1 Introduction

The main subject of the paper is winning strategies for the knotting/unknotting game, which is known as "To Knot or Not to Knot," in which.. (explanation of game).

The knot I will be examining in this paper is what I refer to as the "4-petal Flower Knot" (see image).

For the sake of convenience, I have created a notation system that involves numbering each precrossing as a pair of numbers, either n or $-n$ where n is an integer from 1 to 8. Beginning on a random point on the knot, follow a path along the knot in a particular direction and label each precrossing, beginning from 1. Whenever a precrossing is reached a second time, label it again with the same number, but with sign $-$. Thus, once the initial point is reached a second time, each precrossing should have a pair of labels, namely $(n, -n)$. As precrossings are resolved during the game, either n or $-n$ will be removed for each precrossing, according to which strand is on top, leaving only one label per precrossing. For example, if on the first precrossing, the strand that was passed through first on the initial labeling is on top of the other, then that newly resolved crossing will take on the label 1. If instead on the same precrossing, the strand that was passed through second on the initial labeling is on top of the other, then that newly resovled crossing will take on the label -1. Once the game ends, there will be a sequence of eight labels that correspond to the final knot diagram. The motivation behind this notation is that it simplifies the task of referring to which precrossing is resolved in what way, and allows the reader to visualize the knot diagram without much explanation.

As it does not matter how the knot is labeled, including where the initial point on the knot is and in what direction the knot is followed along, we shall use this notation for the sake of consistency throughout the paper. (see image)

2 Body

Given a shadow of the "4-petal Flower Knot," the knotter has a winning strategy if he/she does not go first. Before exploring this strategy, I will present some definitions.

Definition 1 *Knotting Sequence = A sequence of labels, which if contained in the sequence of the final knot diagram, produces a knot.*

There are 8 unique knotting sequences. Taking into account mirroring, there are 16 sequences. These sequences are:

[2, -4, 5, -8] [6, -7, 8, -5] [-2, 4, -5, 8] [-6, 7, -8, 5]
[1, -3, 5, -6] [1, -3, 8, 4] [-1, 3, -5, 6] [-1, 3, -8, -4]
[1, -7, 8, -2] [2, -3, 5, 7] [-1, 7, -8, 2] [-2, 3, -5, -7]
[4, -6, 7, -3] [2, -4, 6, -1] [-4, 6, -7, 3] [-2, 4, -6, 1]

To prove that these sequences are in fact knotting sequences only requires checking two of the sequences, one in the left column and one in the right column.

The 4 sequences in the left column are essentially the same sequence but rotated along each of the 4 petals of the knot. The 4 sequences in the right column are also essentially the same sequence but with respect to each of the 4 inner strands of the knot. Thus, the symmetry of this particular knot eases the task of checking each of these 16 sequences and reduces it to two. Take any sequence from the left column, and resolve the remaining precrossings in any manner, and the knot diagram will be knotted. Repeat with one sequence from the right column. By symmetry, all of the sequences are shown to create a knot.

Thus, if the knotter can produce a knot diagram with at least one of these sequences contained in the final knot sequence, the knotter wins.

I propose that there exists a strategy so that if the knotter moves second, one of these final knotting sequences is guaranteed at the end of the game, and thus the knotter wins the game. But before I present the strategy, a few more definitions must be given.

Definition 2 *Buddy = A label that exists in at least 2 knotting sequences with another label. For example,* 1 *and* 4 *are buddies,* 2 *and* -8 *are buddies, etc.*

Lemma 1 *A pair of buddies will always have the same set of signs + or − relative to each other. For example, because* 1 *and* 4 *are buddies,* 1 *and* -4 *can never be buddies, and vise versa. (Proof requires explanation of how the knotting sequences were formed. Based on the idea of trefoil within the 4-petal flower knot, and the fact that there are 8 ways to obtain it. The buddies are obtained by creating trefoils, which require a certain sequence within a knotting sequence (over under over or under over under). Thus, by changing the sign of a lable within a sequence, this strict trefoil pattern is ruined and the trefoil becomes the unknot.)*

Definition 3 *Buddy System = The set of buddies for a given label.*

Lemma 2 *A buddy system for a given label will never have both labels of a precrossing (i.e. positive and negative values of a label). This is an extension of the previous lemma.*

The buddy system is motivated by the strategy to build up as many knotting sequences as the knotter can, or to prevent as many as the unknotter can. It will play an important part in the strategy.

2.1 Strategy

The Unknotter moves first on any crossing of his/her choosing. There is no loss of generality here because each label has the same number of elements in its buddy system so it is not more advantageous for one player or the other to move on a particular precrossing on the first turn. The symmetrical nature of this knot also contributes to this as well.

The Knotter moves on any label within the buddy system of the first move. Already, there are two knotting sequences with two labels already made.

The Unknotter moves in such a way as to make one of the knotting sequences impossible. (If the Unknotter does not prevent one of the two knotting sequences halfway made, then the Knotter can move on one of those two sequences, in effect being one label away from a knot, as well as build at least one other knotting sequence so that it is halfway made (this can be done by moving on a label that is within both buddy systems of the first two labels). At this point, the Unknotter is forced to prevent the knotting sequence that is one away from completion. The Knotter can then move on a label that is within the buddy systems of the 1st move and the two moves made previously by the Knotter. This builds up at least two knotting sequences with one label away from completion, so that regardless of where the Unknotter moves next, the Knotter can complete at least one knotting sequence. (For example, [1, -3, 7, -6, -5, 4, -8, -2].) Therefore, it is advantageous to prevent this easy scenario for the Knotter and to move in a way as to prevent one of the two knotting sequences built up by the Knotter's 1st move.)

The Knotter moves on the knotting sequence that the Unknotter did not prevent so that there exists one knotting sequence that is one label away from completion. Of the two possible labels, the Knotter must move on the label whose opposite does not lie in the buddy system of the Unknotter's previous move. (From this point on, the Unknotter's moves are forced.)

The Unknotter is forced to move so that this sequence is impossible.

The Knotter will have at least two knotting sequences left available for completion, two of which willl be halfway complete. Within those two knotting sequences, there will be a common label. The Knotter moves on that label. (As this label is common to two halfway made knotting sequences, there are now two knotting sequences with one label away from completion.) (This scenario is an assumption that is explained later in the general notes.)

The Unknotter is forced to prevent one of the knotting sequences.

The Knotter moves on the other knotting sequence and completes it. The Knotter wins.

2.2 General Notes

The first two moves do not affect generality, which is preserved. It has already been shown why the first move does not affect generality. To show that the second move does not cause a loss of generality involves the buddy system. Due to the symmetrical nature of this particular knot, each label has a buddy system with the same number of buddies. Therefore, since the second move requires that the Knotter move within the buddy system (which each label has of equal size) of the label of the first move (which does not matter in terms of generality), the second move does not lose generality. Essentially, this allows one to consider every possible sequence of moves in every possible game with the first two moves already determined. By checking every possible sequence with the starting label as 1*, one can easily verify that the same properties/outcome hold with any other starting label (as long as the proceeding moves are adjusted accordingly). Additionally, by checking every possible sequence with some starting label n and* *second move within the buddy system of n, one can easily verify that the same properties/outcome hold with any other move within the buddy system of n (as long as the proceeding moves are adjusted accordingly).*

Since the first two moves are not particularly strategic and do not cause a loss of generality, and the last 4 moves are determined, only the third and fourth move are variable. Thus, it reduces the number of possibilities of moves significantly, so that it is easily verifiable that after the 5th move, there will be at least two halfway complete sequences with a label in common. It is the Knotter's move on this label that forces a win in his/her favor.