m-1}\lambda_{1,m}}{f_0(x)}$
+	$f_{2m-1}(x) = \lambda_{1,2m-1} - f_1(x)$
−	$f_2(x) = \frac{\lambda_{1,2m-2}\lambda_{2,2m-1}}{f_{2m-1}(x)}$
+	$f_{2m-2}(x) = \lambda_{2,2m-2} - f_2(x)$
.	
.	
.	
−	$f_{m-2}(x) = \frac{\lambda_{m-3,m+2}\lambda_{m-2,m+3}}{f_{m+3}(x)}$
+	$f_{m+2}(x) = \lambda_{m-2,m+2} - f_{m-2}(x)$
−	$f_{m-1}(x) = \frac{\lambda_{m-2,m+1}\lambda_{m-1,m+2}}{f_{m+2}(x)}$
+	$f_{m+1}(x) = \lambda_{m-1,m+1} - f_{m-1}(x)$
−	$f_m(x) = \frac{\lambda_{m-1,m}\lambda_{0,m+1}}{f_{m+1}(x)}$

Thus, $\Sigma(x) = f_0(x) + f_m(x) = x + f_m(x) = \lambda_{0,m}$. Assume $f_m(x)$ is a linear term over a linear term. Thus, Thus, $\lim_{x\to\infty} \Sigma(x) = \lim_{x\to\infty} x + f_7(x) = \infty$. Similarly, $\lim_{x\to-\infty} \Sigma(x) = -\infty$. Due to the assumption, a horizontal line can only cross $\Sigma(x)$ 0, 1, or 2 times throughout the *whole* graph.

Denote the singularity of $f_m(x)$ as y_0. Any singularity of $f_m(x)$ is also a singularity of $\Sigma(x)$. It is explained in [1] why $\Sigma(x)$ must have a positive singularity, and since we only have y_0 as the singularity for $\Sigma(x)$, y_0 must be positive.

$\Sigma(x)$ is heavily dominated by $f_m(x)$ near its singularity, y_0. Since $f_m(x)$ has a negative derivative, $\Sigma(x)$ must have a negative slope close to y_0. Thus, we have the following possible graph for $\Sigma(x)$.

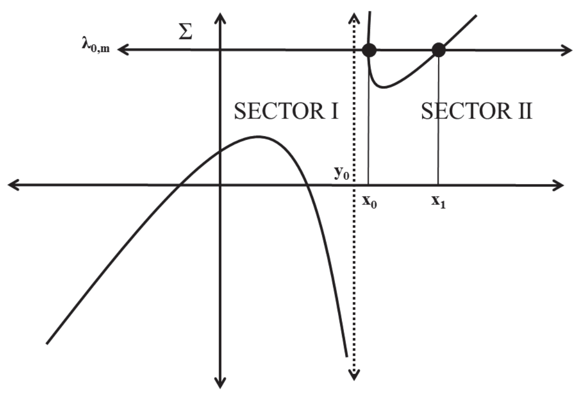


Behavior of Sigma

We call the area to the right of y_0 Sector II and the area to the left Sector I. Note that, at this point, this may not be the exact graph of $\Sigma(x)$. We do not know if $\Sigma(x)$ is ever actually positive in Sector I. There is also the possibility that $\Sigma(x)$ may have some negative values in Sector II.

To prove that the graph giving rise to the R-Multigraph with only one cycle is 2 to 1, we must show that there exists a positive $\lambda_{0,m}$ (represented by a horizonal line) which crosses $\Sigma(x)$ exactly two times and both times within the *same* sector. The crossings must also occur in the same sector for which all $f_j(x)$'s are positive. ([2])

From observation of the graph of $\Sigma(x)$ above, one can draw a *positive* horizontal line representing $\lambda_{0,m}$ in such a way that it crosses $\Sigma(x)$ only twice and in the same sector. We will suppose that these two crossings occur at x_0 and x_1 where $x_0 < x_1$.



Behavior of Sigma

Note that in Sector I, we have no guarantee that the positive horizontal line represeting $\lambda_{0,m}$ would ever cross $\Sigma(x)$ because we don't know if $\Sigma(x)$ will ever actually be positive in Sector I.

What remains left to show is that all $f_j(x)$'s are positive in Sector II.
Obviously, $f_0(x) = x$ is positive in Sector II since the x's in Sector II are positive (recall y_0 is positive). If $f_0(x)$ is positive in Sector II, it follows that $f_1(x) = \frac{\lambda_{0,2m-1}\lambda_{1,m}}{f_0(x)}$ is positive in Sector II also since λ_{ij}'s are positive.

The remaining equations take two alternating forms: a fraction form and a subtraction form. If the λ_{ij}'s in the subtraction equation forms can be chosen in a way such that each subtraction equation is positive in Sector II, the fraction equations will be positive also in Sector II due to the dependency of the equations.

Take an arbitrary equation in subtraction form. We will, for now, ignore the last two equations in the cycle.

$$f_q(x) = \lambda_{i,j} - f_r(x)$$

Note that $f_r(x)$ must be an equation in fraction form, so its derivative sign is negative. Let $\lambda_{i,j} = f_r(y_0) + C_{i,j}$ where $C_{i,j}$ is some positive number. Since $f_r(x)$ has a negative derivative, it must have a negative slope. Thus, $f_r(y_0) > f_r(x)$ if $y_0 < x$. Thus, $f_q(x)$ is positive in Sector II.

Now we consider the last two equations in our cycle. Our choice of y_0 as the singularity will determine $\lambda_{m-1,m+1}$. Since y_0 is the singularity of $f_m(x) = \frac{\lambda_{m-1,m}\lambda_{0,m+1}}{f_{m+1}(x)}$, $f_{m+1}(y_0) = 0$. Now $f_{m+1}(x) = \lambda_{m-1,m+1} - f_{m-1}(x)$. So $f_{m+1}(y_0) = \lambda_{m-1,m+1} - f_{m-1}(y_0) = 0$. Thus, $\lambda_{m-1,m+1} = f_{m-1}(y_0)$. By substitution, $f_{m+1}(x) = f_{m-1}(y_0) - f_{m-1}(x)$. So, in order for $f_{m+1}(x) > 0$, we need $f_{m-1}(y_0) > f_{m-1}(x)$. Now $f_{m-1}(x)$ has a negative slope because it has a negative derivative. Thus, $f_{m-1}(y_0) > f_{m-1}(x)$ if $y_0 < x$. Thus, $f_{m+1}(x)$ is positive in Sector II. It follows then that $f_m(x) = \frac{\lambda_{m-1,m}\lambda_{0,m+1}}{f_{m+1}(x)}$ is positive in Sector II also since λ_{ij}'s are positive.

Thus, all $f_j(x)$'s are positive in Sector II. The graph giving rise to the R-Multigraph with only 1 cycle is 2 to 1.

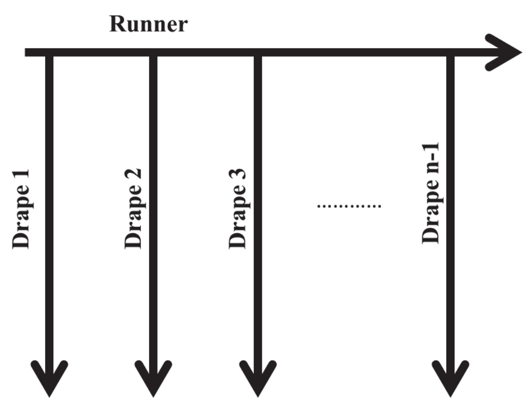
□

It will be demonstrated that this theorem can be generalized. If we can construct the graph such that its R-Multigraph consists of exactly n independent cycles, then the graph is 2^n to 1.

We will begin by showing various ways to construct R-Multigraphs with n independent cycles.

2 The Curtain Algorithm

To construct a R-Multigraph with n independent cycles using the curtain algorithm, the form of the R-Multigraph will be that of a curtain with a runner and $n-1$ drapes.

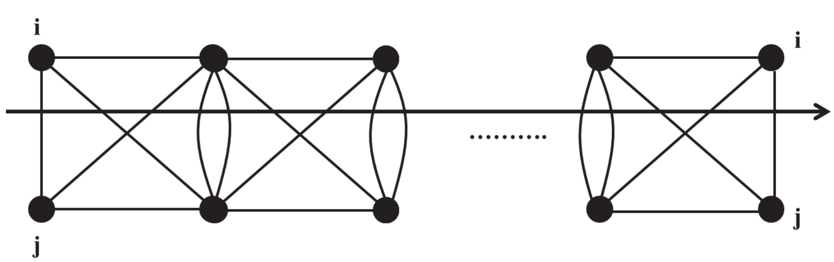


The runner is one cycle and the remaining cycles are drapes.

Suppose n is the number of independent cycles desired for the R-Multigraph.

Case 1: $n = 1$

There is only one runner and no drapes. The runner length must be at least 3. Labeling on the left and right ends of the runner must be identical. All other vertices must be labeled differently.

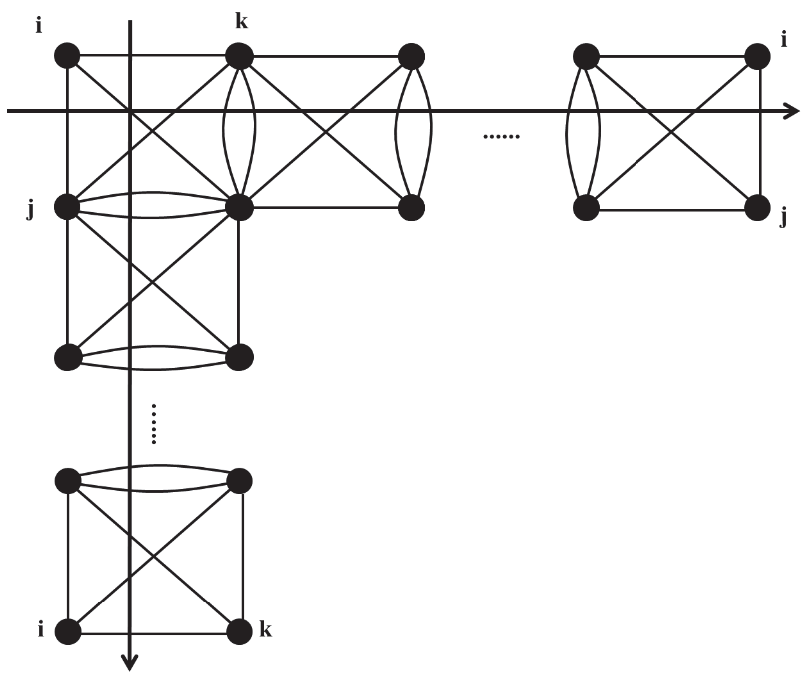


The one cycle is the runner.

The graph giving rise to this R-Multigraph is 2 to 1 by Theorem 1.

Case 2: $n = 2$

There is one runner and one drape. The runner length must be at least 3. The drape length must be at least 3 also (including the quadrilateral at the intersection of the drape and runner). Labeling on the left and right ends of the runner must be identical. Labeling on the top and bottom of the drape must be identical. All other vertices must be labeled differently.

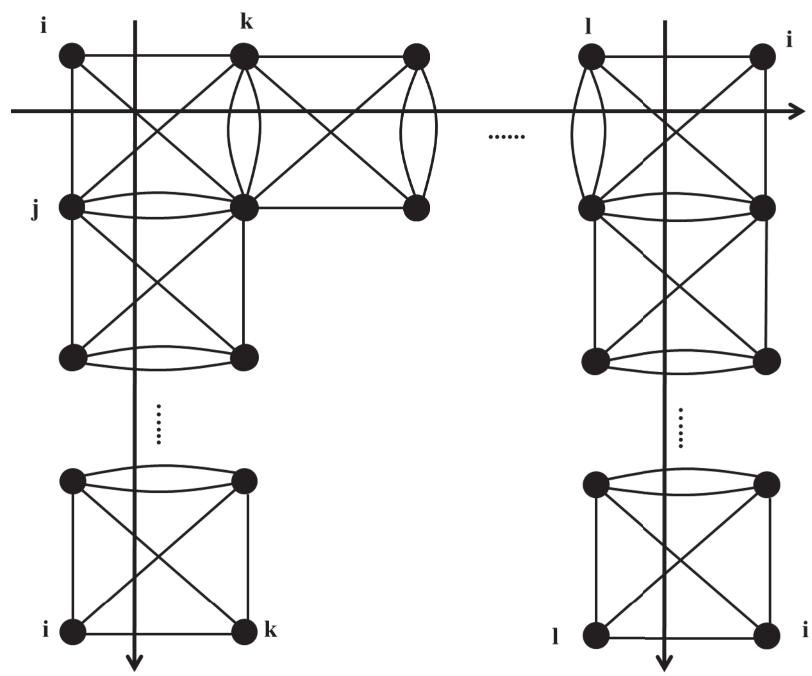


Although the drape intersects the runner at the leftmost quadrilateral in this picture, the drape may be placed anywhere along the runner.

Note that the runner cycle and the drape cycle are independent cycles since there would be no f_j's in common if we were to write equations for each cycle. By Theorem 1, the graph giving rise to this R-Multigraph is 2^2 to 1 because there are two sets of positive conductivities per cycle and there are two independent cycles to consider.

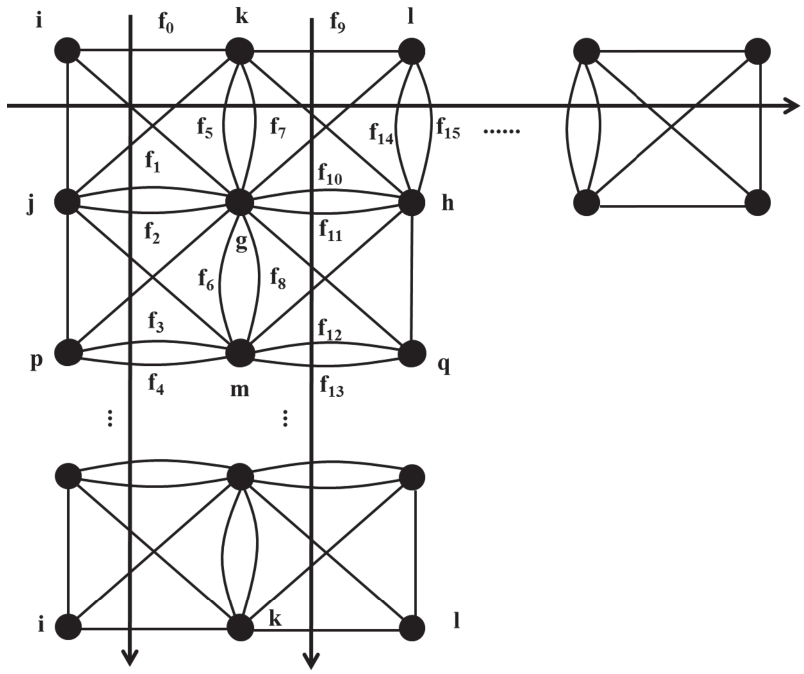
Case 3: $n = 3$

There is one runner and 2 drapes. The runner length must be at length at least 3. Each drape must be of length at least 3 (including the quadrilateral at the intersection of the drape and runner). Labeling on the left and right ends of the runner must be identical. Labeling on the top and bottom of each drape must be identical. All other vertices must be labeled differently.



Although the drapes intersect the runner at the leftmost and rightmost quadrilateral in this picture, the drapes may be placed anywhere along the runner. Also, although this picture depicts the drapes to be of the same length, drapes may be of different lengths.

If there is at least one quadrilateral separating the drapes, it is obvious that the runner and drape cycles are independent. However, if the drapes are connected to each other, it is not so obvious if they are independent.



The drapes are connected together.

At first, it appears as if the drape cycles may not be independent due to the f_j's on the edges that connect the drapes such as f_6 and f_8. However, upon careful examination, it is realized that f_6 and f_8 are completely determined by our choices for $\lambda_{j,m}$, $\lambda_{p,g}$, $\lambda_{j,p}$, $\lambda_{h,m}$, $\lambda_{g,q}$, and $\lambda_{h,q}$.

By the quadrilateral rule,

$$f_6 = \frac{\lambda_{j,m}\lambda_{p,g}}{\lambda_{j,p}}$$

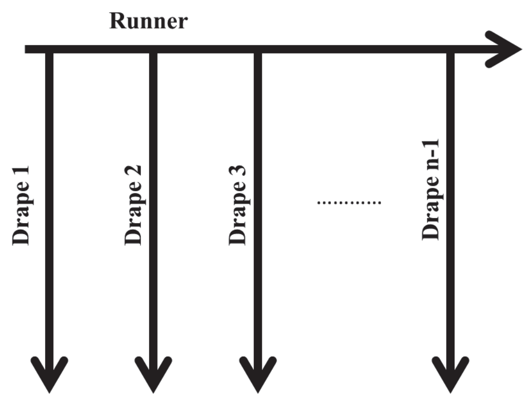
and

$$f_8 = \frac{\lambda_{h,m}\lambda_{g,q}}{\lambda_{h,q}}$$

Similarly, all the f_j's on the edges that connect the drapes are completely determined by our choices for certain λ's. Thus, the drape cycles are independent since they share no f_j's. By Theorem 1, the graph giving rise to this R-Multigraph is 2^3 to 1 because there are two sets of positive conductivities per cycle and there are three independent cycles to consider.

Case 4: $n>3$

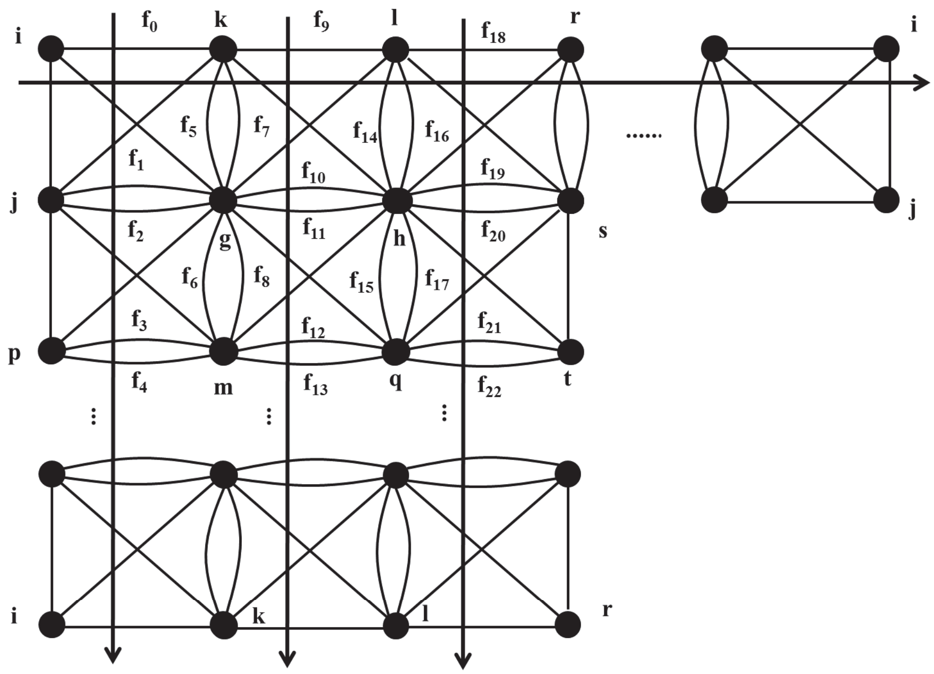
There is one runner and $n-1$ drapes. The runner length must be at least $n-1$. Each drape must be of length at least 3 (including the quadrilateral at the intersection of the drape and runner). Labeling on the left and right ends of the runner must be identical. Labeling on the top and bottom of each drape must be identical. All other vertices must be labeled differently.



There may be gaps between drapes or there may not.

If there is at least one quadrilateral separating the drapes, it is obvious that the runner and drape cycles are independent. By the previous case, we know that if two drapes are connected to each other, they are still independent. But what if more than two drapes are connected? Are the drapes still independent?

We will consider the case where three drapes are connected together. The process of showing that these drapes are independent is similar for any number of drapes that are connected together.



Drapes may be of different length.

f_6 and f_{17} are completely determined by our choices for $\lambda_{j,m}$, $\lambda_{p,g}$, $\lambda_{j,p}$, $\lambda_{h,t}$, $\lambda_{q,s}$, and $\lambda_{s,t}$.

By the quadrilateral rule,

$$f_6 = \frac{\lambda_{j,m}\lambda_{p,g}}{\lambda_{j,p}}$$

and

$$f_{17} = \frac{\lambda_{h,t}\lambda_{q,s}}{\lambda_{s,t}}$$

Once f_6 and f_{17} are determined, however, it is easy to determine f_8 and f_{17} by the response matrix conditions.

$$f_8 = \lambda_{g,m} - f_6$$

and

$$f_{15} = \lambda_{h,q} - f_{17}$$

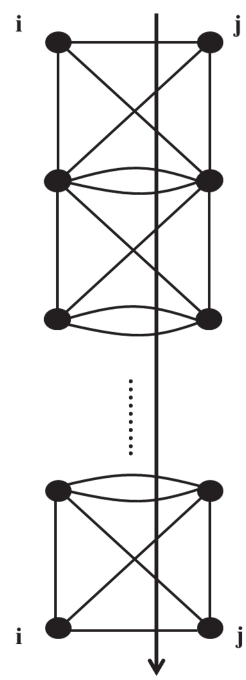
Similarly, all the f_j's on the edges that connect the drapes are completely determined by our choices for certain λ's. Thus, the drape cycles are independent since they share no f_j's. By Theorem 1, the graph giving rise to this R-Multigraph is 2^n to 1 because there are two sets of positive conductivities per cycle and there are n independent cycles to consider.

3 The Layering Algorithm

Although the layering algorithm is similar to the curtain algorithm, the main difference is the lack of a runner. Suppose n is the number of independent cycles desired for the R-Multigraph.

Case 1: $n = 1$

There is only one layer. The layer length must be at least 3. Labeling on the top and bottom ends of the layer must be identical. All other vertices must be labeled differently.

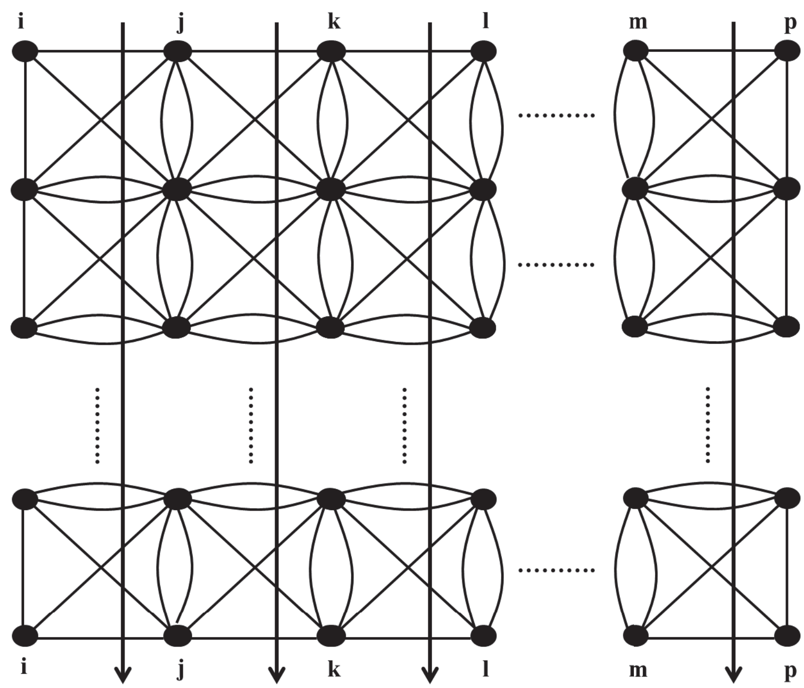


There is one layer for one cycle.

The graph giving rise to this R-Multigraph is 2 to 1 by Theorem 1.

Case 2: $n \geq 2$

There are n layers. Each layer must have length at least 3. Labeling on the top and bottom ends of the layers must be identical. All other vertices must be labeled differently.



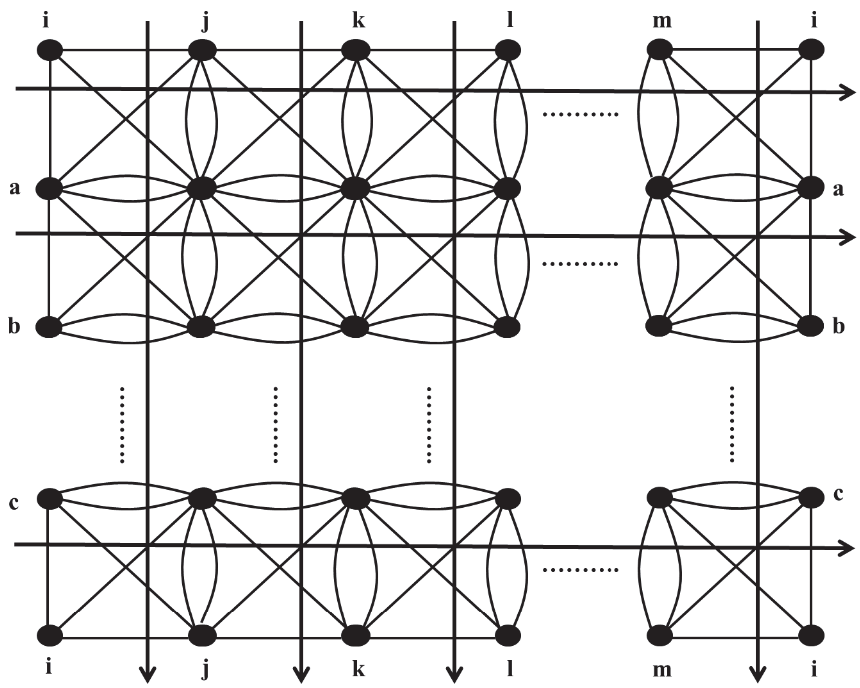
n layers for n independent cycles

At first, it appears as if the layer cycles may not be independent due to the f_j's on the edges that connect the layers. However, these layer cycles are independent from one another by the same reasoning provided in the curtain algorithm. By Theorem 1, the graph giving rise to this R-Multigraph is 2^n to 1 because there are two sets of positive conductivities per cycle and there are n independent cycles to consider.

4 The Compact Rectangle Algorithm

It is important to note that this algorithm can only be applied in the case that we consider R-Multigraphs with 6 or more cycles. Suppose n is the number of independent cycles desired for the R-Multigraph.

The R-Multigraph is a s x t rectangle where $n = s + t$ and $s, t \geq 3$. The labeling on the left and right ends of rows are identical. The labeling on the top and bottom ends of columns are identical. All other vertices must be labeled differently. See the (3,3)-torus in [4] for an example of a R-Multigraph with 6 independent cycles constructed using this algorithm.



The number of rows is greater than or equal to 3. The number of columns is greater than or equal to 3 also.

Although many λ's are shared between cycles, these λ's only show up in the numerator portion of all fractional equations, thus not affecting the positivity of the f_j's. Although the cycles do not appear to be independent, they are independent by the same reasoning provided in the curtain algorithm.

5 Further Research

We have demonstrated various ways to construct R-Multigraphs with n independent cycles so that the graphs giving rise to these R-Multigraphs are 2^n to 1. It appears that there should be a similar way to construct R-Multigraphs so that the graphs giving rise to them are 3^n to 1, 4^n to 1, and so forth.

References

[1] French, Pan, 2^n *to 1 Graphs*, University of Washington Math REU (2004).

[2] Kempton, *n-1 Graphs*, University of Washington Math REU (2011).

[3] Curtis, Morrow, *Inverse Problems for resistor Networks*, World Scientific, v.13, Series on Applied Mathematics, Singapore, (2000).

[4] Wu, *n to 1 Graphs*, University of Washington Math REU (2012).