n to 1 Graphs

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Abstract. Proofs that the Race Track graph in [1] is 2 to 1, the Pseudo 2 to 1 graph is 1 to 1, and the (3,3)-torus in [1] is 64 to 1 are provided so as to demonstrate the procedure of showing that a graph's response matrix can have a certain number of sets of postivie conductivities.

Contents

1 Preliminaries

Definition 1. When referring to a *graph*, G, we mean a connected, undirected, finite graph with no loops along with a vertex set V that can be partitioned into two disjoint subsets, V_{int} and ∂V, where vertices in V_{int} are represented by an open dot and vertices in ∂V are represented by a black dot. ∂V is always nonempty.

Definition 2. Suppose E is the set of edges in a graph. A *conductivity* on a graph G is a function γ: E$\rightarrow\mathbb{R}^+$ that assigns to each edge, e, in a graph to a positive real number, $\gamma(e)$.

Definition 3. A *resistor network*, $\Gamma = (G, \gamma)$, is a graph, G, with a conductivity function, γ.

Definition 4. Suppose $\Gamma = (G, \gamma)$ is a resistor network with n vertices $(v_1, v_2, ...v_n)$. Then the $Kirchhoff\ Matrix$ of $\Gamma = (G, \gamma)$ is an n x n matrix, K, with entries defined as

$$K_{i,j} = \begin{cases} \gamma_{ij} & \text{if } i \neq j, \\ -\sum_{i \neq j} \gamma_{ij} & \text{if } i = j, \end{cases}$$

where

$$\gamma_{ij} = \sum_{\text{all edges e joining } v_i \text{ to } v_j} \gamma(e)$$

Note that if there doesn't exist a direct edge between v_i and v_j in G, then $\gamma_{ij} = 0$.

Here are some notable characteristics of the Kirchhoff Matrix.
1. $\gamma_{ij} \geq 0$, $\forall i \neq j$ (i.e. off-diagonal entries are always positive or 0)
2. Row sums are 0.
3. K is symmetric.

Definition 5. Suppose $\Gamma = (G, \gamma)$ is a resistor network with n vertices $(v_1, v_2, ...v_n)$ where m of them are boundary vertices. The *response matrix* of $\Gamma = (G, \gamma)$ is an m x m matrix, Λ, defined as

$$\Lambda = A - BC^{-1}B^T$$

where A, B, and C are submatrices obtained from the Kirchhoff Matrix in the following fashion.

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B^T} & \mathbf{C} \end{bmatrix}$$

Submatrices of the Kirchhoff Matrix used in obtaining the Response Matrix

A is an m x m submatrix, and C is an $(n-m)$ x $(n-m)$ submatrix. C is shown to be invertible in [3]. Entries of Λ are designated as λ_{ij}.

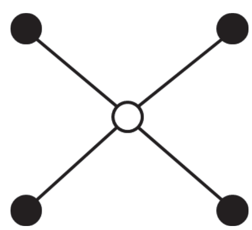
Here are some notable characteristics of the Response Matrix. Note how they are identical to the characteristics of the Kirchhoff Matrix.
1. $\lambda_{ij} \geq 0$, $\forall i \neq j$ (i.e. off-diagonal entries are always positive or 0)
2. Row sums are 0.
3. Λ is symmetric.

Given a response matrix, Λ, and a graph, G, is it possible to recover all the conductivities of the edges, γ, in the graph, G, that created the response matrix? This is known as the inverse problem for resistor networks.

Definition 6. Let $n \geq 1$. A graph is *n to* 1 if there are n unique sets of positive conductivities for the edges that correspond to its response matrix. Note that if G is 1 to 1, then G is uniquely recoverable (i.e. there is only one set of positive conductivities for the edges that corresponds to the response matrix).

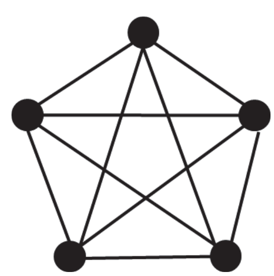
In this paper, we will specifically focus on graphs built out of 4-stars and a technique for determining whether or not a graph built out of 4-stars is uniquely recoverable. If it is not uniquely recoverable, we will be able to tell how many sets of positive conductivities for the edges are valid for its response matrix.

Definition 7. An $n-star$ is a graph with n boundary vertices and one interior vertex, where each boundary vertex is connected by a single edge to the interior vertex and there are no other edges.



A 4-star

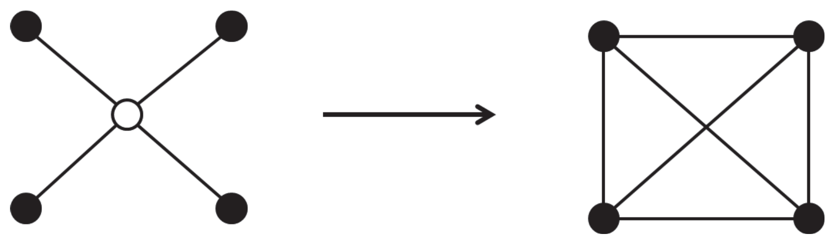
Definition 8. A *complete graph* on n vertices, denoted K_n (so as to distinguish it from the Kirchhoff Matrix, K), is a graph with n boundary vertices, no interior vertices, and a single edge connecting every pair of vertices.



A complete graph on 5 vertices, K_5

Definition 9. A $Star - K\ Transformation$ (also known as "interiorizing") takes a n-star and creates a complete graph from it with the interior vertex removed.

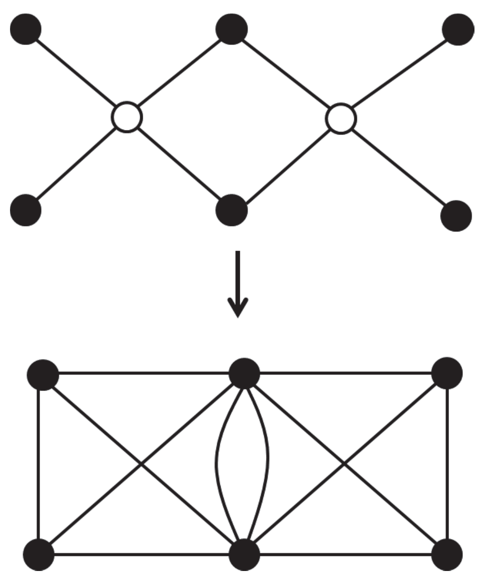
Example 1. Here is an example of a 4-star going through a Star-K Transformation to form a complete graph.



A 4-star undergoes a Star-K Transformation to form the complete graph, K_4.

How would we apply a Star-K Transformation to a graph composed of multiple n-stars? Perform a Star-K Transformation on each n-star in the graph and then connect the results.

Example 2. Here is an example of two 4-stars going through a Star-K Transformation.

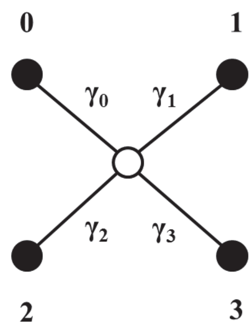


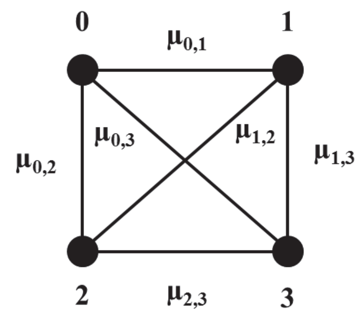
A double edge is formed by connecting two 4-stars that have undergone Star-K Transformations.

Definition 10. The $R - Multigraph$ is the complete graph formed after performing a Star-K Transformation on a graph composed of n-stars. Edges in the R-Multigraph also have their own conductivities labeled as μ_{ij}.

Definition 11. The $R - Matrix$ is a matrix that stores the values of the μ_{ij}'s on the R-Multigraph. If a multiple edge occurs in the R-Multigraph, the R-Matrix separates the multiple edges by storing both values in a set as the entry in the matrix. This differs from the response matrix since a multiple edge in the R-Multigraph results in a sum of their conductivities as the entry in the response matrix. When there is only a single edge in the R-Multigraph, the response matrix and the R-Matrix will share the same entry. See [2] for an example.

Definition 12. Given the following 4-star with its conductivities, we perform a Star-K Transformation on it.





γ_i is the conductivitiy of the edge connecting i to the interior vertex in the 4-star. μ_{ij} is the conductivity of the edge connecting i and j in the R-Multigraph.

The *Quadrilateral Rule* states that

$$\mu_{0,1}\mu_{2,3} = \mu_{0,2}\mu_{1,3} = \mu_{0,3}\mu_{1,2}$$

According to [3], $\mu_{ij} = \frac{\gamma_i \gamma_j}{\sigma}$ where σ is the sum of all conductivities of edges in the 4-star. Thus, the quadrilateral rule holds since

$$\mu_{0,1}\mu_{2,3} = (\frac{\gamma_0\gamma_1}{\sigma})(\frac{\gamma_2\gamma_3}{\sigma}) = (\frac{\gamma_0\gamma_2}{\sigma})(\frac{\gamma_1\gamma_3}{\sigma}) = \mu_{0,2}\mu_{1,3} = (\frac{\gamma_0\gamma_3}{\sigma})(\frac{\gamma_1\gamma_2}{\sigma}) = \mu_{0,3}\mu_{1,2}$$

According to [4], if the resistor network Γ is already its own complete graph, then the response matrix of Γ is easily calculated. An entry λ_{ij} of the response matrix is directly equal to the conductivity of the edge connecting vertices v_i and v_j in the resistor network.

If the resistor network is not a complete graph, then there exists some complete graph, say K_n, whose edge conductivities, μ_{ij}'s, can be used to calculate the entries, λ_{ij} of the response matrix of the resistor network. We say that the resistor network is *response equivalent* to the complete graph K_n.

Theorem 1. Suppose that the resistor network Γ is a graph composed of n-stars. Let K_n be the complete graph obtained by performing a Star-K transformation on Γ. Then Γ is response equivalent to the complete graph K_n iff the conductivities on K_n satisfy the quadrilateral rule.

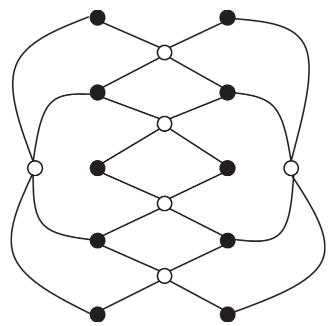
Proof. See [4]. □

Suppose we are given a resistor network composed of n-star(s) and wish to find the conductivities of its edges. We apply a Star-K transformation to obtain the complete graph. If we can parametrize our response matrix (choosing certain values for some λ_{ij}'s) such that the conductivities in the complete graph satisfy the quadrilateral rule, then, by Theorem 1, our resistor network would be response equivalent to the complete graph. By [2], this means that if there only exists one edge between two vertices i and j in the complete graph, the λ_{ij} entry in the response matrix is directly equal to the conductivity of the edge joining i and j in the complete graph. However, we must exercise caution in the cases where are multiple edges between two vertices. Suppose there are multiple edges joining the vertices i and j in the complete graph. In this case, the λ_{ij} entry in the reponse matrix is the sum of the conductivities of all multiples edges connecting i and j in the complete graph.

Knowing only the sum of edges in the complete graph is not enough to recover conductivities of the edges of the resistor network. It is necessary to obtain the conductivities of each separate edge in the complete graph. See equation 2 in [2] for a formula that acquires the conductivities of the edges in the graph of the resistor network from the conductivities of edges on the complete graph. However, this equation is more difficult to use if the edges of the complete graph have negative conductivities. Thus, we will use the general formula in [5] in these instances. If our response matrix can only recover 1 set of positive conductivities for the resistor network, our resistor network graph is said to be uniquely recoverable (1 to 1). However, if our response matrix can recover mutiple working sets of positive conductivities, the resistor network is not uniquely recoverable. We say that a graph for a resistor network is *n to* 1 if there are n sets of positive conductivities valid for a single response matrix.

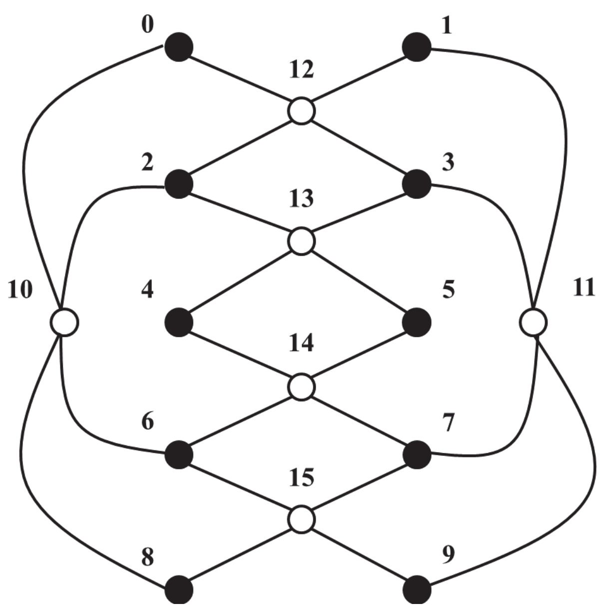
2 The Race Track graph

Example 3. Consider the Race Track graph in [1].

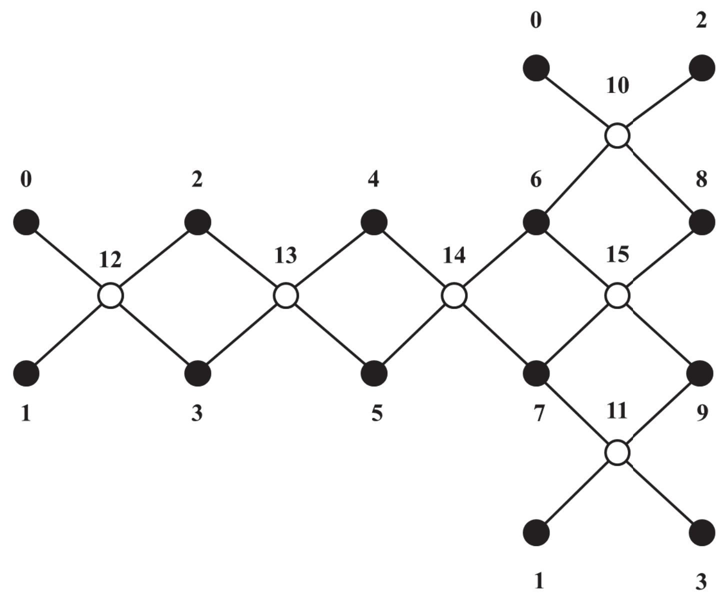


We shall prove that it is is 2 to 1. In other words, we want to show that there exists two sets of positive conductivities that are valid for its response matrix.

Proof. We begin by labeling the graph's vertices.

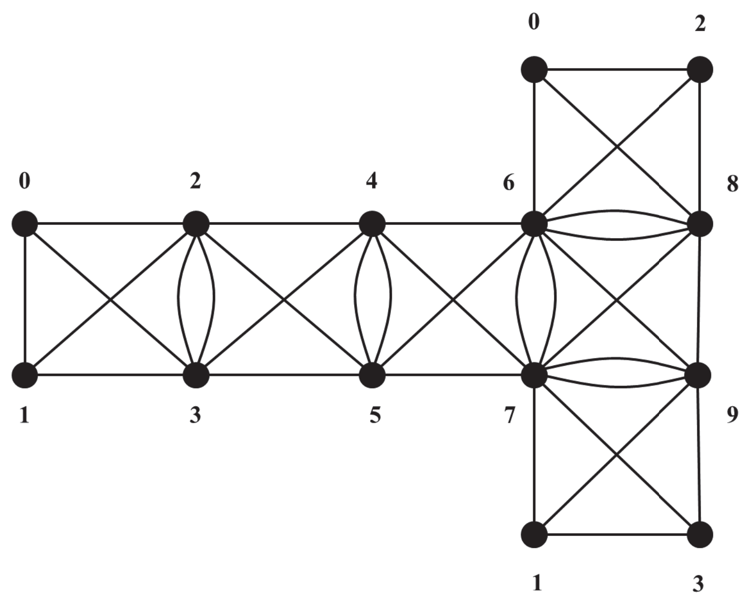


A redrawing of the Race Track graph by careful examination of the hidden stars might be beneficial before a Star-K Tranformation. For example, there is a 4-star on the left formed by the interior vertex 10 and the boundary vertices 0, 2, 6, and 8.



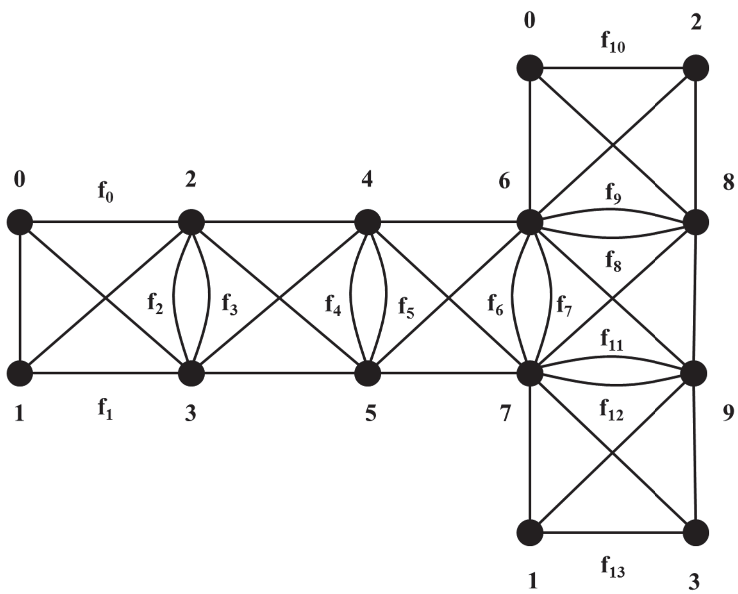
The Race Track graph Redrawn

We perform a Star-K transformation on the redrawn Race Track graph.



The R-Multigraph of the Race Track graph

We will assume that the conductivities of the edges in the R-Multigraph satsify the quadrilateral rule. By Theorem 1, the resistor network is response equivalent to the R-Multigraph. By the definition of response equivalent, this would mean that the λ_{ij} entry in the response matrix is directly equal to the conductivity of the edge joining i and j in the complete graph if there only exists one edge between two vertices i and j. If there exists two edges connecting i and j, the λ_{ij} entry in the reponse matrix is the sum of the conductivities of the two edges connecting i and j in the complete graph. This is what the author of [2] refers to as the "response matrix condition."
To find the conducitivities of edges in the resistor network, we need the conductivities of each edge in the R-Multigraph. We shall label the edges in the double edges with f_j's.

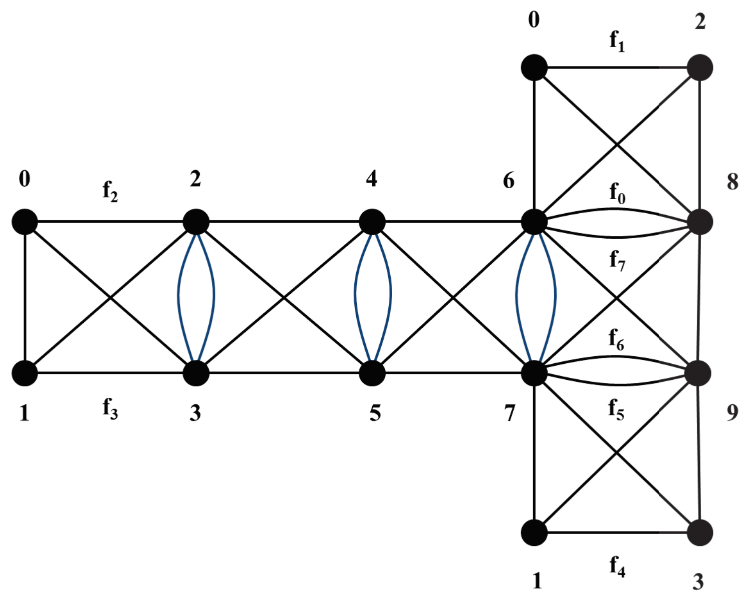


R-Multigraph

To start, assume $f_0(x) = x$. From there, one can determine equations for all the $f_j(x)$'s knowing that the response matrix condition and quadrilateral rule must be satisfied by our assumption. We obtain the following system of equations

$f_0(x) = x$
$f_1(x) = \frac{\lambda_{1,2}\lambda_{0,3}}{f_0(x)}$ by quadrilateral rule ($\lambda_{0,3}\lambda_{1,2} = f_0(x)f_1(x)$)
$f_2(x) = \frac{\lambda_{1,2}\lambda_{0,3}}{\lambda_{0,1}}$
$f_3(x) = \lambda_{2,3} - f_2(x)$ by response matrix condition ($\lambda_{2,3} = f_3(x) + f_2(x)$)
$f_4(x) = \frac{\lambda_{2,4}\lambda_{3,5}}{f_3(x)}$
$f_5(x) = \lambda_{4,5} - f_4(x)$
$f_6(x) = \frac{\lambda_{4,6}\lambda_{5,7}}{f_5(x)}$
$f_7(x) = \lambda_{6,7} - f_6(x) = \frac{\lambda_{7,8}\lambda_{6,9}}{\lambda_{8,9}}$
.
.
.
At this point, it is realized that $f_2(x)$ is completely determined by our choices for $\lambda_{1,2}$, $\lambda_{0,3}$, and $\lambda_{0,1}$. Similarly, $f_7(x)$ is completely determined by our choices for $\lambda_{7,8}$, $\lambda_{6,9}$, and $\lambda_{8,9}$. However, once $f_7(x)$ is known, it is easy to obtain $f_6(x)$ by using the response matrix condition $f_7(x) = \lambda_{6,7} - f_6(x)$. Once $f_6(x)$ is known, it is easy to obtain $f_5(x)$ using the quadrilateral rule $f_6(x) = \frac{\lambda_{4,6}\lambda_{5,7}}{f_5(x)}$ and our choices for $\lambda_{4,6}$ and $\lambda_{5,7}$. We continue this process to obtain $f_4(x)$ and $f_3(x)$. Thus, $f_2(x)$ to $f_7(x)$ can be easily determined after certain λ_{ij}'s

are chosen. These edges will be relabeled in blue. f_j's will also be completely relabeled.



Relabeled R-Multigraph

Assume $f_0(x) = x$ and determine equations for all the $f_j(x)$'s knowing that the response matrix condition and quadrilateral rule must be satisfied. Note that although it may not be clear from the picture, there are 2 edges between vertices 0 and 2. We will also include the sign of the derivative for each $f_j(x)$. The purpose of this will be made clear later.

Sign of Derivative	Equation
$+$	$f_0(x) = x$
$-$	$f_1(x) = \frac{\lambda_{0,6}\lambda_{2,8}}{f_0(x)}$
$+$	$f_2(x) = \lambda_{0,2} - f_1(x)$
$-$	$f_3(x) = \frac{\lambda_{0,3}\lambda_{1,2}}{f_2(x)}$
$+$	$f_4(x) = \lambda_{1,3} - f_3(x)$
$-$	$f_5(x) = \frac{\lambda_{1,7}\lambda_{3,9}}{f_4(x)}$
$+$	$f_6(x) = \lambda_{7,9} - f_5(x)$
$-$	$f_7(x) = \frac{\lambda_{7,8}\lambda_{6,9}}{f_6(x)}$

Suppose x_0 is a value such that $f_j(x_0)$ is non-positive. Using $f_j(x_0)$ in the equation for obtaining conductivities of the edges in the resistor network from the conductivities of edges in the complete graph may yield negative values or

0. But γ_i>0. Thus, *all* $f_j(x)$'s must be positive. This restricts the range of values that can be chosen for certain λ_{ij}'s. This will be examined in greater detail.

Recall we want to prove that there exists two sets of positive conductivities for a single response matrix. Due to the equation for obtaining conductivities of edges in the resistor network from the conductivities of edges in the complete graph, there needs to be two sets of positive $f_j(x)$'s. To find these two sets of positive $f_j(x)$'s, consider

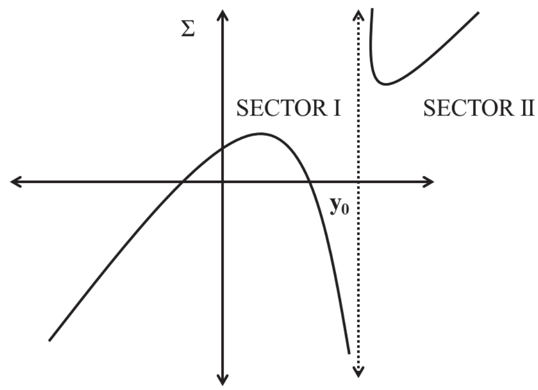
$$\Sigma(x) = f_0(x) + f_7(x) = x + f_7(x) = \lambda_{6,8}$$

Examining the behavior of $\Sigma(x)$ will reveal whether or not there are two sets of positive conductivities.

Assume that $f_7(x)$ is a linear term over a linear term. Thus, $\lim_{x\to\infty} \Sigma(x) = \lim_{x\to\infty} x + f_7(x) = \infty$. Similarly, $\lim_{x\to-\infty} \Sigma(x) = -\infty$. Due to the assumption, a horizontal line can only cross $\Sigma(x)$ 0, 1, or 2 times throughout the *whole* graph.

Denote the singularity of $f_7(x)$ as y_0. Any singularity of $f_7(x)$ is also a singularity of $\Sigma(x)$. It is explained in [1] why $\Sigma(x)$ must have a positive singularity, and since we only have y_0 as the singularity for $\Sigma(x)$, y_0 must be positive.

$\Sigma(x)$ is heavily dominated by $f_7(x)$ near its singularity, y_0. Since $f_7(x)$ has a negative derivative, $\Sigma(x)$ must have a negative slope close to y_0. Thus, we have the following possible graph for $\Sigma(x)$.

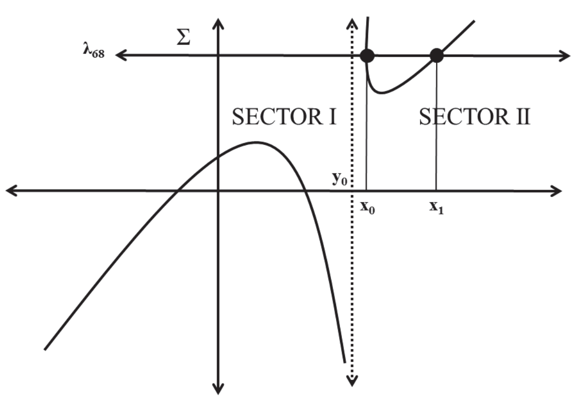


Behavior of Sigma

We call the area to the right of y_0 Sector II and the area to the left Sector I. Note that, at this point, this may not be the exact graph of $\Sigma(x)$. We do not know if $\Sigma(x)$ is ever actually positive in Sector I. There is also the possibility that $\Sigma(x)$ may have some negative values in Sector II.

To prove that the Race Track graph is 2 to 1, we must show that there exists a positive $\lambda_{6,8}$ (represented by a horizonal line) which crosses $\Sigma(x)$ exactly two times and both times within the *same* sector. The crossings must also occur in the same sector for which all $f_j(x)$'s are positive. ([2])

From observation of the graph of $\Sigma(x)$ above, one can draw a *positive* horizontal line representing $\lambda_{6,8}$ in such a way that it crosses $\Sigma(x)$ only twice and in the same sector. We will suppose that these two crossings occur at x_0 and x_1 where $x_0<x_1$.



Behavior of Sigma

Note that in Sector I, we have no guarantee that the positive horizontal line represeting $\lambda_{6,8}$ would ever cross $\Sigma(x)$ because we don't know if $\Sigma(x)$ will ever actually be positive in Sector I.

What remains left to show is that all $f_j(x)$'s are positive in Sector II.

Obviously, $f_0(x) = x$ is positive in Sector II since the x's in Sector II are positive (recall y_0 is positive). If $f_0(x)$ is positive in Sector II, it follows that $f_1(x) = \frac{\lambda_{0,6}\lambda_{2,8}}{f_0(x)}$ is positive in Sector II also since λ_{ij}'s are positive.

Recall that $f_2(x) = \lambda_{0,2} - f_1(x)$. We can choose our $\lambda_{0,2}$ in such a way that $f_2(x)$ will be positive in Sector II. Suppose z_0 is the root of $f_2(x) = \lambda_{0,2} - f_1(x)$. Then $f_2(z_0) = \lambda_{0,2} - f_1(z_0) = 0$. In order for $f_2(z_0)>0$, we need $\lambda_{0,2}>f_1(z_0)$. Let $\lambda_{0,2} = f_1(z_0) + C_{0,2}$ where $C_{0,2}$ is some positive number. By substitution, $f_2(x) = \lambda_{0,2} - f_1(x) = (f_1(z_0) + C_{0,2}) - f_1(x)$. Now $f_1(x)$ has a negative slope because of its negative derivative. Thus, $f_1(z_0)>f_1(x)$ if $z_0<x$. Let's choose $z_0 = y_0$, the singularity of $\Sigma(x)$. Then $f_1(y_0)>f_1(x)$ if $y_0<x$. So, if we restrict our x to be on the right of y_0, then $f_2(x) = f_1(y_0) - f_1(x) + C_{0,2}>0$ (recall $C_{0,2}$ is positive). So, by choosing $\lambda_{0,2}$ to be $f_1(y_0) + C_{0,2}$, we have guaranteed that $f_2(x)$ will be positive in Sector II.

It might be questionable to the reader as to why one might need to add $C_{0,2}$ into $\lambda_{0,2}$ since $f_2(x)$ would be positive in Sector II anyway without $C_{0,2}$. However, if $\lambda_{0,2} = f_1(y_0)$, this will create a problem in calculating the first term, $f_5(y_0)$, in $f_6(x) = f_5(y_0) - f_5(x)$. To calculate $f_5(y_0)$, one would need $f_4(y_0)$ since $f_5(x) = \frac{\lambda_{1,7}\lambda_{3,9}}{f_4(x)}$. To calculate $f_4(y_0)$, we would need $f_3(y_0)$ since $f_4(x) = \lambda_{1,3} - f_3(x)$. To obtain $f_3(y_0)$, $f_2(y_0)$ is needed since $f_3(x) = \frac{\lambda_{0,3}\lambda_{1,2}}{f_2(x)}$. However, if $\lambda_{0,2} = f_1(y_0)$, then $f_2(y_0) = \lambda_{0,2} - f_1(y_0) = f_1(y_0) - f_1(y_0) = 0$. But, if $f_2(y_0) = 0$ then $f_3(y_0)$ would have a 0 in its denominator. Thus, to avoid such an issue, we let $\lambda_{0,2} = f_1(z_0) + C_{0,2}$ where $C_{0,2}$ is a positive number.

In general, we add a $C_{i,j}$ to a λ_{ij} when choosing the λ_{ij} in such a way to make a $f_{ij}(x)$ positive in a certain sector. However, we do *not* add a C_{ij} to a λ_{ij} when we are creating a singularity for a f_{ij}.

If $f_2(x)$ is positive in Sector II, it follows that $f_3(x) = \frac{\lambda_{0,3}\lambda_{1,2}}{f_2(x)}$ is positive in Sector II also since λ_{ij}'s are positive.

Recall that $f_4(x) = \lambda_{1,3} - f_3(x)$. We can choose our $\lambda_{1,3}$ in such a way that $f_4(x)$ is positive in Sector II. Suppose z_1 is the root of $f_4(x) = \lambda_{1,3} - f_3(x)$. Then $f_4(z_1) = \lambda_{1,3} - f_3(z_1) = 0$. In order for $f_4(z_1)>0$, we need $\lambda_{1,3}>f_3(z_1)$. Let $\lambda_{1,3} = f_3(z_1) + C_{1,3}$ where $C_{1,3}$ is some positive number. By substitution, $f_4(x) = \lambda_{1,3} - f_3(x) = (f_3(z_1) + C_{1,3}) - f_3(x)$. Now $f_3(x)$ has a negative slope because of its negative derivative. Thus, $f_3(z_1)>f_3(x)$ if $z_1<x$. Let's choose $z_1 = y_0$, the singularity of $\Sigma(x)$. Then $f_3(y_0)>f_3(x)$ if $y_0<x$. So, if we restrict

our x to be on the right of y_0, then $f_4(x) = f_3(y_0) - f_3(x) + C_{1,3}>0$ (recall $C_{1,3}$ is positive). So, by choosing $\lambda_{1,3}$ to be $f_3(y_0) + C_{1,3}$, we have guaranteed that $f_4(x)$ will be positive in Sector II. The reasoning as to why we choose to add $C_{1,3}$ into $\lambda_{1,3}$ is similar to the reasoning as to why we added $C_{0,2}$ into $\lambda_{0,2}$.

If $f_4(x)$ is positive in Sector II, it follows that $f_5(x) = \frac{\lambda_{1,7}\lambda_{3,9}}{f_4(x)}$ is positive in Sector II also since λ_{ij}'s are positive.

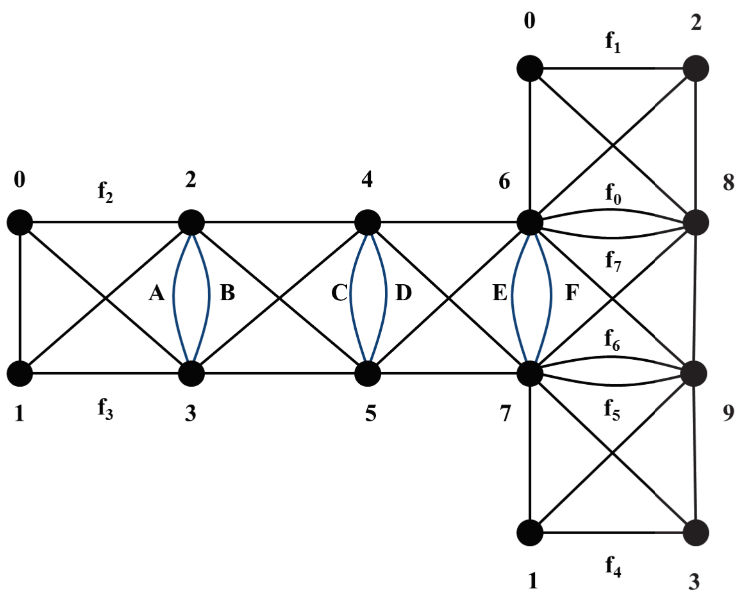
Our choice of y_0 as the singularity of $\Sigma(x)$ will determine $\lambda_{7,9}$. Since y_0 is the singularity of $f_7(x) = \frac{\lambda_{7,8}\lambda_{6,9}}{f_6(x)}$, $f_6(y_0) = 0$. Now $f_6(x) = \lambda_{7,9} - f_5(x)$. So $f_6(y_0) = \lambda_{7,9} - f_5(y_0) = 0$. Thus, $\lambda_{7,9} = f_5(y_0)$. By substitution, $f_6(x) = f_5(y_0) - f_5(x)$. So, in order for $f_6(x)>0$, we need $f_5(y_0)>f_5(x)$. Now $f_5(x)$ has a negative slope because it has a negative derivative. Thus, $f_5(y_0)>f_5(x)$ if $y_0<x$. Thus, $f_6(x)$ is positive in Sector II. It follows then that $f_7(x) = \frac{\lambda_{7,8}\lambda_{6,9}}{f_6(x)}$ is positive in Sector II also since λ_{ij}'s are positive.

Thus, all $f_j(x)$'s are positive in Sector II. The Race Track graph is 2 to 1. $\square$

Let's attempt to create a 2 to 1 graph by choosing appropriate λ_{ij}'s using the Race Track graph. We will use the algorithm outlined in [2].

Step 1: Pick a positive value, y_0, to be the singularity of $f_7(x)$. This will also be the singularity of $\Sigma(x)$. Let y_0 be 3.

Step 2: Choose values of the λ_{ij}'s in the quadrilateral to uphold the quadrilateral rule. We will label the blue edges as A, B, C, D, E, and F. By using our choices for λ_{ij}'s, we can obtain values for these edges. After substitution, we obtain a new set of equations for our $f_j(x)$'s.



R-Multigraph of the Race Track

By the quadrilateral rule

$$f_0(x)f_1(x) = \lambda_{0,6}\lambda_{2,8} = \lambda_{2,6}\lambda_{0,8}$$

Choose $\lambda_{0,6} = 1$, $\lambda_{2,8} = 1$, and $\lambda_{2,6} = 1$. This forces $\lambda_{0,8} = 1$.

By the quadrilateral rule

$$f_7(x)f_6(x) = F\lambda_{8,9} = \lambda_{6,9}\lambda_{7,8}$$

Choose $\lambda_{8,9} = 1$, $\lambda_{6,9} = 1$, and $\lambda_{7,8} = 1$. This forces $F = 1$.

By the quadrilateral rule

$$f_4(x)f_5(x) = \lambda_{1,7}\lambda_{3,9} = \lambda_{3,7}\lambda_{1,9}$$

Choose $\lambda_{1,7} = 1$, $\lambda_{3,9} = 1$, and $\lambda_{3,7} = 1$. This forces $\lambda_{1,9} = 1$.

By the quadrilateral rule

$$f_2(x)f_3(x) = \lambda_{0,1}A = \lambda_{0,3}\lambda_{1,2}$$

Choose $\lambda_{0,1} = 1$, $\lambda_{0,3} = 1$, and $\lambda_{1,2} = 1$. This forces $A = 1$.

Now $F = 1$, and by the response matrix condition, $E = \lambda_{6,7} - F = \lambda_{6,7} - 1$. If we choose $\lambda_{6,7} = 1$, then $E = 0$, but we must have positive conductivities. So, let's choose $\lambda_{6,7} = 2$. Thus, $E = 1$.

By the quadrilateral rule, $D = \frac{\lambda_{5,6}\lambda_{4,7}}{E}$. Since $E = 1$, $D = \frac{\lambda_{5,6}\lambda_{4,7}}{1} = \lambda_{5,6}\lambda_{4,7}$. Let's choose $\lambda_{5,6} = 1$ and $\lambda_{4,7} = 1$. Thus, $D = 1$.

Now $D = 1$, and by the response matrix condition, $C = \lambda_{4,5} - D = \lambda_{4,5} - 1$. In order to have positive conductivities, $\lambda_{4,5}>1$. So, let's choose $\lambda_{4,5} = 2$. Thus, $C = 1$.

By the quadrilateral rule, $B = \frac{\lambda_{2,5}\lambda_{3,4}}{C} = \lambda_{2,5}\lambda_{3,4}$ since $C = 1$. Let's choose $\lambda_{2,5} = 1$ and $\lambda_{3,4} = 1$. Thus, $B = 1$.

Now, by the response matrix condition, $A = \lambda_{2,3} - B$. We already know that $A = 1$ and $B = 1$. This forces $\lambda_{2,3} = 2$.

By the quadrilateral rule

$$DE = \lambda_{4,6}\lambda_{5,7} = \lambda_{5,6}\lambda_{4,7}$$

We already know that $D = 1$, $E = 1$, $\lambda_{5,6} = 1$, and $\lambda_{4,7} = 1$. This forces $\lambda_{4,6}\lambda_{5,7} = 1$. Let's choose $\lambda_{4,6} = 1$ and $\lambda_{5,7} = 1$.

By the quadrilateral rule

$$BC = \lambda_{2,4}\lambda_{3,5} = \lambda_{3,4}\lambda_{2,5}$$

We already know that $B = 1$, $C = 1$, $\lambda_{3,4} = 1$, and $\lambda_{2,5} = 1$. This forces $\lambda_{2,4}\lambda_{3,5} = 1$. Let's choose $\lambda_{2,4} = 1$ and $\lambda_{3,5} = 1$.

Now we can substitute and obtain a new set of equations for our $f_j(x)$'s.

Sign of Derivative	Equation
$+$	$f_0(x) = x$
$-$	$f_1(x) = \frac{1}{f_0(x)}$
$+$	$f_2(x) = \lambda_{0,2} - f_1(x)$
$-$	$f_3(x) = \frac{1}{f_2(x)}$
$+$	$f_4(x) = \lambda_{1,3} - f_3(x)$

$-$	$f_5(x) = \frac{1}{f_4(x)}$
$+$	$f_6(x) = \lambda_{7,9} - f_5(x)$
$-$	$f_7(x) = \frac{1}{f_6(x)}$

Step 3: Obtain the other λ_{ij} values that were determined by our choice of singularity for $\Sigma(x)$ and by making $f_j(x)$'s positive in Sector II (the region to the right of $y_0 = 3$).

We have established that in order for $f_2(x)$ to be positive we need $\lambda_{0,2} = f_1(y_0) + C_{0,2}$ where $C_{0,2}$ is some positive number. Let $C_{0,2} = 1$. Since $y_0 = 3$, $\lambda_{0,2} = f_1(3) + 1 = \frac{1}{3} + 1 = \frac{4}{3}$.

We have established that in order for $f_4(x)$ to be positive we need $\lambda_{1,3} = f_3(y_0) + C_{1,3}$ where $C_{1,3}$ is some positive number. Let $C_{1,3} = 1$. Since $y_0 = 3$, $\lambda_{1,3} = f_3(3) + 1 = 1 + 1 = 2$.

We have established that in order for y_0 to be the singularity of $\Sigma(x)$, $\lambda_{7,9} = f_5(y_0)$. Since $y_0 = 3$, $\lambda_{7,9} = f_5(3) = 1$.

After substitution, we obtain a new set of equations for our $f_j(x)$'s.

Sign of Derivative	Equation
$+$	$f_0(x) = x$
$-$	$f_1(x) = \frac{1}{x}$
$+$	$f_2(x) = \frac{4}{3} - \frac{1}{x} = \frac{4x-3}{3x}$
$-$	$f_3(x) = \frac{3x}{4x-3}$
$+$	$f_4(x) = 2 - \frac{3x}{4x-3} = \frac{5x-6}{4x-3}$
$-$	$f_5(x) = \frac{4x-3}{5x-6}$
$+$	$f_6(x) = 1 - \frac{4x-3}{5x-6} = \frac{x-3}{5x-6}$
$+$	$f_7(x) = \frac{5x-6}{x-3}$

Thus, $\Sigma(x) = x + \frac{5x-6}{x-3}$

Step 4: Choose an x-coordinate for one of the two crossings between $\lambda_{6,8}$ and $\Sigma(x)$ in Sector II to determine $\lambda_{6,8}$. Check if any f_j's are negative by this choice of x. Create the response matrix. For the specific x-coordinate chosen, create the R-Matrix. Find the conductivities of edges in the Race Track graph corresponsding to the choice of x-coordinate.

We will choose $x = 4$ for the x-coordinate for a crossing in Sector II. All f_j's are positive at $x = 4$.

Equation
$f_0(4) = 4$

$f_1(4) = \frac{1}{4}$

$f_2(4) = \frac{13}{12}$

$f_3(4) = \frac{12}{13}$

$f_4(4) = \frac{14}{13}$

$f_5(4) = \frac{13}{14}$

$f_6(4) = \frac{1}{14}$

$f_7(4) = 14$

Thus, $\Sigma(4) = 4 + 14 = 18 = \lambda_{6,8}$. This is a positive $\lambda_{6,8}$ as required. Now we have all the λ_{ij}'s we need, and using the fact that row sums are 0 and response matrices are symmetrical, we can create the response matrix, Λ. Note that $\lambda_{ij} = 0$ if there doesn't exist a direct edge between vertices i and j in the R-Multigraph.

$$\Lambda = \begin{bmatrix} \frac{-16}{3} & 1 & \frac{4}{3} & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -6 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 1 \\ \frac{4}{3} & 1 & \frac{-22}{3} & 2 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & -8 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -6 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & -6 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & -25 & 2 & 18 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 2 & -8 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 18 & 1 & -22 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & -5 \end{bmatrix}$$

The response matrix of the Race Track graph

For $x = 4$, we have the following R-Matrix.

$$R = \begin{bmatrix} \frac{-16}{3} & 1 & \{\frac{1}{4}, \frac{13}{12}\} & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -6 & 1 & \{\frac{12}{13}, \frac{14}{13}\} & 0 & 0 & 0 & 1 & 0 & 1 \\ \{\frac{1}{4}, \frac{13}{12}\} & 1 & \frac{-22}{3} & \{1,1\} & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & \{\frac{12}{13}, \frac{14}{13}\} & \{1,1\} & -8 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -6 & \{1,1\} & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & \{1,1\} & -6 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & -25 & \{1,1\} & \{4,14\} & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & \{1,1\} & -8 & 1 & \{\frac{13}{14}, \frac{1}{14}\} \\ 1 & 0 & 1 & 0 & 0 & 0 & \{4,14\} & 1 & -22 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & \{\frac{13}{14}, \frac{1}{14}\} & 1 & -5 \end{bmatrix}$$

The R-Matrix corresponding to $x = 4$

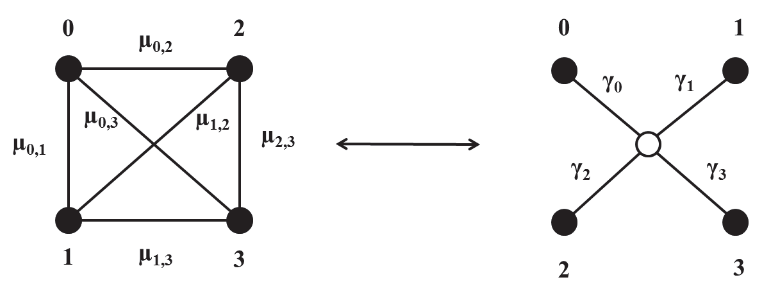
Since the conductivities of all edges in the complete graph are positive for $x = 4$, we can easily use equation 2 in [2] to obtain the conductivities of edges (γ_i's) in the Race Track graph.

$$\gamma_i = \alpha_i \sum_m \alpha_m$$

where

$$\alpha_i = \sqrt{\frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}}}$$

Note that i, j, and k are vertices all within the *same* quadrilateral, and $\sum_m \alpha_m$ is the sum of all the α_m's in the quadrilateral. Let us figure out the conductivities of edges in the Race Track graph using the first quadrilateral.



Quadrilateral One and the 4-star compared to it

$\alpha_0 = \sqrt{\frac{\mu_{0,2}\mu_{0,3}}{\mu_{2,3}}} = \sqrt{\frac{\frac{13}{12}*1}{1}} = \sqrt{\frac{13}{12}}$

Note that $\alpha_0 = \sqrt{\frac{\mu_{0,2}\mu_{0,1}}{\mu_{2,1}}}$ and $\alpha_0 = \sqrt{\frac{\mu_{0,3}\mu_{0,1}}{\mu_{3,1}}}$ all work too. Also, $\mu_{0,2} = f_2(4)$ and $\mu_{2,3} = A = 1$ since we are looking at only the first quadrilateral.

$\alpha_1 = \sqrt{\frac{\mu_{0,1}\mu_{1,3}}{\mu_{0,3}}} = \sqrt{\frac{12}{13}}$

$\alpha_2 = \sqrt{\frac{\mu_{0,2}\mu_{2,3}}{\mu_{0,3}}} = \sqrt{\frac{13}{12}}$

$\alpha_3 = \sqrt{\frac{\mu_{1,3}\mu_{2,3}}{\mu_{1,2}}} = \sqrt{\frac{12}{13}}$

Since $\sum_m \alpha_m$ is the sum of all the α_m's in the quadrilateral,

$$\sum_m \alpha_m = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \frac{50}{\sqrt{13}\sqrt{12}}$$

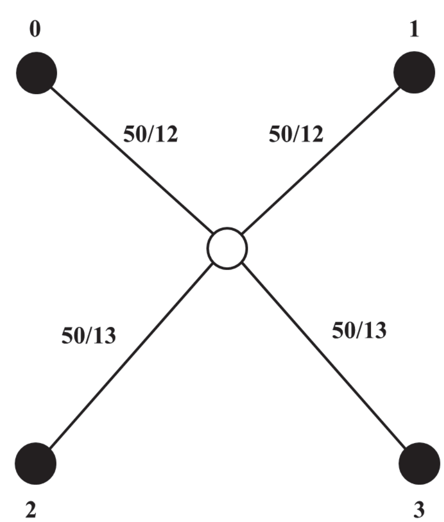
Thus, we have the following conductivities for the edges in the Race Track graph.

$\gamma_0 = \alpha_0 \sum_m \alpha_m = \sqrt{\frac{13}{12}}(\frac{50}{\sqrt{13}\sqrt{12}}) = \frac{50}{12}$

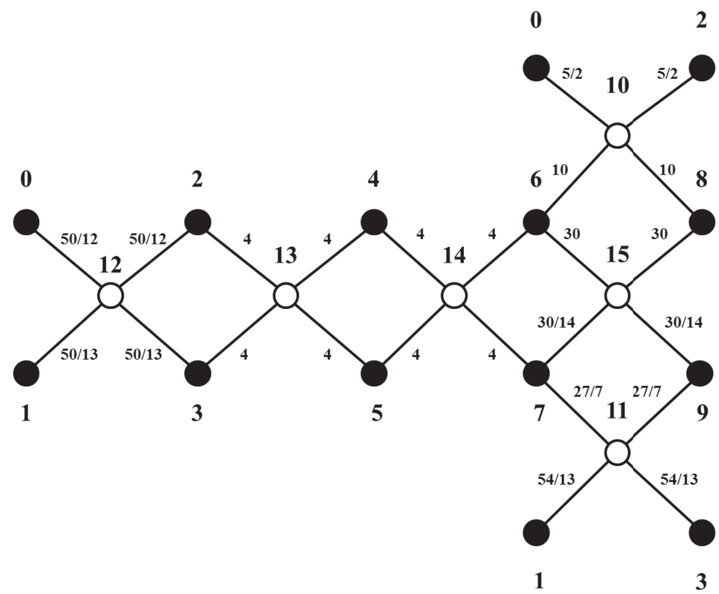
$\gamma_1 = \alpha_1 \sum_m \alpha_m = \sqrt{\frac{12}{13}}(\frac{50}{\sqrt{13}\sqrt{12}}) = \frac{50}{13}$

$\gamma_2 = \alpha_2 \sum_m \alpha_m = \sqrt{\frac{13}{12}}(\frac{50}{\sqrt{13}\sqrt{12}}) = \frac{50}{12}$

$\gamma_3 = \alpha_3 \sum_m \alpha_m = \sqrt{\frac{12}{13}}(\frac{50}{\sqrt{13}\sqrt{12}}) = \frac{50}{13}$



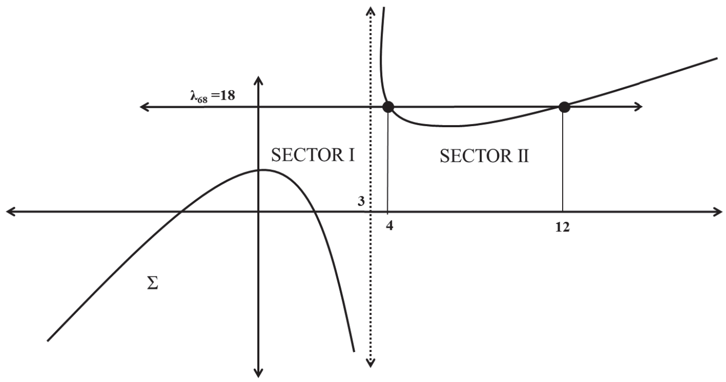
We repeat this process for all the quadrilaterals (six total) and obtain all the conductivities of the edges in the Race Track graph corresponding to $x = 4$.



The Race Track graph with conductivities corresponding to $x = 4$

Step 5: Determine the x-coordinate for the other crossing. Plug this x into the equations for the $f_j(x)$'s and note if any are negative. For this 2nd x-coordinate, create the 2nd R-Matrix. Find the conductivities of edges in the Race Track graph corresponsding to this x-coordinate.

When is $\Sigma(x) = \lambda_{6,8}$? Solving for x in $\Sigma(x) = x + \frac{5x-6}{x-3} = \lambda_{6,8} = 18$ reveals two solutions: 4, as expected, and 12.



Graph of Sigma crossing $\lambda_{6,8}$ at 4 and 12

Plugging 12 into the $f_j(x)$'s shows that all the $f_j(x)$'s are positive at this 2nd crossing.

Equation
$f_0(12) = 12$

$f_1(12) = \frac{1}{12}$

$f_2(12) = \frac{5}{4}$

$f_3(12) = \frac{4}{5}$

$f_4(12) = \frac{6}{5}$

$f_5(12) = \frac{5}{6}$

$f_6(12) = \frac{1}{6}$

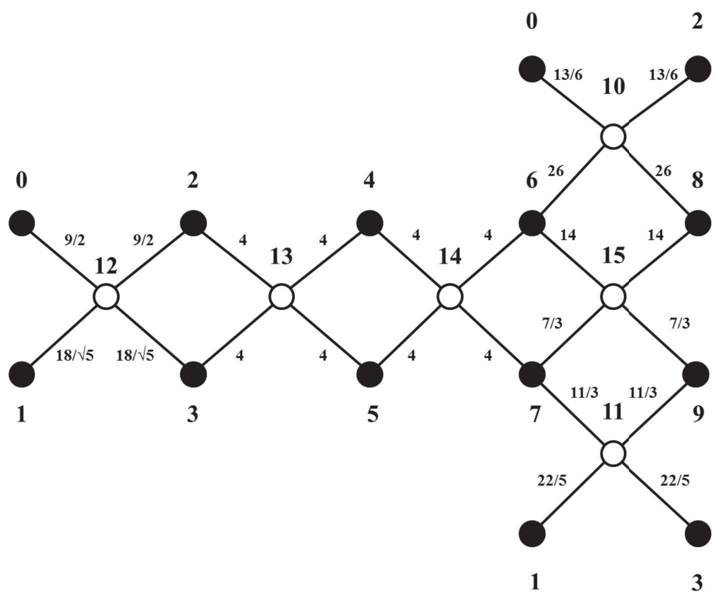
$f_7(12) = 6$

For $x = 12$, we have the following R-Matrix.

$$R = \begin{bmatrix} \frac{-16}{3} & 1 & \{\frac{1}{12}, \frac{5}{4}\} & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -6 & 1 & \{\frac{4}{5}, \frac{6}{5}\} & 0 & 0 & 0 & 1 & 0 & 1 \\ \{\frac{1}{12}, \frac{5}{4}\} & 1 & \frac{-22}{3} & \{1,1\} & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & \{\frac{4}{5}, \frac{6}{5}\} & \{1,1\} & -8 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -6 & \{1,1\} & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & \{1,1\} & -6 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & -25 & \{1,1\} & \{12,6\} & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & \{1,1\} & -8 & 1 & \{\frac{5}{6}, \frac{1}{6}\} \\ 1 & 0 & 1 & 0 & 0 & 0 & \{12,6\} & 1 & -22 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & \{\frac{5}{6}, \frac{1}{6}\} & 1 & -5 \end{bmatrix}$$

The R-Matrix corresponding to $x = 12$

We obtain the conductivities for the edges in the Race Track graph corresponding to $x = 12$ using equation 2 in [2].

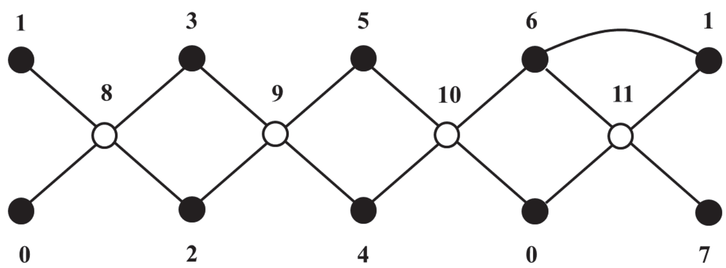


The Race Track graph with conductivities corresponding to $x = 12$

Thus, with one single response matrix, we can obtain two sets of positive conductivities for the edges in the Race Track graph.

3 The Pseudo 2 to 1 graph

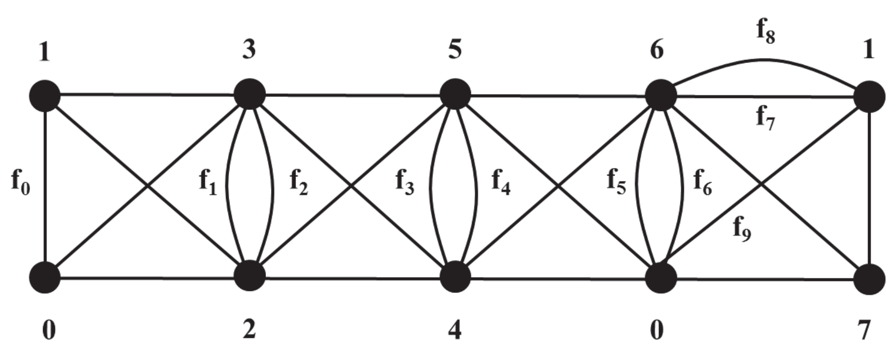
Example 4. Consider the Pseudo 2 to 1 graph.



The Pseudo 2 to 1 graph
Note the edge connecting vertices 6 and 1.

We will show that there exists a way to draw a positive horizontal line representing $\lambda_{0,1}$ such that it crosses the graph for $\Sigma(x)$ twice but in two different sectors where in one sector, all $f_j(x)$'s are positive, and in the other sector, at least one $f_j(x)$ is negative. In addition, the x-coordinates of the crossing points are both positive.

We begin by performing a Star-K Transformation on the Pseudo 2 to 1 graph to obtain its R-Multigraph.



The R-Multigraph of the Pseudo 2 to 1 graph

Once again, assume that $f_0(x) = x$ and determine equations for all the $f_j(x)$'s knowing that the response matrix condition and quadrilateral rule must be satisfied.

Sign of Derivative	Equation
$+$	$f_0(x) = x$
$-$	$f_1(x) = \frac{\lambda_{1,3}\lambda_{0,2}}{f_0(x)}$
$+$	$f_2(x) = \lambda_{2,3} - f_1(x)$
$-$	$f_3(x) = \frac{\lambda_{3,5}\lambda_{2,4}}{f_2(x)}$
$+$	$f_4(x) = \lambda_{4,5} - f_3(x)$
$-$	$f_5(x) = \frac{\lambda_{5,6}\lambda_{0,4}}{f_4(x)}$
$+$	$f_6(x) = \lambda_{0,6} - f_5(x)$
$+$	$f_7(x) = \frac{f_6(x)\lambda_{1,7}}{\lambda_{0,7}}$
$-$	$f_8(x) = \lambda_{1,6} - f_7(x)$
$+$	$f_9(x) = \frac{f_6(x)\lambda_{1,7}}{\lambda_{6,7}}$

Consider

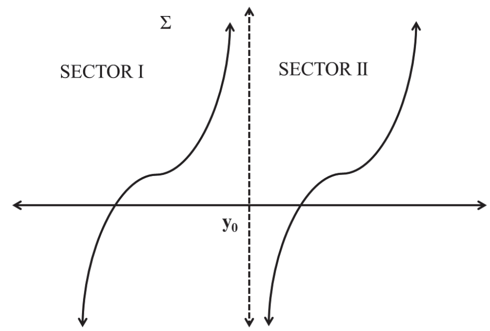
$$\Sigma(x) = f_0(x) + f_9(x) = x + f_9(x) = \lambda_{0,1}$$

Examining the behavior of $\Sigma(x)$ will reveal why there is only one set of conductivities that can be considered.

Assume that $f_9(x)$ is a linear term over a linear term. Thus $\lim_{x\to\infty} \Sigma(x) = \lim_{x\to\infty} x + f_9(x) = \infty$. Similarly, $\lim_{x\to-\infty} \Sigma(x) = -\infty$. Due to the assumption, a horizontal line can only cross $\Sigma(x)$ 0, 1, or 2 times throughout the *whole* graph.

Suppose y_0 is the singularity of $f_9(x)$. Thus, it is also a singularity of $\Sigma(x)$. Since this is the only singularity of $\Sigma(x)$ and there must always exist a positive singularity for $\Sigma(x)$ by [1], y_0 is positive.

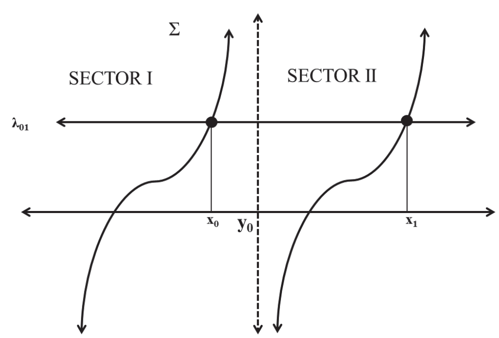
$\Sigma(x)$ is heavily dominated by $f_9(x)$ near its singularity, y_0. Since $f_9(x)$ has a positive derivative, $\Sigma(x)$ must have a positive slope close to y_0. Armed with this knowledge of $\Sigma(x)$, let's try to graph it.



Graph of $\Sigma(x)$
Note that this may not be an exact graph of $\Sigma(x)$.

For now, we shall call the area to the right of y_0 Sector II and the area to the left Sector I.

From observation of the graph of $\Sigma(x)$, there exists a way to draw a positive horizontal line representing $\lambda_{0,1}$ such that it crosses the graph for $\Sigma(x)$ twice. However, every such horizontal line will always cross $\Sigma(x)$ in two different sectors. So, we know that the Pseudo 2 to 1 graph is at most 1 to 1. To prove that the Pseudo 2 to 1 graph is 1 to 1, we need to show that all $f_j(x)$'s are positive in a sector where a crossing occurs.



Graph of $\Sigma(x)$ with $\lambda_{0,1}$
Note that this may not be an exact graph of $\Sigma(x)$.

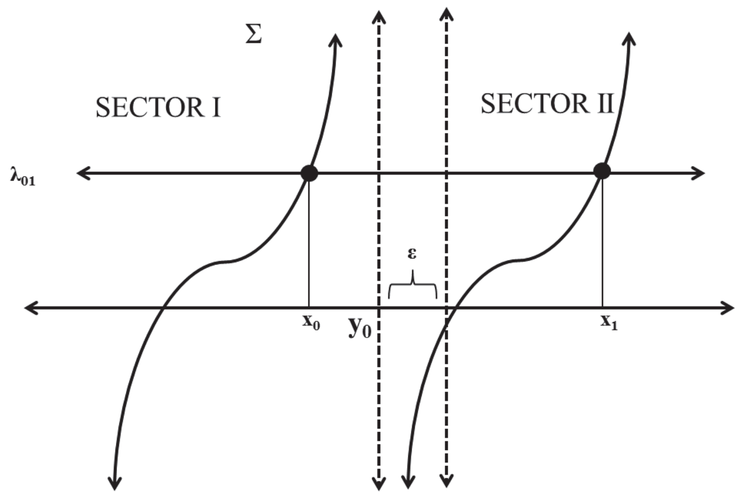
Recall that y_0 is positive since at least one singularity must be positive by [1]. Thus, $f_0(x) = x$ is positive also to the right of y_0. It follows then that $f_1(x) = \frac{\lambda_{1,3}\lambda_{0,2}}{f_0(x)}$ is positive to the right of y_0 since λ_{ij}'s are positive. Thus, both $f_0(x)$ and $f_1(x)$ are positive in Sector II.

Recall that $f_2(x) = \lambda_{2,3} - f_1(x)$. We can choose our $\lambda_{2,3}$ in such a way that

$f_2(x)$ will be positive in Sector II. Let z_0 be the root of $f_2(x)$. So $f_2(z_0) = \lambda_{2,3} - f_1(z_0) = 0$. So, in order for $f_2(z_0)>0$, we need $\lambda_{2,3}>f_1(z_0)$. Let $\lambda_{2,3} = f_1(z_0) + C_{2,3}$ where $C_{2,3}$ is some positive number. By substitution, $f_2(x) = \lambda_{2,3} - f_1(x) = (f_1(z_0) + C_{2,3}) - f_1(x)$. Now $f_1(x)$ has a negative slope because of its negative derivative. Thus, $f_1(z_0)>f_1(x)$ if $z_0<x$. Let's choose $z_0 = y_0$, the singularity of $\Sigma(x)$. Then $f_1(y_0)>f_1(x)$ if $y_0<x$. So if we restrict our x to be on the right of y_0 then $f_2(x) = f_1(y_0) - f_1(x) + C_{2,3}>0$ (recall $C_{2,3}$ is positive). So, by choosing $\lambda_{2,3}$ to be $f_1(y_0)+C_{2,3}$, we have guaranteed that $f_2(x)$ will be positive in Sector II. It follows then that $f_3(x) = \frac{\lambda_{3,5}\lambda_{2,4}}{f_2(x)}$ is positive to the right of y_0 also since λ_{ij}'s are positive. Thus, both $f_2(x)$ and $f_3(x)$ are positive in Sector II.

Our choice of y_0 as the singularity of $\Sigma(x)$ forces $\lambda_{4,5}$ to be a certain value. Note that $f_9(x) = \frac{f_6(x)\lambda_{1,7}}{\lambda_{6,7}}$ has a singularity when $f_6(x) = \lambda_{0,6} - f_5(x)$ has a singularity when $f_5(x) = \frac{\lambda_{5,6}\lambda_{0,4}}{f_4(x)}$ has a singularity when $f_4(x) = 0$. Since y_0 is the value such that $f_4(y_0) = \lambda_{4,5} - f_3(y_0) = 0$, $\lambda_{4,5} = f_3(y_0)$. By substitution, $f_4(x) = f_3(y_0) - f_3(x)$. So, in order for $f_4(x)>0$, we need $f_3(y_0)>f_3(x)$. Now $f_3(x)$ has a negative slope because it has a negative derivative. Thus, $f_3(y_0)>f_3(x)$ if $y_0<x$. Thus, $f_4(x)$ is positive to the right of y_0. It follows then that $f_5(x) = \frac{\lambda_{5,6}\lambda_{0,4}}{f_4(x)}$ is positive to the right of y_0 also since λ_{ij}'s are positive. Thus, both $f_4(x)$ and $f_5(x)$ are positive in Sector II.

Recall that $f_6(x) = \lambda_{0,6} - f_5(x)$. We can choose our $\lambda_{0,6}$ in such a way that $f_6(x)$ will be positive in a slightly smaller portion of Sector II. Let z_1 be the root of $f_6(x)$. So $f_6(z_1) = \lambda_{0,6} - f_5(z_1) = 0$. So, in order for $f_6(z_1)>0$, we need $\lambda_{0,6}>f_5(z_1)$. Let $\lambda_{0,6} = f_5(z_1) + C_{0,6}$ where $C_{0,6}$ is some positive number. By substitution, $f_6(x) = \lambda_{0,6} - f_5(x) = (f_5(z_1) + C_{0,6}) - f_5(x)$. Now $f_5(x)$ has a negative slope because of its negative derivative. Thus, $f_5(z_1)>f_5(x)$ if $z_1<x$. Let's choose $z_1 = y_0 + \epsilon$. Then $f_5(y_0 + \epsilon)>f_5(x)$ if $y_0 + \epsilon<x$. So if we restrict our x to be on the right of $y_0 + \epsilon$ then $f_6(x) = f_5(y_0 + \epsilon) - f_5(x) + C_{0,6}>0$ (recall $C_{0,6}$ is positive). Let's readjust Sector II so that it is the area to the right of $y_0 + \epsilon$. So, by choosing $\lambda_{0,6}$ to be $f_5(y_0 + \epsilon) + C_{0,6}$, we have guaranteed that $f_6(x)$ will be positive in Sector II. It follows then that $f_7(x) = \frac{f_6(x)\lambda_{1,7}}{\lambda_{0,7}}$ and $f_9(x) = \frac{f_6(x)\lambda_{1,7}}{\lambda_{6,7}}$ are positive to the right of $y_0 + \epsilon$ also since λ_{ij}'s are positive. Thus, $f_6(x)$, $f_7(x)$, and $f_9(x)$ are positive in Sector II. Note that $f_0(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_4(x)$, and $f_5(x)$ will still be positive in Sector II despite this readjustment since $y_0<y_0 + \epsilon$.



Graph of $\Sigma(x)$ with readjusted Sector II

It might be questionable to the reader as to why one might need $z_1 = y_0 + \epsilon$ instead of $z_1 = y_0$. Suppose $z_1 = y_0$. Then $\lambda_{0,6} = f_5(z_1) + C_{0,6} = f_5(y_0) + C_{0,6}$. But in order to calculate $f_5(y_0) = \frac{\lambda_{5,6}\lambda_{0,4}}{f_4(y_0)}$, we need $f_4(y_0)$. But $f_4(y_0) = \lambda_{4,5} - f_3(y_0) = f_3(y_0) - f_3(y_0) = 0$ (recall that $\lambda_{4,5} = f_3(y_0)$). But that would mean that $f_5(y_0)$ would have a 0 in the denominator. To avoid this issue we add an ϵ to z_1.

Typically, we will need to add or subtract the root of a $f_j(x)$ by an epsilon if the $f_j(x)$ is after another $f_h(x)$ that contains a λ_{ij} that is forced to be a certain value by our choice of singularity for $\Sigma(x)$ and before another $f_l(x)$ that is a term of $\Sigma(x)$. In this specific example, epsilon is added to z_1, the root of $f_6(x)$, and $f_6(x)$ is after $f_4(x)$, an equation that contains $\lambda_{4,5}$ which was forced to be $f_3(y_0)$ due to our choice of y_0 as the singularity of $\Sigma(x)$ and before $f_9(x)$ which is a term in $\Sigma(x) = x + f_9(x)$.

By the quadrilateral rule,

$$f_6(x)\lambda_{1,7} = f_7(x)\lambda_{0,7} = f_9(x)\lambda_{6,7}$$

Choosing $f_6(x)$, $\lambda_{1,7}$, $\lambda_{0,7}$, $f_9(x)$, and $\lambda_{6,7}$ will force $f_7(x)$ to be a certain value. Thus, one can easily choose a big enough $\lambda_{1,6}$ such that $f_8(x) = \lambda_{1,6} - f_7(x)$ is positive to the right of $y_0 + \epsilon$.

Thus, all $f_j(x)$'s are positive to the right of $y_0 + \epsilon$. So all $f_j(x)$'s are positive in a sector where a crossing can occur, specifically Sector II. Thus, the Pseudo 2 to 1 graph is 1 to 1.

Let's attempt to create a 1 to 1 graph by choosing appropriate λ_{ij}'s using the Pseudo 2 to 1 graph.

Step 1: Pick a positive value, y_0, to be the singularity of $f_9(x)$. This is also the singularity of $\Sigma(x)$. Let y_0 be 1.

Step 2: Choose values of the λ_{ij}'s in the quadrilateral to uphold the quadrilateral rule. After substitution, we obtain a new set of equations for our $f_j(x)$'s.

By the quadrilateral rule,

$$f_0(x)f_1(x) = \lambda_{1,3}\lambda_{0,2} = \lambda_{0,3}\lambda_{1,2}$$

Choose $\lambda_{1,3}$, $\lambda_{0,2}$, and $\lambda_{0,3} = 1$. This forces $\lambda_{1,2} = 1$ too.

By the quadrilateral rule,

$$f_2(x)f_3(x) = \lambda_{3,5}\lambda_{2,4} = \lambda_{3,4}\lambda_{2,5}$$

Choose $\lambda_{3,5}$, $\lambda_{2,4}$, and $\lambda_{3,4} = 1$. This forces $\lambda_{2,5} = 1$ too.

By the quadrilateral rule,

$$f_4(x)f_5(x) = \lambda_{5,6}\lambda_{0,4} = \lambda_{0,5}\lambda_{4,6}$$

Choose $\lambda_{5,6}$, $\lambda_{0,4}$, and $\lambda_{0,5} = 1$. This forces $\lambda_{4,6} = 1$ too.

By the quadrilateral rule,

$$f_6(x)\lambda_{1,7} = f_7(x)\lambda_{0,7} = \lambda_{6,7}f_9(x)$$

Choose $\lambda_{1,7}$ and $\lambda_{6,7} = 1$.

Now we subsitute to obtain a new set of equations for the $f_j(x)$'s

Sign of Derivative	Equation
$+$	$f_0(x) = x$
$-$	$f_1(x) = \frac{1}{f_0(x)}$
$+$	$f_2(x) = \lambda_{2,3} - f_1(x)$
$-$	$f_3(x) = \frac{1}{f_2(x)}$
$+$	$f_4(x) = \lambda_{4,5} - f_3(x)$
$-$	$f_5(x) = \frac{1}{f_4(x)}$
$+$	$f_6(x) = \lambda_{0,6} - f_5(x)$
$+$	$f_7(x) = f_6(x)$
$-$	$f_8(x) = \lambda_{1,6} - f_7(x)$
$+$	$f_9(x) = f_6(x)$

Step 3: Obtain the other λ_{ij} values that were determined by our choice of singularity for $\Sigma(x)$ and by making $f_j(x)$'s positive in Sector II (the region to the right of $y_0 = 1 + \epsilon$). Also, after we have determined $f_7(x)$, choose a $\lambda_{1,6}$ such that $f_8(x) = \lambda_{1,6} - f_7(x)$ will be positive in Sector II.

We have established that in order for $f_2(x)$ to be positive we need $\lambda_{2,3} = f_1(y_0) + C_{2,3}$ where $C_{2,3}$ is some positive number. Let $C_{2,3} = 1$. Since $y_0 = 1$, $\lambda_{2,3} = f_1(1) + 1 = 1 + 1 = 2$.

We have established that in order for y_0 to be the singularity of $\Sigma(x)$, $\lambda_{4,5} = f_3(y_0)$. Since $y_0 = 1$, $\lambda_{4,5} = f_3(1) = 1$.

We have established that in order for $f_6(x)$ to be positive we need $\lambda_{0,6} = f_5(y_0 + \epsilon) + C_{0,6}$ where $C_{0,6}$ is some positive number. Let $C_{0,6} = 1$ and $\epsilon = 0.1$. Since $y_0 = 1$, $\lambda_{0,6} = f_5(1 + 0.1) + 1 = 1 + 1 = 13$.

After substitution, we obtain a new set of equations for our $f_j(x)$'s.

Sign of Derivative	Equation
$+$	$f_0(x) = x$
$-$	$f_1(x) = \frac{1}{x}$
$+$	$f_2(x) = 2 - \frac{1}{x} = \frac{2x-1}{x}$
$-$	$f_3(x) = \frac{x}{2x-1}$
$+$	$f_4(x) = 1 - \frac{x}{2x-1} = \frac{x-1}{2x-1}$
$-$	$f_5(x) = \frac{2x-1}{x-1}$
$+$	$f_6(x) = 13 - \frac{2x-1}{x-1} = \frac{11x-12}{x-1}$
$+$	$f_7(x) = f_6(x) = \frac{11x-12}{x-1}$
$-$	$f_8(x) = \lambda_{1,6} - f_7(x) = \lambda_{1,6} - \frac{11x-12}{x-1}$
$+$	$f_9(x) = f_6(x) = \frac{11x-12}{x-1}$

Let $\lambda_{1,6} = 100$. So, $f_8(x) = \lambda_{1,6} - \frac{11x-12}{x-1} = 100 - \frac{11x-12}{x-1}$. Note that this choice keeps $f_8(x)$ positive in Sector II. Thus, we have the following equations for the $f_j(x)$'s.

Sign of Derivative	Equation
$+$	$f_0(x) = x$
$-$	$f_1(x) = \frac{1}{x}$
$+$	$f_2(x) = \frac{2x-1}{x}$
$-$	$f_3(x) = \frac{x}{2x-1}$
$+$	$f_4(x) = \frac{x-1}{2x-1}$

$-$ $\qquad f_5(x) = \frac{2x-1}{x-1}$

$+$ $\qquad f_6(x) = \frac{11x-12}{x-1}$

$+$ $\qquad f_7(x) = \frac{11x-12}{x-1}$

$-$ $\qquad f_8(x) = 100 - \frac{11x-12}{x-1}$

$+$ $\qquad f_9(x) = \frac{11x-12}{x-1}$

Thus, $\Sigma(x) = x + \frac{11x-12}{x-1}$.

Step 4: Choose an x-coordinate for the crossing between $\lambda_{0,1}$ and $\Sigma(x)$ in Sector II to determine $\lambda_{0,1}$. Check if any f_j's are negative by this choice of x. Create the response matrix. For the specific x-coordinate chosen, create the R-Matrix. Find the conductivities of edges in the Pseudo 2 to 1 graph corresponsding to the choice of x-coordinate.

We will choose $x = 6$ for the x-coordinate for the crossing in Sector II. All f_j's are positive at $x = 6$.

Equation
$f_0(6) = 6$

$f_1(6) = \frac{1}{6}$

$f_2(6) = \frac{11}{6}$

$f_3(6) = \frac{6}{11}$

$f_4(6) = \frac{5}{11}$

$f_5(6) = \frac{11}{5}$

$f_6(6) = \frac{54}{5}$

$f_7(6) = \frac{54}{5}$

$f_8(6) = \frac{446}{5}$

$f_9(6) = \frac{54}{5}$

Thus, $\Sigma(6) = 6 + \frac{11*6-12}{6-1} = \frac{84}{5} = \lambda_{0,1}$. This is a positive $\lambda_{0,1}$ as required.

Recall that by the quadrilateral rule,

$$f_6(x)\lambda_{1,7} = f_7(x)\lambda_{0,7} = \lambda_{6,7}f_9(x)$$

We chose in Step 2 for $\lambda_{1,7}$ and $\lambda_{6,7} = 1$. Since $f_6(6)$, $f_7(6)$, and $f_9(6) = \frac{54}{5}$, by substitution, we have

$$\frac{54}{5} * 1 = \frac{54}{5} * \lambda_{0,7} = 1 * \frac{54}{5}$$

This forces $\lambda_{0,7} = 1$.

Now we have all the λ_{ij}'s we need, and using the fact that row sums are 0 and response matrices are symmetrical, we can create the response matrix, Λ. Note that $\lambda_{ij} = 0$ if there doesn't exist a direct edge between vertices i and j in the R-Multigraph.

$$\Lambda = \begin{bmatrix} -33.8 & \frac{84}{5} & 1 & 1 & 1 & 1 & 13 & 1 \\ \frac{84}{5} & -119.8 & 1 & 1 & 0 & 0 & 100 & 1 \\ 1 & 1 & -6 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & -6 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & -5 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & -4 & 1 & 0 \\ 13 & 100 & 0 & 0 & 1 & 1 & -116 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

The response matrix of the Pseudo 2 to 1 graph

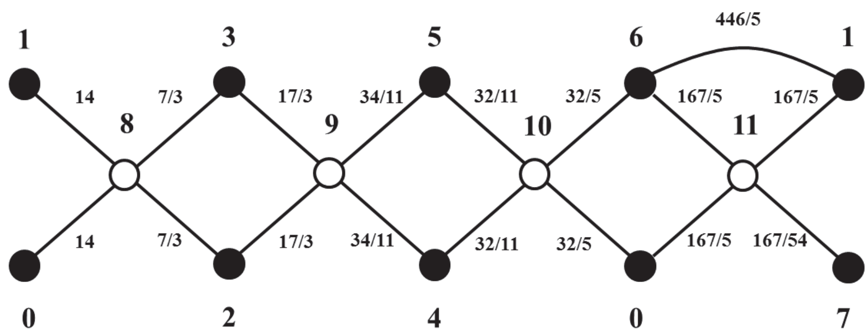
For $x = 6$, we have the following R-Matrix.

$$R = \begin{bmatrix} -33.8 & \{6, \frac{54}{5}\} & 1 & 1 & 1 & 1 & \{\frac{11}{5}, \frac{54}{5}\} & 1 \\ \{6, \frac{54}{5}\} & -119.8 & 1 & 1 & 0 & 0 & \{\frac{54}{5}, \frac{446}{5}\} & 1 \\ 1 & 1 & -6 & \{\frac{1}{6}, \frac{11}{6}\} & 1 & 1 & 0 & 0 \\ 1 & 1 & \{\frac{1}{6}, \frac{11}{6}\} & -6 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & -5 & \{\frac{6}{11}, \frac{5}{11}\} & 1 & 0 \\ 1 & 0 & 1 & 1 & \{\frac{6}{11}, \frac{5}{11}\} & -4 & 1 & 0 \\ \{\frac{11}{5}, \frac{54}{5}\} & \{\frac{54}{5}, \frac{446}{5}\} & 0 & 0 & 1 & 1 & -116 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

The R-Matrix corresponding to $x = 6$

We obtain the conductivities for the edges in the Pseudo 2 to 1 graph corresponding to $x = 6$ using equation 2 in [2].

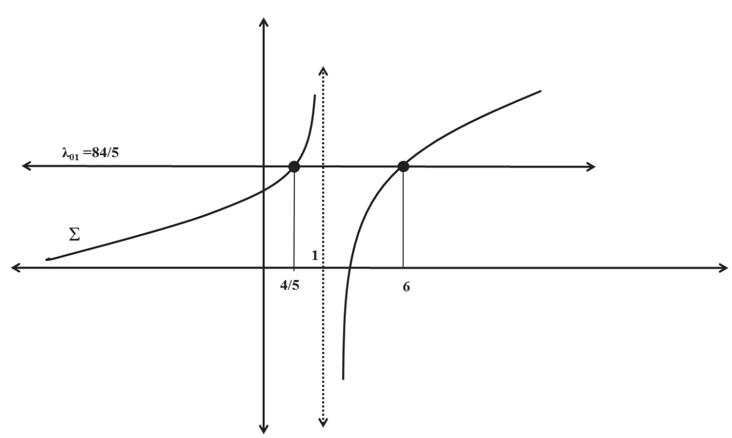
The conductivity of the edge connecting vertex 1 to vertex 6 in the Pseudo 2 to 1 graph is directly equal to $f_8(6) = \frac{446}{5}$.



The Pseudo 2 to 1 graph with conductivities corresponding to $x = 6$

Step 5: Determine the x-coordinate for the other crossing. Plug this x into the equations for the $f_j(x)$'s and note if any are negative. For this 2nd x-coordinate, create the 2nd R-Matrix. Find the conductivities of edges in the Pseudo 2 to 1 graph corresponsding to this x-coordinate.

When is $\Sigma(x) = \lambda_{0,1}$? Solving for x in $\Sigma(x) = x + \frac{11x-12}{x-1} = \lambda_{0,1} = 84/5$ reveals two solutions: 6, as expected, and $\frac{4}{5}$.



Graph of $\Sigma(x)$ crossing $\lambda_{0,1}$ at $\frac{4}{5}$ and 6

Plugging $\frac{4}{5}$ into the $f_j(x)$'s shows that $f_4(\frac{4}{5})$ and $f_5(\frac{4}{5})$ are negative.

Equation
$f_0(\frac{4}{5}) = \frac{4}{5}$

$f_1(\frac{4}{5}) = \frac{5}{4}$

$f_2(\frac{4}{5}) = \frac{3}{4}$

$f_3(\frac{4}{5}) = \frac{4}{3}$

$f_4(\frac{4}{5}) = -\frac{1}{3}$

$f_5(\frac{4}{5}) = -3$

$f_6(\frac{4}{5}) = 16$

$f_7(\frac{4}{5}) = 16$

$f_8(\frac{4}{5}) = 84$

$f_9(\frac{4}{5}) = 16$

For $x = \frac{4}{5}$, we have the following R-Matrix.

$$R = \begin{bmatrix} -33.8 & \{\frac{4}{5}, 16\} & 1 & 1 & 1 & 1 & \{-3, 16\} & 1 \\ \{\frac{4}{5}, 16\} & -119.8 & 1 & 1 & 0 & 0 & \{16, 84\} & 1 \\ 1 & 1 & -6 & \{\frac{5}{4}, \frac{3}{4}\} & 1 & 1 & 0 & 0 \\ 1 & 1 & \{\frac{5}{4}, \frac{3}{4}\} & -6 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & -5 & \{\frac{4}{3}, \frac{-1}{3}\} & 1 & 0 \\ 1 & 0 & 1 & 1 & \{\frac{4}{3}, \frac{-1}{3}\} & -4 & 1 & 0 \\ \{-3, 16\} & \{16, 84\} & 0 & 0 & 1 & 1 & -116 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

The R-Matrix corresponding to $x = \frac{4}{5}$

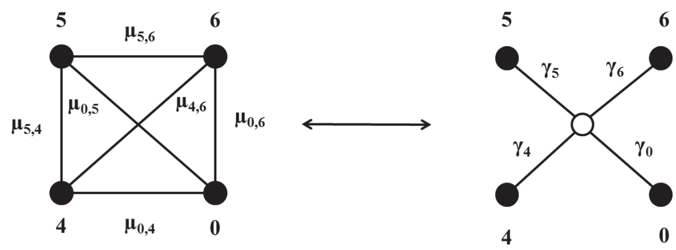
Because of the negative conductivities, we will not use equation 2 in [2] to obtain conductivities for the edges in the Pseudo 2 to 1 graph corresponding to $x = \frac{4}{5}$ due to the difficulty of determining the α's (see [5]). We will use the general formula provided in [3].

$$\gamma_i = -\frac{\det \begin{bmatrix} \mu_{i,i} & \mu_{i,j} \\ \mu_{i,k} & \mu_{j,k} \end{bmatrix}}{\mu_{j,k}}$$

where

$$\mu_{i,i} = -(\mu_{i,j} + \mu_{i,k} + \mu_{i,l})$$

Note that i, j, k, and l are vertices in the same quadrilateral. Let's determine the conductivities of edges in the Pseudo 2 to 1 graph using the third quadrilateral and this general formula.



Quadrilateral Three and the 4-star compared to it

According to the general formula,

$$\mu_{0,0} = -(\mu_{0,6} + \mu_{0,5} + \mu_{0,4}) = -(-3 + 1 + 1) = 1$$

Note that $\mu_{0,6} = f_5(\frac{4}{5}) = -3$ since we are looking only in the third quadrilateral.

So

$$\gamma_0 = -\frac{\det \begin{bmatrix} \mu_{0,0} & \mu_{0,6} \\ \mu_{0,4} & \mu_{4,6} \end{bmatrix}}{\mu_{4,6}} = -\frac{\det \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}}{1} = -4$$

Similarly,

$$\mu_{4,4} = -(\mu_{4,5} + \mu_{4,6} + \mu_{4,0}) = -(-\frac{1}{3} + 1 + 1) = -\frac{5}{3}$$

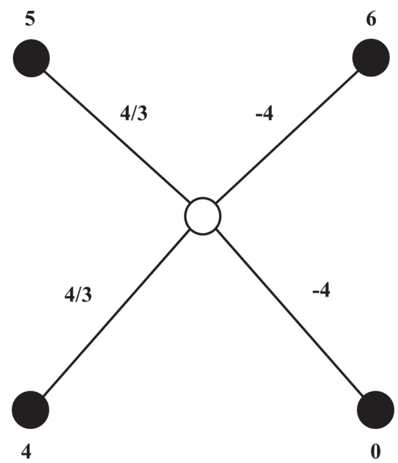
$$\gamma_4 = -\frac{\det \begin{bmatrix} \mu_{4,4} & \mu_{4,5} \\ \mu_{4,0} & \mu_{0,5} \end{bmatrix}}{\mu_{0,5}} = -\frac{\det \begin{bmatrix} -\frac{5}{3} & -\frac{1}{3} \\ 1 & 1 \end{bmatrix}}{1} = \frac{4}{3}$$

$$\mu_{5,5} = -(\mu_{5,6} + \mu_{5,0} + \mu_{5,4}) = -(1 + 1 - \frac{1}{3}) = -\frac{5}{3}$$

$$\gamma_5 = -\frac{\det \begin{bmatrix} \mu_{5,5} & \mu_{5,6} \\ \mu_{5,4} & \mu_{4,6} \end{bmatrix}}{\mu_{4,6}} == -\frac{\det \begin{bmatrix} -\frac{5}{3} & 1 \\ -\frac{1}{3} & 1 \end{bmatrix}}{1} = \frac{4}{3}$$

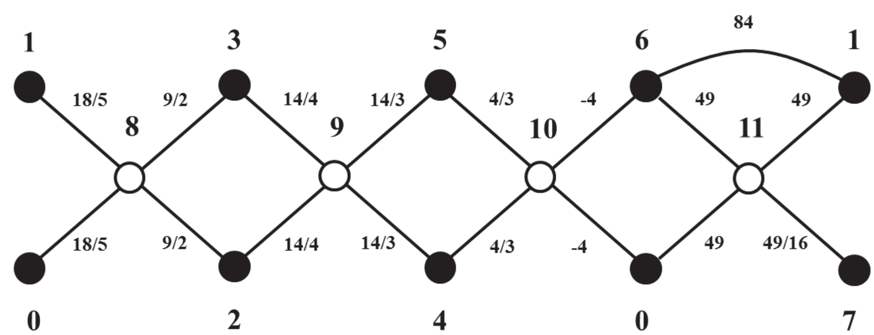
$$\mu_{6,6} = -(\mu_{6,5} + \mu_{6,4} + \mu_{6,0}) = -(1 + 1 - 3) = 1$$

$$\gamma_6 = -\frac{\det \begin{bmatrix} \mu_{6,6} & \mu_{6,5} \\ \mu_{6,0} & \mu_{5,0} \end{bmatrix}}{\mu_{5,0}} = -\frac{\det \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}}{1} = -4$$



We repeat this process for all the quadrilaterals (4 total) and obtain all the conductivities of the edges in the Pseudo 2 to 1 corresponding to $x = \frac{4}{5}$.

The conductivity of the edge connecting vertex 1 to vertex 6 in the Pseudo 2 to 1 graph is directly equal to $f_8(\frac{4}{5}) = 84$.

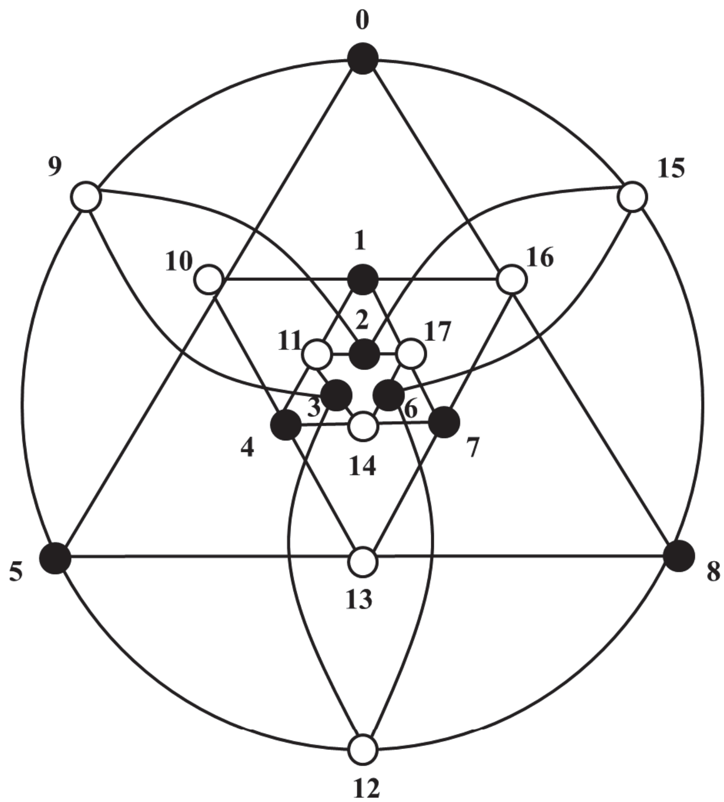


The Pseudo 2 to 1 graph with conductivities corresponding to $x = \frac{4}{5}$

Thus, with a single response matrix, we can only obtain one set of positive conductivities for the edges in the Pseudo 2 to 1 graph.

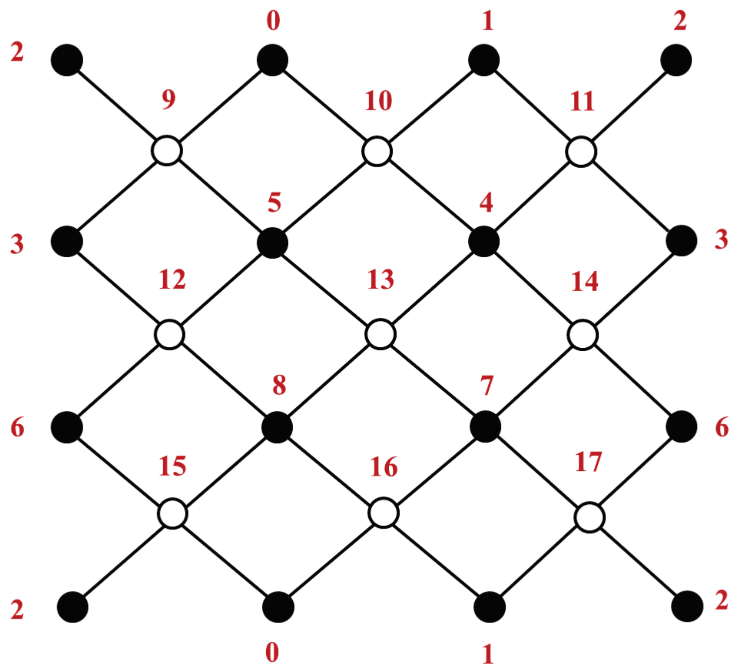
4 The (3,3)-Torus

Example 5. Consider the following resistor network, the (3,3)-torus. We will show that the (3,3)-torus is 64 to 1.



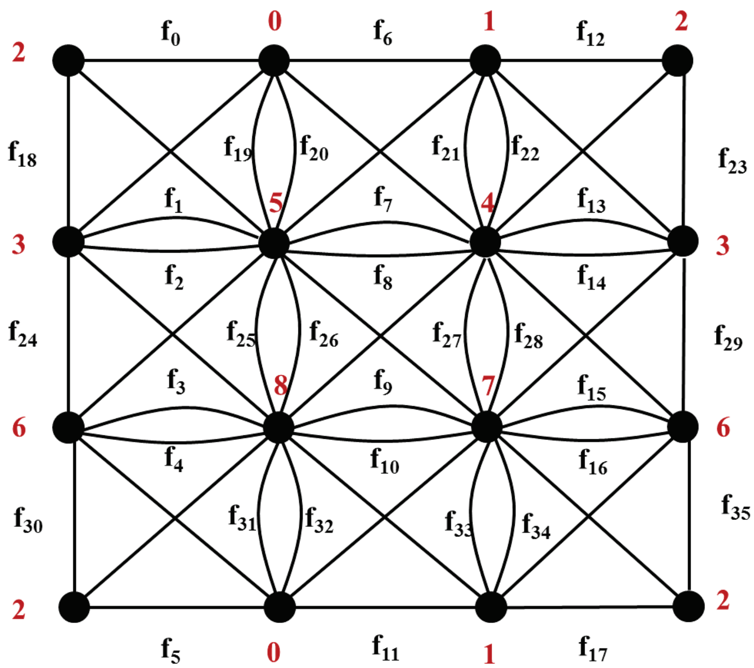
A (3,3)-torus labeled

We redraw the (3,3)-torus first so it will be easier to perform a Star-K Transformation.



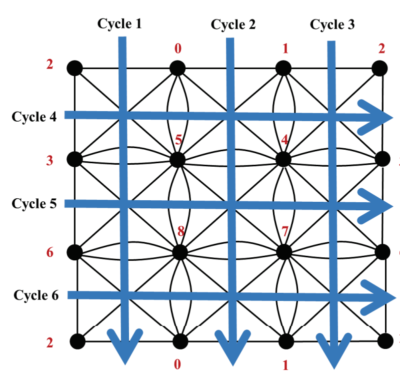
The (3,3)-torus Redrawn

We now perform a Star-K Transformation on the (3,3)-torus.



The R-Multigraph of the (3,3)-torus
Boundary vertices are labeled in red.

Like the previous examples, we determine equations for the $f_j(x)$'s. However, we will need to perform this a total of six times, for each row and column in the R-Multigraph. Essentially, there will be 6 cycles, or 6 sets of $f_j(x)$ equations to consider.



We begin with the first cycle. Assume we know that $f_0(x_1) = x_1$ (the subscript of x represents what cycle the equations represent). Determine equations for all the $f_j(x_1)$'s knowing that the response matrix condition and quadrilateral rule must be satisfied.

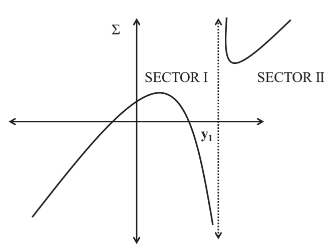
Sign of Derivative	Equation
$+$	$f_0(x_1) = x_1$
$-$	$f_1(x_1) = \frac{\lambda_{2,5}\lambda_{0,3}}{f_0(x_1)}$
$+$	$f_2(x_1) = \lambda_{3,5} - f_1(x_1)$
$-$	$f_3(x_1) = \frac{\lambda_{3,8}\lambda_{5,6}}{f_2(x_1)}$
$+$	$f_4(x_1) = \lambda_{6,8} - f_3(x_1)$
$-$	$f_5(x_1) = \frac{\lambda_{0,6}\lambda_{2,8}}{f_4(x_1)}$

Thus, $\Sigma_{x_1}(x_1) = x_1 + f_5(x_1) = \lambda_{0,2}$.

Assume that $f_5(x_1)$ is a linear term over a linear term. Thus $\lim_{x_1\to\infty} \Sigma_{x_1}(x_1) = \lim_{x_1\to\infty} x_1 + f_5(x_1) = \infty$. Similarly, $\lim_{x_1\to-\infty} \Sigma_{x_1}(x_1) = -\infty$. Due to the assumption, a horizontal line can only cross $\Sigma_{x_1}(x_1)$ 0, 1, or 2 times throughout the whole graph.

Denote the singularity of $f_5(x_1)$ as y_1. Note that y_1 is also the singularity of $\Sigma_{x_1}(x_1)$. Since this is the only singularity of $\Sigma_{x_1}(x_1)$, by [1], y_1 is positive.

$\Sigma_{x_1}(x_1)$ is heavily dominated by $f_5(x_1)$ near y_1. Since $f_5(x_1)$ has a negative derivative, $\Sigma_{x_1}(x_1)$ has a negative slope near y_1.



Denote the area to the right of the singularity as Sector II and the area to the left as Sector I.

From observation of the graph, it is possible to draw a positive horizontal line representing $\lambda_{0,2}$ such that it crosses $\Sigma_{x_1}(x_1)$ twice in the same sector. We need to show that all f_j's in the first cycle are positive in the sector where both crossings occur.

$f_0(x_1)$ is positive in Sector II since y_1 is positive. It follows that $f_1(x_1)$ is positive in Sector II since λ_{ij}'s are always positive.

We can force $f_2(x_1)$ to be positive in Sector II by choosing $\lambda_{3,5} = f_1(y_1) + C_{3,5}$ where $C_{3,5}$ is some positive number. It follows that $f_3(x_1)$ is positive in Sector II.

The choice of y_1 as the singularity forces $\lambda_{6,8} = f_3(y_1)$. Since $f_4(x_1) = \lambda_{6,8} - f_3(x_1)$ and $\lambda_{6,8} = f_3(y_1)$, $f_4(x_1) = f_3(y_1) - f_3(x_1)$. Since $f_3(x_1)$ has a negative derivative, $f_4(x_1)$ is positive in Sector II. It follows that $f_5(x_1)$ is positive in Sector II.

Thus, all f_j's in the first cycle are positive in Sector II.

For the second cycle,

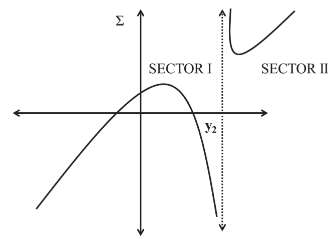
Sign of Derivative	Equation
$+$	$f_6(x_2) = x_2$
$-$	$f_7(x_2) = \frac{\lambda_{0,4}\lambda_{1,5}}{f_6(x_2)}$
$+$	$f_8(x_2) = \lambda_{4,5} - f_7(x_2)$
$-$	$f_9(x_2) = \frac{\lambda_{4,8}\lambda_{5,7}}{f_8(x_2)}$
$+$	$f_{10}(x_2) = \lambda_{7,8} - f_9(x_2)$
$-$	$f_{11}(x_2) = \frac{\lambda_{1,8}\lambda_{0,7}}{f_{10}(x_2)}$

Thus, $\Sigma_{x_2}(x_2) = x_2 + f_{11}(x_2) = \lambda_{0,1}$.

Assume that $f_{11}(x_2)$ is a linear term over a linear term. Thus $\lim_{x_2\to\infty} \Sigma_{x_2}(x_2) = \lim_{x_2\to\infty} x_2 + f_{11}(x_2) = \infty$. Similarly, $\lim_{x_2\to-\infty} \Sigma_{x_2}(x_2) = -\infty$. Due to the assumpion, a horizontal line can only cross $\Sigma_{x_2}(x_2)$ 0, 1, or 2 times throughout the whole graph.

Denote the singularity of $f_{11}(x_2)$ as y_2. Note that y_2 is also the singularity of $\Sigma_{x_2}(x_2)$. Since this is the only singularity of $\Sigma_{x_2}(x_2)$, by [1], y_2 is positive.

$\Sigma_{x_2}(x_2)$ is heavily dominated by $f_{11}(x_2)$ near y_2. Since $f_{11}(x_2)$ has a negative derivative, $\Sigma_{x_2}(x_2)$ has a negative slope near y_2.



Denote the area to the right of the singularity as Sector II and the area to the left as Sector I.

From observation of the graph, it is possible to draw a positive horizontal line representing $\lambda_{0,1}$ such that it crosses $\Sigma_{x_2}(x_2)$ twice in the same sector. We need to show that all f_j's in the second cycle are positive in the sector where both crossings occur.

$f_6(x_2)$ is positive in Sector II since y_2 is positive. It follows that $f_7(x_2)$ is positive in Sector II since λ_{ij}'s are always positive.

We can force $f_8(x_2)$ to be positive in Sector II by choosing $\lambda_{4,5} = f_7(y_2) + C_{4,5}$ where $C_{4,5}$ is some positive number. It follows that $f_9(x_2)$ is positive in Sector II.

The choice of y_2 as singularity forces $\lambda_{7,8} = f_9(y_2)$. Since $f_{10}(x_2) = \lambda_{7,8} - f_9(x_2)$ and $\lambda_{7,8} = f_9(y_2)$ due to the choice of singularity, $f_{10}(x_2) = f_9(y_2) - f_9(x_2)$. Since $f_9(x_2)$ has a negative derivative, $f_{10}(x_2)$ is positive in Sector II. It follows that $f_{11}(x_2)$ is positive in Sector II.

Thus, all f_j's in the second cycle are positive in Sector II.

For the third cycle,

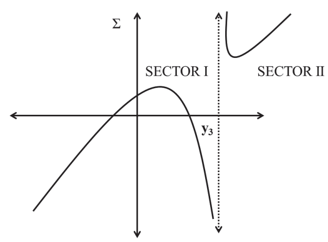
Sign of Derivative	Equation
$+$	$f_{12}(x_3) = x_3$
$-$	$f_{13}(x_3) = \frac{\lambda_{1,3}\lambda_{2,4}}{f_{12}(x_3)}$
$+$	$f_{14}(x_3) = \lambda_{3,4} - f_{13}(x_3)$
$-$	$f_{15}(x_3) = \frac{\lambda_{4,6}\lambda_{3,7}}{f_{14}(x_3)}$
$+$	$f_{16}(x_3) = \lambda_{6,7} - f_{15}(x_3)$
$-$	$f_{17}(x_3) = \frac{\lambda_{1,6}\lambda_{2,7}}{f_{16}(x_3)}$

Thus, $\Sigma_{x_3}(x_3) = x_3 + f_{17}(x_3) = \lambda_{1,2}$.

Assume that $f_{17}(x_3)$ is a linear term over a linear term. Thus $\lim_{x_3\to\infty} \Sigma_{x_3}(x_3) = \lim_{x_3\to\infty} x_3 + f_{17}(x_3) = \infty$. Similarly, $\lim_{x_3\to-\infty} \Sigma_{x_3}(x_3) = -\infty$. Due to the assumption, a horizontal line can only cross $\Sigma_{x_3}(x_3)$ 0, 1, or 2 times throughout the whole graph.

Denote the singularity of $f_{17}(x_3)$ as y_3. Note that y_3 is also the singularity of $\Sigma_{x_3}(x_3)$. Since this is the only singularity of $\Sigma_{x_3}(x_3)$, by [1], y_3 is positive.

$\Sigma_{x_3}(x_3)$ is heavily dominated by $f_{17}(x_3)$ near y_3. Since $f_{17}(x_3)$ has a negative derivative, $\Sigma_{x_3}(x_3)$ has a negative slope near y_3.



Denote the area to the right of the singularity as Sector II and the area to the left as Sector I.

From observation of the graph, it is possible to draw a positive horizontal line representing $\lambda_{1,2}$ such that it crosses $\Sigma_{x_3}(x_3)$ twice in the same sector. We need to show that all f_j's in the third cycle are positive in the sector where both crossings occur.

$f_{12}(x_3)$ is positive in Sector II since y_3 is positive. It follows that $f_{13}(x_3)$ is positive in Sector II since λ_{ij}'s are always positive.

We can force $f_{14}(x_3)$ to be positive in Sector II by choosing $\lambda_{3,4} = f_{13}(y_3)+C_{3,4}$ where $C_{3,4}$ is some positive number. It follows that $f_{15}(x_3)$ is positive in Sector II.

The choice of y_3 as singularity forces $\lambda_{6,7} = f_{15}(y_3)$. Since $f_{16}(x_3) = \lambda_{6,7} - f_{15}(x_3)$ and $\lambda_{6,7} = f_{15}(y_3)$ due to the choice of singularity, $f_{16}(x_3) = f_{15}(y_3) - f_{15}(x_3)$. Since $f_{15}(x_3)$ has a negative derivative, $f_{16}(x_3)$ is positive in Sector II. It follows that $f_{17}(x_3)$ is positive in Sector II.

Thus, all f_j's in the third cycle are positive in Sector II.

For the fourth cycle,

Sign of Derivative	Equation
$+$	$f_{18}(x_4) = x_4$
$-$	$f_{19}(x_4) = \frac{\lambda_{0,3}\lambda_{2,5}}{f_{18}(x_4)}$
$+$	$f_{20}(x_4) = \lambda_{0,5} - f_{19}(x_4)$
$-$	$f_{21}(x_4) = \frac{\lambda_{0,4}\lambda_{1,5}}{f_{20}(x_4)}$

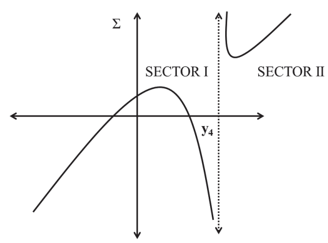
$+$	$f_{22}(x_4) = \lambda_{1,4} - f_{21}(x_4)$
$-$	$f_{23}(x_4) = \frac{\lambda_{2,4}\lambda_{1,3}}{f_{22}(x_4)}$

Thus, $\Sigma_{x_4}(x_4) = x_4 + f_{23}(x_4) = \lambda_{2,3}$.

Assume that $f_{23}(x_4)$ is a linear term over a linear term. Thus $\lim_{x_4\to\infty} \Sigma_{x_4}(x_4) = \lim_{x_4\to\infty} x_4 + f_{23}(x_4) = \infty$. Similarly, $\lim_{x_4\to-\infty} \Sigma_{x_4}(x_4) = -\infty$. Due to the assumption, a horizontal line can only cross $\Sigma_{x_4}(x_4)$ 0, 1, or 2 times throughout the whole graph.

Denote the singularity of $f_{23}(x_4)$ as y_4. Note that y_4 is also the singularity of $\Sigma_{x_4}(x_4)$. Since this is the only singularity of $\Sigma_{x_4}(x_4)$, by [1], y_4 is positive.

$\Sigma_{x_4}(x_4)$ is heavily dominated by $f_{23}(x_4)$ near y_4. Since $f_{23}(x_4)$ has a negative derivative, $\Sigma_{x_4}(x_4)$ has a negative slope near y_4.



Denote the area to the right of the singularity as Sector II and the area to the left as Sector I.

From observation of the graph, it is possible to draw a positive horizontal line representing $\lambda_{2,3}$ such that it crosses $\Sigma_{x_4}(x_4)$ twice in the same sector. We need to show that all f_j's in the fourth cycle are positive in the sector where both crossings occur.

$f_{18}(x_4)$ is positive in Sector II since y_4 is positive. It follows that $f_{19}(x_4)$ is positive in Sector II since λ_{ij}'s are always positive.

We can force $f_{20}(x_4)$ to be positive in Sector II by choosing $\lambda_{0,5} = f_{19}(y_4)+C_{0,5}$ where $C_{0,5}$ is some positive number. It follows that $f_{21}(x_4)$ is positive in Sector II.

The choice of y_4 as the singularity forces $\lambda_{1,4} = f_{21}(y_4)$. Since $f_{22}(x_4) = \lambda_{1,4} - f_{21}(x_4)$ and $\lambda_{1,4} = f_{21}(y_4)$ due to the choice of singularity, $f_{22}(x_4) = f_{21}(y_4) - f_{21}(x_4)$. Since $f_{21}(x_4)$ has a negative derivative, $f_{22}(x_4)$ is positive in Sector II. It follows that $f_{23}(x_4)$ is positive in Sector II.

Thus, all f_j's in the fourth cycle are positive in Sector II.

For the fifth cycle,

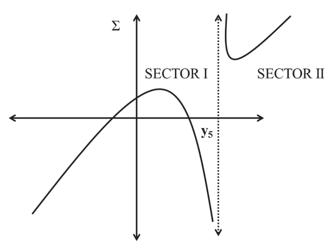
Sign of Derivative	Equation
$+$	$f_{24}(x_5) = x_5$
$-$	$f_{25}(x_5) = \frac{\lambda_{5,6}\lambda_{3,8}}{f_{24}(x_5)}$
$+$	$f_{26}(x_5) = \lambda_{5,8} - f_{25}(x_5)$
$-$	$f_{27}(x_5) = \frac{\lambda_{4,8}\lambda_{5,7}}{f_{26}(x_5)}$
$+$	$f_{28}(x_5) = \lambda_{4,7} - f_{27}(x_5)$
$-$	$f_{29}(x_5) = \frac{\lambda_{4,6}\lambda_{3,7}}{f_{28}(x_5)}$

Thus, $\Sigma_{x_5}(x_5) = x_5 + f_{29}(x_5) = \lambda_{3,6}$.

Assume that $f_{29}(x_5)$ is a linear term over a linear term. Thus $\lim_{x_5 \to \infty} \Sigma_{x_5}(x_5) = \lim_{x_5 \to \infty} x_5 + f_{29}(x_5) = \infty$. Similarly, $\lim_{x_5 \to -\infty} \Sigma_{x_5}(x_5) = -\infty$. Due to the assumption, a horizontal line can only cross $\Sigma_{x_5}(x_5)$ 0, 1, or 2 times throughout the whole graph.

Denote the singularity of $f_{29}(x_5)$ as y_5. Note that y_5 is also the singularity of $\Sigma_{x_5}(x_5)$. Since this is the only singularity of $\Sigma_{x_5}(x_5)$, by [1], y_5 is positive.

$\Sigma_{x_5}(x_5)$ is heavily dominated by $f_{29}(x_5)$ near y_5. Since $f_{29}(x_5)$ has a negative derivative, $\Sigma_{x_5}(x_5)$ has a negative slope near y_5.



Denote the area to the right of the singularity as Sector II and the area to the left as Sector I.

From observation of the graph, it is possible to draw a positive horizontal line representing $\lambda_{3,6}$ such that it crosses $\Sigma_{x_5}(x_5)$ twice in the same sector. We need to show that all f_j's in the fifth cycle are positive in the sector where both crossings occur.

$f_{24}(x_5)$ is positive in Sector II since y_5 is positive. It follows that $f_{25}(x_5)$ is positive in Sector II since λ_{ij}'s are always positive.

We can force $f_{26}(x_5)$ to be positive in Sector II by choosing $\lambda_{5,8} = f_{25}(y_5) + C_{5,8}$

where $C_{5,8}$ is some positive number. It follows that $f_{27}(x_5)$ is positive in Sector II.

The choice of y_5 as the singularity forces $\lambda_{4,7} = f_{27}(y_5)$. Since $f_{28}(x_5) = \lambda_{4,7} - f_{27}(x_5)$ and $\lambda_{4,7} = f_{27}(y_5)$ due to the choice of singularity, $f_{28}(x_5) = f_{27}(y_5) - f_{27}(x_5)$. Since $f_{27}(x_5)$ has a negative derivative, $f_{28}(x_5)$ is positive in Sector II. It follows that $f_{29}(x_5)$ is positive in Sector II.

Thus, all f_j's in the fifth cycle are positive in Sector II.

For the sixth cycle,

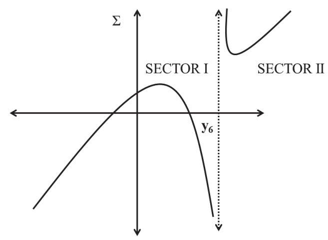
Sign of Derivative	Equation
$+$	$f_{30}(x_6) = x_6$
$-$	$f_{31}(x_6) = \frac{\lambda_{0,6}\lambda_{2,8}}{f_{30}(x_6)}$
$+$	$f_{32}(x_6) = \lambda_{0,8} - f_{31}(x_6)$
$-$	$f_{33}(x_6) = \frac{\lambda_{1,8}\lambda_{0,7}}{f_{32}(x_6)}$
$+$	$f_{34}(x_6) = \lambda_{1,7} - f_{33}(x_6)$
$-$	$f_{35}(x_6) = \frac{\lambda_{2,7}\lambda_{1,6}}{f_{34}(x_6)}$

Thus, $\Sigma_{x_6}(x_6) = x_6 + f_{35}(x_6) = \lambda_{2,6}$.

Assume that $f_{35}(x_6)$ is a linear term over a linear term. Thus $\lim_{x_6\to\infty} \Sigma_{x_6}(x_6) = \lim_{x_6\to\infty} x_6 + f_{35}(x_6) = \infty$. Similarly, $\lim_{x_6\to-\infty} \Sigma_{x_6}(x_6) = -\infty$. Due to the assumption, a horizontal line can only cross $\Sigma_{x_6}(x_6)$ 0, 1, or 2 times throughout the whole graph.

Denote the singularity of $f_{35}(x_6)$ as y_6. Note that y_6 is also the singularity of $\Sigma_{x_6}(x_6)$. Since this is the only singularity of $\Sigma_{x_6}(x_6)$, by [1], y_6 is positive.

$\Sigma_{x_6}(x_6)$ is heavily dominated by $f_{35}(x_6)$ near y_6. Since $f_{35}(x_6)$ has a negative derivative, $\Sigma_{x_6}(x_6)$ has a negative slope near y_6.



Denote the area to the right of the singularity as Sector II and the area to the left as Sector I.

From observation of the graph, it is possible to draw a positive horizontal line representing $\lambda_{2,6}$ such that it crosses $\Sigma_{x_6}(x_6)$ twice in the same sector. We need to show that all f_j's in the sixth cycle are positive in the sector where both crossings occur.

$f_{30}(x_6)$ is positive in Sector II since y_6 is positive. It follows that $f_{31}(x_6)$ is positive in Sector II since λ_{ij}'s are always positive.

We can force $f_{32}(x_6)$ to be positive in Sector II by choosing $\lambda_{0,8} = f_{31}(y_6)+C_{0,8}$ where $C_{0,8}$ is some positive number. It follows that $f_{33}(x_6)$ is positive in Sector II.

The choice of y_6 as the singularity forces $\lambda_{1,7} = f_{33}(y_6)$. Since $f_{34}(x_6) = \lambda_{1,7} - f_{33}(x_6)$ and $\lambda_{1,7} = f_{33}(y_6)$ due to the choice of singularity, $f_{34}(x_6) = f_{33}(y_6) - f_{33}(x_6)$. Since $f_{33}(x_6)$ has a negative derivative, $f_{34}(x_6)$ is positive in Sector II. It follows that $f_{35}(x_6)$ is positive in Sector II.

Thus, all f_j's in the sixth cycle are positive in Sector II.

As can be seen from the equations, the cycles are independent of one another due to the presence of independent connectors (see [6]). Since each cycle has 2 sets of valid, positive conductivities, and there are 6 cycles, there are $2^6 = 64$ total sets of valid positive conductivities for the (3,3)-torus. Thus, the (3,3)-torus is 64 to 1.

Let's attempt to create a 64 to 1 graph by choosing appropriate λ_{ij}'s using the (3,3)-torus.

Step 1: Pick the singularities for the Σ's of each cycle.

Let $y_1 = 2$, $y_2 = 5$, $y_3 = 8$, $y_4 = 1$, $y_5 = 3$, and $y_6 = 7$.

Step 2: Choose values of the λ_{ij}'s in the quadrilateral to uphold the quadrilateral rule. After substitution, obtain a new set of the equations for the f_j's.

By the quadrilateral rule,

$$f_0(x_1)f_1(x_1) = f_{18}(x_4)f_{19}(x_4) = \lambda_{2,5}\lambda_{0,3}$$

Choose $\lambda_{2,5} = 2$ and $\lambda_{0,3} = 4$.

By the quadrilateral rule,

$$f_2(x_1)f_3(x_1) = f_{24}(x_5)f_{25}(x_5) = \lambda_{3,8}\lambda_{5,6}$$

Choose $\lambda_{3,8} = 1$ and $\lambda_{5,6} = 3$.

By the quadrilateral rule,

$$f_4(x_1)f_5(x_1) = f_{30}(x_6)f_{31}(x_6) = \lambda_{0,6}\lambda_{2,8}$$

Choose $\lambda_{0,6} = 7$ and $\lambda_{2,8} = 6$.

By the quadrilateral rule,

$$f_6(x_2)f_7(x_2) = f_{20}(x_4)f_{21}(x_4) = \lambda_{0,4}\lambda_{1,5}$$

Choose $\lambda_{0,4} = 11$ and $\lambda_{1,5} = 14$.

By the quadrilateral rule,

$$f_8(x_2)f_9(x_2) = f_{26}(x_5)f_{27}(x_5) = \lambda_{5,7}\lambda_{4,8}$$

Choose $\lambda_{5,7} = 13$ and $\lambda_{4,8} = 7$.

By the quadrilateral rule,

$$f_{10}(x_2)f_{11}(x_2) = f_{32}(x_6)f_{33}(x_6) = \lambda_{1,8}\lambda_{0,7}$$

Choose $\lambda_{1,8} = 2$ and $\lambda_{0,7} = 6$.

By the quadrilateral rule,

$$f_{12}(x_3)f_{13}(x_{13}) = f_{22}(x_4)f_{23}(x_4) = \lambda_{1,3}\lambda_{2,4}$$

Choose $\lambda_{1,3} = 17$ and $\lambda_{2,4} = 8$.

By the quadrilateral rule,

$$f_{14}(x_3)f_{15}(x_3) = f_{28}(x_5)f_{29}(x_5) = \lambda_{4,6}\lambda_{3,7}$$

Choose $\lambda_{4,6} = 9$ and $\lambda_{3,7} = 1$.

By the quadrilateral rule,

$$f_{16}(x_3)f_{17}(x_3) = f_{34}(x_6)f_{35}(x_6) = \lambda_{2,7}\lambda_{1,6}$$

Choose $\lambda_{2,7} = 3$ and $\lambda_{1,6} = 5$.

Now we can substitute and obtain a new set of equations for our f_j's.

For the first cycle,

Sign of Derivative	Equation
$+$	$f_0(x_1) = x_1$
$-$	$f_1(x_1) = \frac{8}{f_0(x_1)}$
$+$	$f_2(x_1) = \lambda_{3,5} - f_1(x_1)$
$-$	$f_3(x_1) = \frac{3}{f_2(x_1)}$
$+$	$f_4(x_1) = \lambda_{6,8} - f_3(x_1)$
$-$	$f_5(x_1) = \frac{42}{f_4(x_1)}$

For the second cycle,

Sign of Derivative	Equation
$+$	$f_6(x_2) = x_2$
$-$	$f_7(x_2) = \frac{154}{f_6(x_2)}$
$+$	$f_8(x_2) = \lambda_{4,5} - f_7(x_2)$
$-$	$f_9(x_2) = \frac{91}{f_8(x_2)}$
$+$	$f_{10}(x_2) = \lambda_{7,8} - f_9(x_2)$
$-$	$f_{11}(x_2) = \frac{12}{f_{10}(x_2)}$

For the third cycle,

Sign of Derivative	Equation
$+$	$f_{12}(x_3) = x_3$
$-$	$f_{13}(x_3) = \frac{136}{f_{12}(x_3)}$
$+$	$f_{14}(x_3) = \lambda_{3,4} - f_{13}(x_3)$
$-$	$f_{15}(x_3) = \frac{9}{f_{14}(x_3)}$
$+$	$f_{16}(x_3) = \lambda_{6,7} - f_{15}(x_3)$
$-$	$f_{17}(x_3) = \frac{15}{f_{16}(x_3)}$

For the fourth cycle,

Sign of Derivative	Equation
$+$	$f_{18}(x_4) = x_4$
$-$	$f_{19}(x_4) = \frac{8}{f_{18}(x_4)}$
$+$	$f_{20}(x_4) = \lambda_{0,5} - f_{19}(x_4)$
$-$	$f_{21}(x_4) = \frac{154}{f_{20}(x_4)}$
$+$	$f_{22}(x_4) = \lambda_{1,4} - f_{21}(x_4)$
$-$	$f_{23}(x_4) = \frac{136}{f_{22}(x_4)}$

For the fifth cycle,

Sign of Derivative	Equation
$+$	$f_{24}(x_5) = x_5$
$-$	$f_{25}(x_5) = \frac{3}{f_{24}(x_5)}$
$+$	$f_{26}(x_5) = \lambda_{5,8} - f_{25}(x_5)$
$-$	$f_{27}(x_5) = \frac{91}{f_{26}(x_5)}$
$+$	$f_{28}(x_5) = \lambda_{4,7} - f_{27}(x_5)$
$-$	$f_{29}(x_5) = \frac{9}{f_{28}(x_5)}$

For the sixth cycle,

Sign of Derivative	Equation
$+$	$f_{30}(x_6) = x_6$
$-$	$f_{31}(x_6) = \frac{42}{f_{30}(x_6)}$
$+$	$f_{32}(x_6) = \lambda_{0,8} - f_{31}(x_6)$
$-$	$f_{33}(x_6) = \frac{12}{f_{32}(x_6)}$
$+$	$f_{34}(x_6) = \lambda_{1,7} - f_{33}(x_6)$
$-$	$f_{35}(x_6) = \frac{15}{f_{34}(x_6)}$

Step 3: Obtain the other λ_{ij} values that were determined by our choice of singularities for the Σ's and by keeping the f_j's positive. After substitution, obtain a new set of the equations for the f_j's.

We have established that in order for $f_2(x_1)$ to be positive, we need $\lambda_{3,5} = f_1(y_1) + C_{3,5}$ where $C_{3,5}$ is some positive number. Let $C_{3,5} = 1$. Since $y_1 = 2$, $\lambda_{3,5} = f_1(2) + 1 = 4 + 1 = 5$.

In order for y_1 to be the singularity of $\Sigma_{x_1}(x_1)$, $\lambda_{6,8} = f_3(y_1)$. Since $y_1 = 2$, $\lambda_{6,8} = f_3(2) = 3$.

Thus the equations for the first cycle are

Sign of Derivative	Equation
$+$	$f_0(x_1) = x_1$

$-$	$f_1(x_1) = \frac{8}{x_1}$
$+$	$f_2(x_1) = \frac{5x_1-8}{x_1}$
$-$	$f_3(x_1) = \frac{3x_1}{5x_1-8}$
$+$	$f_4(x_1) = \frac{12x_1-24}{5x_1-8}$
$-$	$f_5(x_1) = \frac{210x_1-336}{12x_1-24}$

We have established that in order for $f_8(x_2)$ to be positive, we need $\lambda_{4,5} = f_7(y_2) + C_{4,5}$ where $C_{4,5}$ is some positive number. Let $C_{4,5} = 1$. Since $y_2 = 5$, $\lambda_{4,5} = f_7(5) + 1 = \frac{154}{5} + 1 = \frac{159}{5}$.

In order for y_2 to be the singularity of $\Sigma_{x_2}(x_2)$, $\lambda_{7,8} = f_9(y_2)$. Since $y_2 = 5$, $\lambda_{7,8} = f_9(5) = 91$.

Thus the equations for the second cycle are

Sign of Derivative	Equation
$+$	$f_6(x_2) = x_2$
$-$	$f_7(x_2) = \frac{154}{x_2}$
$+$	$f_8(x_2) = \frac{159x_2-770}{5x_2}$
$-$	$f_9(x_2) = \frac{455x_2}{159x_2-770}$
$+$	$f_{10}(x_2) = \frac{14014x_2-70070}{159x_2-770}$
$-$	$f_{11}(x_2) = \frac{1908x_2-9240}{14014x_2-70070}$

We have established that in order for $f_{14}(x_3)$ to be positive, we need $\lambda_{3,4} = f_{13}(y_3) + C_{3,4}$ where $C_{3,4}$ is some positive number. Let $C_{3,4} = 1$. Since $y_3 = 8$, $\lambda_{3,4} = f_{13}(8) + 1 = \frac{136}{8} + 1 = 18$.

In order for y_3 to be the singularity of $\Sigma_{x_3}(x_3)$, $\lambda_{6,7} = f_{15}(y_3)$. Since $y_3 = 8$, $\lambda_{6,7} = f_{15}(8) = 9$.

Thus, the equations for the third cycle are

Sign of Derivative	Equation
$+$	$f_{12}(x_3) = x_3$
$-$	$f_{13}(x_3) = \frac{136}{x_3}$

+	$f_{14}(x_3) = \frac{18x_3 - 136}{x_3}$
$-$	$f_{15}(x_3) = \frac{9x_3}{18x_3 - 136}$
+	$f_{16}(x_3) = \frac{153x_3 - 1224}{18x_3 - 136}$
$-$	$f_{17}(x_3) = \frac{270x_3 - 2040}{153x_3 - 1224}$

We have established that in order for $f_{20}(x_4)$ to be positive, we need $\lambda_{0,5} = f_{19}(y_4) + C_{0,5}$ where $C_{0,5}$ is some positive number. Let $C_{0,5} = 1$. Since $y_4 = 1$, $\lambda_{0,5} = f_{19}(1) + 1 = \frac{8}{1} + 1 = 9$.

In order for y_4 to be the singularity of $\Sigma_{x_4}(x_4)$, $\lambda_{1,4} = f_{21}(y_4)$. Since $y_4 = 1$, $\lambda_{1,4} = f_{21}(1) = 154$.

Thus, the equations for the fourth cycle are

Sign of Derivative	Equation
+	$f_{18}(x_4) = x_4$
$-$	$f_{19}(x_4) = \frac{8}{x_4}$
+	$f_{20}(x_4) = \frac{9x_4 - 8}{x_4}$
$-$	$f_{21}(x_4) = \frac{154x_4}{9x_4 - 8}$
+	$f_{22}(x_4) = \frac{1232x_4 - 1232}{9x_4 - 8}$
$-$	$f_{23}(x_4) = \frac{1224x_4 - 1088}{1232x_4 - 1232}$

We have established that in order for $f_{26}(x_5)$ to be positive, we need $\lambda_{5,8} = f_{25}(y_5) + C_{5,8}$ where $C_{5,8}$ is some positive number. Let $C_{5,8} = 1$. Since $y_5 = 3$, $\lambda_{5,8} = f_{25}(3) + 1 = \frac{3}{3} + 1 = 2$.

In order for y_5 to be the singularity of $\Sigma_{x_5}(x_5)$, $\lambda_{4,7} = f_{27}(y_5)$. Since $y_5 = 3$, $\lambda_{4,7} = f_{27}(3) = 91$.

Thus, the equations for the fifth cycle are

Sign of Derivative	Equation
+	$f_{24}(x_5) = x_5$
$-$	$f_{25}(x_5) = \frac{3}{x_5}$
+	$f_{26}(x_5) = \frac{2x_5 - 3}{x_5}$

$-$	$f_{27}(x_5) = \frac{91x_5}{2x_5-3}$
$+$	$f_{28}(x_5) = \frac{91x_5-273}{2x_5-3}$
$-$	$f_{29}(x_5) = \frac{18x_5-27}{91x_5-273}$

We have established that in order for $f_{32}(x_6)$ to be positive, we need $\lambda_{0,8} = f_{31}(y_6) + C_{0,8}$ where $C_{0,8}$ is some positive number. Let $C_{0,8} = 1$. Since $y_6 = 7$, $\lambda_{0,8} = f_{31}(7) + 1 = \frac{42}{7} + 1 = 7$.

In order for y_6 to be the singularity of $\Sigma_{x_6}(x_6)$, $\lambda_{1,7} = f_{33}(y_6)$. Since $y_6 = 7$, $\lambda_{1,7} = f_{33}(7) = 12$.

Thus, the equations for the sixth cycle are

Sign of Derivative	Equation
$+$	$f_{30}(x_6) = x_6$
$-$	$f_{31}(x_6) = \frac{42}{x_6}$
$+$	$f_{32}(x_6) = \frac{7x_6-42}{x_6}$
$-$	$f_{33}(x_6) = \frac{12x_6}{7x_6-42}$
$+$	$f_{34}(x_6) = \frac{72x_6-504}{7x_6-42}$
$-$	$f_{35}(x_6) = \frac{105x_6-630}{72x_6-504}$

Step 4: Choose the x-coordinates of one of the two crossings for each cycle to obtain more λ_{ij}'s. Check if any f_j's are negative with this x. Create the response matrix.

In cycle one, let $x_1 = 3$ be the x-coordinate for one of the two crossings to the right of the singularity, $y_1 = 2$. No f_j's are negative by this choice.

$f_0(3) = 3$

$f_1(3) = \frac{8}{3}$

$f_2(3) = \frac{7}{3}$

$f_3(3) = \frac{9}{7}$

$f_4(3) = \frac{12}{7}$

$f_5(3) = \frac{49}{2}$

Thus, $\Sigma_{x_1}(3) = 3 + \frac{49}{2} = \frac{55}{2} = \lambda_{0,2}$.

In cycle two, let $x_2 = 7$ be the x-coordinate for one of the two crossings to the right of the singularity, $y_2 = 5$. No f_j's are negative by this choice.

$f_6(7) = 7$

$f_7(7) = 22$

$f_8(7) = \frac{49}{5}$

$f_9(7) = \frac{65}{7}$

$f_{10}(7) = \frac{572}{7}$

$f_{11}(7) = \frac{21}{143}$

Thus, $\Sigma_{x_2}(7) = 7 + \frac{21}{143} = \frac{1022}{143} = \lambda_{0,1}$.

In cycle three, let $x_3 = 12$ be the x-coordinate for one of the two crossings to the right of the singularity, $y_3 = 8$. No f_j's are negative by this choice.

$f_{12}(12) = 12$

$f_{13}(12) = \frac{34}{3}$

$f_{14}(12) = \frac{20}{3}$

$f_{15}(12) = \frac{27}{20}$

$f_{16}(12) = \frac{153}{20}$

$f_{17}(12) = \frac{100}{51}$

Thus, $\Sigma_{x_3}(12) = 12 + \frac{100}{51} = \frac{712}{51} = \lambda_{1,2}$.

In cycle four, let $x_4 = 2$ be the x-coordinate for one of the two crossings to the right of the singularity, $y_4 = 1$. No f_j's are negative by this choice.

$f_{18}(2) = 2$

$f_{19}(2) = 4$

$f_{20}(2) = 5$

$f_{21}(2) = \frac{154}{5}$

$f_{22}(2) = \frac{616}{5}$

$f_{23}(x) = \frac{85}{77}$

Thus, $\Sigma_{x_4}(2) = 2 + \frac{85}{77} = \frac{239}{77} = \lambda_{2,3}$.

In cycle five, let $x_5 = 10$ be the x-coordinate for one of the two crossings to the right of the singularity, $y_5 = 3$. No f_j's are negative by this choice.

$f_{24}(10) = 10$

$f_{25}(10) = \frac{3}{10}$

$f_{26}(10) = \frac{17}{10}$

$f_{27}(10) = \frac{910}{17}$

$f_{28}(10) = \frac{637}{17}$

$f_{29}(10) = \frac{153}{637}$

Thus, $\Sigma_{x_5}(10) = 10 + \frac{153}{637} = \frac{6523}{637} = \lambda_{3,6}$.

In cycle six, let $x_6 = 20$ be the x-coordinate for one of the two crossings to the right of the singularity, $y_6 = 7$. No f_j's are negative by this choice.

$f_{30}(20) = 20$

$f_{31}(20) = \frac{21}{10}$

$f_{32}(20) = \frac{49}{10}$

$f_{33}(20) = \frac{120}{49}$

$f_{34}(20) = \frac{468}{49}$

$f_{35}(20) = \frac{245}{156}$

Thus, $\Sigma_{x_6}(20) = 20 + \frac{245}{156} = \frac{3365}{156} = \lambda_{2,6}$.

Now we have all the λ_{ij}'s that we need, and using the fact that row sums are 0 and response matrices are symmetrical along the diagonal, we can create the response matrix. Note that $\lambda_{ij} = 0$ if there doesn't exist a direct edge between vertices i and j in the R-Multigraph.

$$\Lambda = \begin{bmatrix} \frac{-22493}{286} & \frac{1022}{143} & \frac{55}{2} & 4 & 11 & 9 & 7 & 6 & 7 \\ \frac{1022}{143} & \frac{-1641710}{7293} & \frac{712}{51} & 17 & 154 & 14 & 5 & 12 & 2 \\ \frac{55}{2} & \frac{712}{51} & \frac{-17384947}{204204} & \frac{239}{77} & 8 & 2 & \frac{3365}{156} & 3 & 6 \\ 4 & 17 & \frac{239}{77} & \frac{-415824}{7007} & 18 & 5 & \frac{6523}{637} & 1 & 1 \\ 11 & 154 & 8 & 18 & \frac{-1649}{5} & \frac{159}{5} & 9 & 91 & 7 \\ 9 & 14 & 2 & 5 & \frac{159}{5} & \frac{-399}{5} & 3 & 13 & 2 \\ 7 & 5 & \frac{3365}{156} & \frac{6523}{637} & 9 & 3 & \frac{-518345}{7644} & 9 & 3 \\ 6 & 12 & 3 & 1 & 91 & 13 & 9 & -226 & 91 \\ 7 & 2 & 6 & 1 & 7 & 2 & 3 & 91 & -119 \end{bmatrix}$$

The response matrix of the (3,3)-torus

Step 5: Determine the x-coordinate for the other crossing in each cycle. Check if any f_j's are negative with this x.

When is $\Sigma_{x_1}(x_1) = \lambda_{0,2} = \frac{55}{2}$? Solving for x_1 in $\Sigma_{x_1}(x_1) = x_1 + \frac{210x_1 - 336}{12x_1 - 24} = \frac{55}{2}$ reveals two solutions: 3, as expected, and 9.

Plugging 9 into the f_j's in the first cycle shows that all the f_j's in the first cycle are positive at this 2nd crossing.

$f_0(9) = 9$

$f_1(9) = \frac{8}{9}$

$f_2(9) = \frac{37}{9}$

$f_3(9) = \frac{27}{37}$

$f_4(9) = \frac{84}{37}$

$f_5(9) = \frac{37}{2}$

When is $\Sigma_{x_2}(x_2) = \lambda_{0,1} = \frac{1022}{143}$? Solving for x_2 in $\Sigma_{x_2}(x_2) = x_2 + \frac{1908x_2 - 9240}{14014x_2 - 70070} = \frac{1022}{143}$ reveals two solutions: 7, as expected, and $\frac{35110}{7007}$.

Plugging $\frac{35110}{7007}$ into the f_j's in the second cycle shows that all the f_j's in the second cycle are positive at this 2nd crossing.

$f_6(\frac{35110}{7007}) = \frac{35110}{7007}$

$f_7(\frac{35110}{7007}) = \frac{539539}{17555}$

$f_8(\frac{35110}{7007}) = \frac{3742}{3511}$

$f_9(\frac{35110}{7007}) = \frac{319501}{3742}$

$f_{10}(\frac{35110}{7007}) = \frac{99951}{9355}$

$f_{11}(\frac{35110}{7007}) = \frac{14968}{7007}$

When is $\Sigma_{x_3}(x_3) = \lambda_{1,2} = \frac{712}{51}$? Solving for x_3 in $\Sigma_{x_3}(x_3) = x_3 + \frac{270x_3-2040}{153x_3-1224} = \frac{712}{51}$ reveals two solutions: 12, as expected, and $\frac{418}{51}$.

Plugging $\frac{418}{51}$ into the f_j's in the third cycle shows that all the f_j's in the third cycle are positive at this 2nd crossing.

$f_{12}(\frac{418}{51}) = \frac{418}{51}$

$f_{13}(\frac{418}{51}) = \frac{3468}{209}$

$f_{14}(\frac{418}{51}) = \frac{294}{209}$

$f_{15}(\frac{418}{51}) = \frac{627}{98}$

$f_{16}(\frac{418}{51}) = \frac{255}{98}$

$f_{17}(\frac{418}{51}) = \frac{98}{17}$

When is $\Sigma_{x_4}(x_4)$ and $\lambda_{2,3} = \frac{239}{77}$? Solving for x_4 in $\Sigma_{x_4}(x_4) = x_4 + \frac{1224x_4-1088}{1232x_4-1232} = \frac{239}{77}$ reveals two solutions: 2, as expected, and $\frac{171}{154}$.

Plugging $\frac{171}{154}$ into the f_j's in the fourth cycle shows that all the f_j's in the fourth cycle are positive at this 2nd crossing.

$f_{18}(\frac{171}{154}) = \frac{171}{154}$

$f_{19}(\frac{171}{154}) = \frac{1232}{171}$

$f_{20}(\frac{171}{154}) = \frac{307}{171}$

$f_{21}(\frac{171}{154}) = \frac{26334}{307}$

$f_{22}(\frac{171}{154}) = \frac{20944}{307}$

$f_{23}(\frac{171}{154}) = \frac{307}{154}$

When is $\Sigma_{x_5}(x_5)$ and $\lambda_{3,6} = \frac{6523}{637}$? Solving for x_5 in $\Sigma_{x_5}(x_5) = x_5 + \frac{18x_5-27}{91x_5-273} = \frac{6523}{637}$ reveals two solutions: 10, as expected, and $\frac{1938}{637}$.

Plugging $\frac{1938}{637}$ into the f_j's in the fifth cycle shows that all the f_j's in the fifth cycle are positive at this 2nd crossing.

$f_{24}(\frac{1938}{637}) = \frac{1938}{637}$

$f_{25}(\frac{1938}{637}) = \frac{637}{646}$

$f_{26}(\frac{1938}{637}) = \frac{655}{646}$

$f_{27}(\frac{1938}{637}) = \frac{58786}{655}$

$f_{28}(\frac{1938}{637}) = \frac{819}{655}$

$f_{29}(\frac{1938}{637}) = \frac{655}{91}$

When is $\Sigma_{x_6}(x_6)$ and $\lambda_{2,6} = \frac{3365}{156}$? Solving for x_6 in $\Sigma_{x_6}(x_6) = x_6 + \frac{105x_6-630}{72x_6-504} = \frac{3365}{156}$ reveals two solutions: 20, as expected, and $\frac{2219}{312}$.

Plugging $\frac{2219}{312}$ into the f_j's in the sixth cycle shows that all the f_j's in the sixth cycle are positive at this 2nd crossing.

$f_{30}(\frac{2219}{312}) = \frac{2219}{312}$

$f_{31}(\frac{2219}{312}) = \frac{1872}{317}$

$f_{32}(\frac{2219}{312}) = \frac{347}{317}$

$f_{33}(\frac{2219}{312}) = \frac{3804}{347}$

$f_{34}(\frac{2219}{312}) = \frac{360}{347}$

$f_{35}(\frac{2219}{312}) = \frac{347}{24}$

Since there are 2 solutions for each cycle and each cycle is independent, there are $2^6 = 64$ solutions. For a single response matrix, we can obtain 64 sets of positive conductivities.

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