Finding Broken Resistors in a Network

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Abstract

This article will focus on the z-sequence and certain determinants of the Λ matrix as a means of retrieving broken resistors in a square network. We attempt to find the broken resistors given only the Λ matrix for the broken network. Some of the results can be extended to rectangular networks and circular planar networks.

1 Introduction

The problems presented in this paper are similar to those in [3], except we only consider the case in which the connection between a resistor and node has been completely broken: no current is allowed to pass the break, so the conductivity of the resistor is effectively zero.

A *graph with boundary* Γ is a triple $(V, \partial V, E)$ where V denotes the set of nodes, E denotes the set of edges, and ∂V is a non-empty subset of V denoting the set of boundary nodes. A *circular planar graph* is Γ embedded in a disc in a plane such that all boundary nodes lie on the circle C which bounds the disc and the rest of Γ lies within the interior of the disc. The boundary nodes will be numbered $v_1, \dots, v_N$ in clockwise order around C and a *circular pair* of boundary nodes $(P;Q) = (p_1, \dots, p_k; q_1, \dots, q_k)$ will be a sequence of boundary nodes such that $(p_1, \dots, p_k, q_k, \dots, q_1)$ is in clockwise order around C. The *medial graph* of Γ, $\mathcal{M}(\Gamma)$, *geodesics*, *lenses*, and the *z-sequence* will be defined as in [1].

A *path* between boundary nodes a and b of Γ is either an edge ab, or a sequence of interior nodes $n_1, n_2, \dots, n_m$ such that $an_1, n_1n_2, \dots, n_{m-1}n_m, n_mb$ are edges in Γ. A circular pair of boundary nodes $(P;Q) = (p_1, \dots, p_k; q_1, \dots, q_k)$ is *connected through* Γ if there are paths from p_i to q_i for each i and the paths are disjoint.

An edge of Γ may be removed by either deleting it or contracting it to a single node. An edge connecting two boundary nodes may not be contracted to a single node. If, after removing an edge from Γ, the resultant graph Γ' has at least one circular pair which was connected through Γ but is not connected through Γ', and if this property holds for all edges, the graph is said to be *critical*. A graph is critical if and only if its $\mathcal{M}(\Gamma)$ is lensless ([1]).

The following transformation from figures 1a to 1b is called a Y–Δ transformation. A Δ–Y transformation is the transformation from figures 1b to 1a.

Two graphs are said to be Y–Δ equivalent if a sequence of Y–Δ or Δ–Y transformations can be made on each graph to obtain the other.

A *circular planar resistor network* is Γ with a function γ which assigns to each $e \in E$ a positive real number $\gamma(e)$ called the conductance. The Dirichlet to Neumann map $\Lambda_{\Gamma,\gamma}$ for

(a) Y (b) Δ

Figure 1: Y–Δ transformation

a circular planar resistor network will be defined as in [1]. The notation $\Lambda(P;Q)$, where P and Q form a circular pair, will denote the sub-matrix of Λ formed from the intersection of rows P and columns Q. A *square network of resistors* will be defined as in [2]. A *square network* Ω will also be defined as in [2], but we will call a square network a *square graph* in this paper.

Whenever we remove edges from a graph, we do not change the boundary nodes on both the graph and its respective medial graph. So given a graph Γ and obtaining another graph Γ' by removing edges, we will make two important assumptions throughout this paper:

1. The numbering of the boundary nodes will start from the same node on both graphs.

2. The numbering of the z-sequence for their medial graphs, $\mathcal{M}(\Gamma)$ and $\mathcal{M}(\Gamma')$, will start at the same node for both medial graphs.

In the case of broken resistors, we consider both the forward problem—break resistors and find the resulting changes in the Λ matrix—and the inverse problem—given only the Λ for the broken network, find the broken resistors. The focus will be on the inverse case. For suitably chosen conductivity functions, [1] tells us two Y–Δ equivalent graphs cannot be distinguished from just the Λ matrix, so we will attempt recovery up to Y–Δ equivalence. We will also analyze both critical and non-critical graphs and attempt to extend the results to a general circular planar resistor network.

2 Single Resistors

There are two ways to determine the position of a single broken resistor in a square network. The first uses the z-sequence and the second uses determinants of the Λ matrix.

2.1 The z-sequence

Given the Λ matrix for a circular planar resistor network (Γ,γ), we can find the z-sequence from the Λ matrix using a Mathematica package written by Chris Staskewicz. If we have two square resistor networks and we know only one resistor was removed, only two geodesics in $\mathcal{M}(\Gamma)$ can change since only two geodesics cross at one resistor in Γ. However, the z-sequence for the new graph may change in many places, so we will define a change in the z-sequence to mean the following: if the numbers j and j' occur in the same position in the original z-sequence Z and altered z-sequence Z', respectively, then the other occurence of j and j' do not occur in the same position.

By the way $\mathcal{M}(\Omega)$ is constructed, removing any edge from Ω can affect at most two geodesics in $\mathcal{M}(\Omega)$, since only two geodesics can intersect at a single edge in Ω. In the case of a single broken edge in a square graph, the new graph is still critical.

With these facts, we can make the following theorem.

THEOREM 2.1–1: *If a single edge has been broken in a square graph Ω to obtain another graph Ω', the geodesics associated with the first number j in the z-sequence Z which changes and the number j' in the new z-sequence Z' in the same position as j intersect at the broken resistor in the original graph Ω.*

2.2 Determinants

For a single broken resistor in a square network, determinants of the Λ matrix may be used to find the position of the broken resistor. We consider the following two cases separately:

1. A broken boundary spike
2. Any other broken resistor which is not a boundary spike.

In the first case, the broken resistor can be easily located from the Λ matrix.

THEOREM 2.2–1: *Given a square network of resistors and another square network with the i^{th} boundary resistor broken, the i^{th} row and column of Λ will contain all zeros.*

We now give the algorithm for the second case. Given an $n \times n$ rectangular network, we number two adjacent sides 0 through $n+1$, starting from the upper left corner. Then each node in the network can be indentified by the ordered pair (a, b) where a denotes the row of the node and b its column. In addition, we can identify each edge σ in Γ by the two nodes it connects. Thus, $\sigma_{(0,1);(1,1)}$ is the edge that connects the node at $(0,1)$ to the node $(1,1)$. Finally, we denote the set of nodes on a side as A, B, C, or D, going clockwise from the left side.

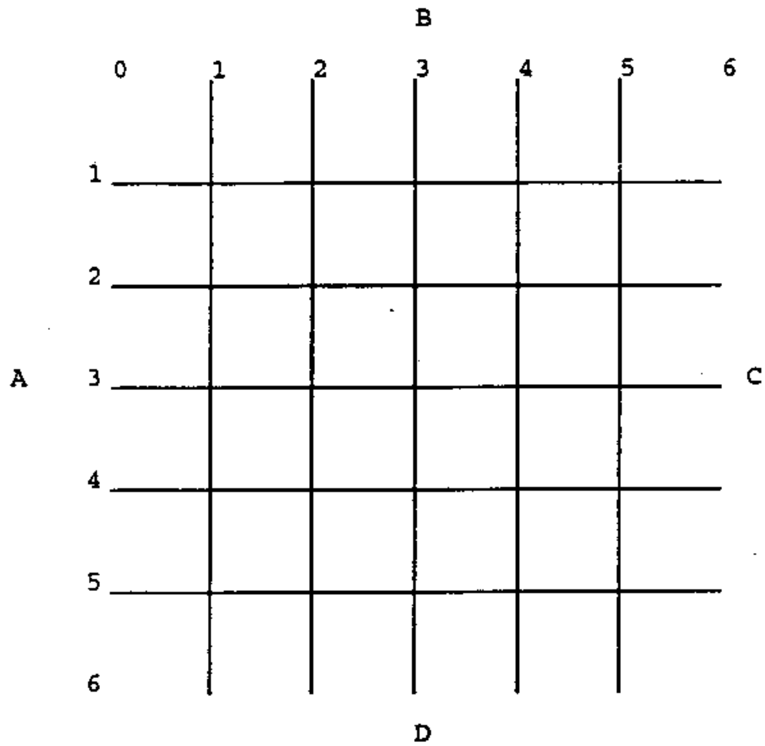


Figure 2: In this diagram, $A = \{(a,b) : a = 0,1,\ldots,6; b = 0\}, B = \{(a,b) : a = 0; b = 0,1,\ldots,6\}, C = \{(a,b) : a = 0,1,\ldots,6; b = 6\}, D = \{(a,b) : a = 6; b = 0,1,\ldots,6\}$

If we draw a line from the node $(0,0)$ to $(n+1,n+1)$, and one from $(0,n+1)$ to $(n+1,0)$, we have divided Γ into four sectors.

Now, suppose that exactly one of the edges in Γ is broken. Assuming it is not an edge connecting a node in A, B, C, or D to an interior node, we examine sub-determinants of Λ. First, we look at $det(\Lambda_{A,C})$. Clearly, the only way to connect the circular pairs A and C is to use all the horizontal edges. Thus if $det(\Lambda_{A,C}) = 0$, one of these edges must be broken. Similarly, if $det(\Lambda_{B,D}) = 0$, the broken edge must be one of the vertical ones. (Notice that we can't have both $det(\Lambda_{A,C})$ and $det(\Lambda_{B,D})$ zero, because of the assumption that only one edge is broken.) Next, we examine the sub-determinants corresponding to the adjacent circular pairs: $det(\Lambda_{A,B})$, $det(\Lambda_{B,C})$, $det(\Lambda_{C,D})$, and $det(\Lambda_{A,D})$. Clearly, there is only one way to connect these circular pairs. Next, because one edge is broken, two of these determinants will be zero. If we look at the intersection of the set of edges (call it β) that connect the cirular pairs whose determinants vanish, we have the broken edge restricted to one sector of Γ.

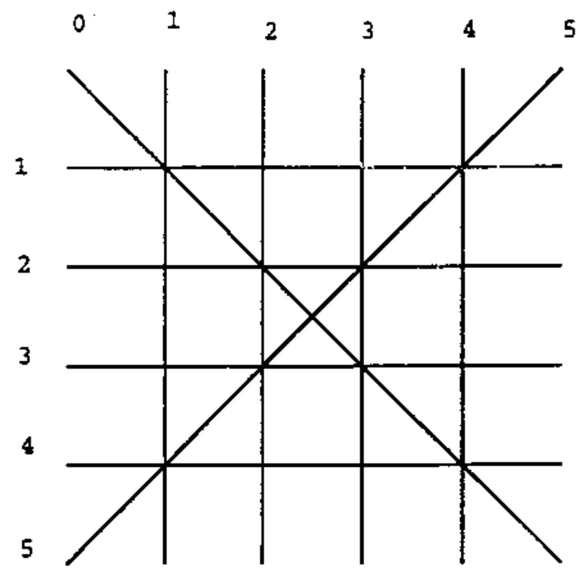


Figure 3: If $\sigma_{(1,2);(1,3)}$ is broken, $det(\Lambda_{A,C}) = 0$ and $det(\Lambda_{C,D}) = det(\Lambda_{A,D}) = 0$ (Common side B)

Notice that whichever two of $det(\Lambda_{A,B})$; $det(\Lambda_{B,C})$; $det(\Lambda_{C,D})$; and $det(\Lambda_{A,D})$ is zero, they have one side in common. To establish the broken edge's position, we examine the sub-determinants corresponding to connecting nodes in the common side to nodes in either adjacent side.

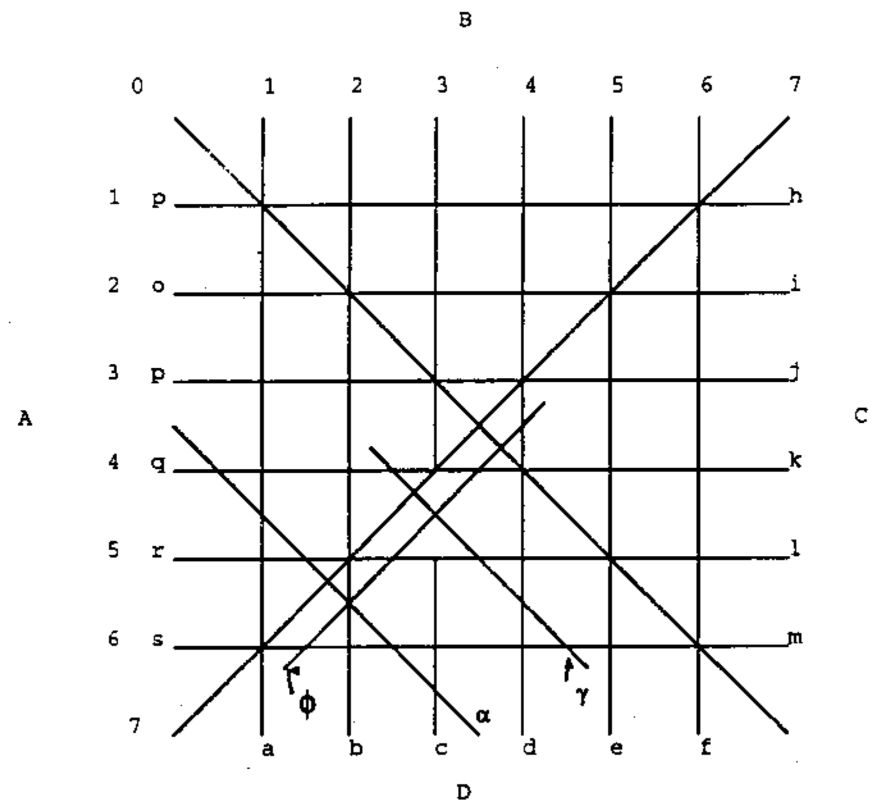


Figure 4: In this diagram $\sigma_{(4,3);(5,3)}$ is broken. Thus $det(\Lambda_{B,D}) = 0$ and $det(\Lambda_{A,D}) = det(\Lambda_{C,D}) = 0$. Here, the common side D.)

First, we connect subsets of the common side to subsets of the adjacent counterclockwise side. We connect the four farthest nodes to one another (two from each side). In the above diagram, the four farthest nodes between C and D are marked a, b, h and i. Then there is clearly one edge in β that must traversed if the nodes are to connected. Thus if $det(\Lambda_{a,b;h,i})$ $= 0$, that edge must be broken. If $det(\Lambda_{a,b;h,i}) \neq 0$, then that edge is not broken. Next, we connect the six farthest nodes (a,b,c and h,i,j in the diagram). Then there are new edges in β that must be traversed. Notice that these edges lie on the same line (marked α in the diagram). If $det(\Lambda_{a,b,c;h,i,j}) = 0$, one of these must be broken; if this determinant is non-zero, they are whole. We continue in this fashion until we get a vanishing sub-dterminant. Then, one of the members of β that is in the set of edges traversed must be broken. (In the above diagram, $det(\Lambda_{a,b,c,d,e;h,i,j,k,l}) = 0$. Notice that the set of edges in β that are traversed lie on a line, labeled γ.)

To finally establish the position of the broken resistor, we repeat the process, only now with the clockwise adjacent side. When we find a vanishing sub-determinant with this new side, we see where the lines that the questionable resistors lie on cross. This intersection is where the broken resistor lies. (In the diagram, we already have the broken resistor on γ.) When we examine determinants corresponding to A and D, we see that $det(\Lambda_{a-f;n-s})$ $= det(\Lambda_{A,D}) = 0$. The questionable edges are on the line ϕ. Notice that the intersection of ϕ and γ is the broken resistor.

We can also generalize the process of finding a single broken edge to a rectangular network.

3 Multiple Resistor Removal

In this section, we remove multiple resistors. Now we need to not only find the location of the removed edges, but also how many were removed. As it turns out, we were unable to completely accomplish the first for even the seemingly simple case of two connected resistors. We were also unable to answer the second question, but we were able to find a relationship between changes in the medial graph, the number of edges removed and the relative position of the set of edges to be removed.

3.1 Geodesics

In particular, we have a formula that gives the number of changed geodesics in the medial graph as a function of how many edges are removed and how many geodesics cross the set of edges to be removed.

Definition. Any geodesic that crosses any of the edges in the set to be removed is said to be changed.

It follows from the definition that once changed, a geodesic can not be un-changed.

Notation:

$\begin{pmatrix} \# \\ \text{changes} \end{pmatrix}_k$ = the total number of geodesics that are changed by the removal of k edges.

$\begin{pmatrix} \# \\ n\ \times \end{pmatrix}_k$ = the total number of geodesics in the original graph that cross exactly n of the k edges to be removed.

Useful Lemma: *After removing k edges, consider removing a $(k+1)^{st}$ edge. Then:*

$$\begin{pmatrix} \# \\ \text{changes} \end{pmatrix}_{k+1} = \begin{cases} \begin{pmatrix} \# \\ \text{changes} \end{pmatrix}_k + 2, & \text{if the } (k+1)^{st} \text{ edge has geodesics crossing it which did not cross any of the first } k, \\ \begin{pmatrix} \# \\ \text{changes} \end{pmatrix}_k + 1, & \text{if the } (k+1)^{st} \text{ edge has exactly one geodesic crossing it which also crossed at least one of the first } k, \\ \begin{pmatrix} \# \\ \text{changes} \end{pmatrix}_k, & \text{if the } (k+1)^{st} \text{ edge has exactly two geodesics crossing it which both crossed at least one of the first } k. \end{cases}$$

Proof. Follows from the definition of changed geodesics, and the fact that any edge is crossed by exactly two geodesics.

THEOREM: *Given any critical circular planar graph and any set of n edges. Then, after removing the set of edges and before removing the lenses,*

$$\binom{\#}{changes}_n = 2(n) - \left[\sum_{i=1}^{n}(i-1)\binom{\#}{i\times}_n\right].$$

Proof. We proceed by induction on the number of edges removed.

Consider $n = 1$. Because each edge is crossed by two geodesics, the removal of one edge will change two geodesics:

$$\binom{\#}{\text{changes}}_1 = 2 = 2(1) - \left[\sum_{i=1}^{1}(i-1)\binom{\#}{i\times}_1\right].$$

Next we assume the result for $n = k$:

$$\binom{\#}{\text{changes}}_k = 2(k) - \left[\sum_{i=1}^{k}(i-1)\binom{\#}{i\times}_k\right].$$

Now, if we remove one more edge ($(k+1)$ total), we have, by the Useful Lemma, three cases:

Case 1: $\binom{\#}{\text{changes}}_{k+1} = \binom{\#}{\text{changes}}_k + 2$, if the $(k+1)^{st}$ edge has geodesics crossing it which did not cross any of the previous k,

Case 2: $\binom{\#}{\text{changes}}_{k+1} = \binom{\#}{\text{changes}}_k + 1$, if the $(k+1)^{st}$ edge has exactly one geodesic crossing it which also crossed at least one of the previous k,

Case 3: $\binom{\#}{\text{changes}}_{k+1} = \binom{\#}{\text{changes}}_k$, if the $(k+1)^{st}$ edge has exactly two geodesics crossing it which both crossed at least one of the previous k.

Case 1:
$$\begin{aligned}\binom{\#}{\text{changes}}_{k+1} &= \binom{\#}{\text{changes}}_k + 2 \\ &= 2(k) - \left[\sum_{i=1}^{k}(i-1)\binom{\#}{i\times}_k\right] + 2 \\ &= 2(k+1) - \left[\sum_{i=1}^{k+1}(i-1)\binom{\#}{i\times}_{k+1}\right]\end{aligned}$$

The last step is done because none of the terms in the summation are changed.

Case 2: Now the $(k+1)^{st}$ edge has exactly one geodesic crossing it which also crossed at least one of the previous k. We assume WLOG that the removal of the $(k+1)^{st}$ edge causes a geodesic that had previously crossed j to-be-removed edges to now cross $j+1$ to-be-removed edges. Next, we look at

$$\Delta = \left[2(k+1) - \left[\sum_{i=1}^{k+1}(i-1)\begin{pmatrix}\#\\ i\times\end{pmatrix}_{k+1}\right]\right] - \left[2(k) - \left[\sum_{i=1}^{k}(i-1)\begin{pmatrix}\#\\ i\times\end{pmatrix}_{k}\right]\right]$$

(We want to show $\Delta = 1$.)

$\Delta = 2 - [(j-1)(-1) + (j)(+1)]$ (because, for all $i \neq j$ and $i \neq j+1$, we have $\begin{pmatrix}\#\\ i\times\end{pmatrix}_k = \begin{pmatrix}\#\\ i\times\end{pmatrix}_{k+1}$)

$= 2 - [-j + 1 + j]$

$= 1.$

Case 3: Now the $(k+1)^{st}$ edge has exactly two geodesics crossing it which both crossed at least one of the previous k. We assume WLOG that the removal of the $(k+1)^{st}$ edge causes one geodesic that previously crossed j to-be-removed edges to now cross $j+1$ to-be-removed edges, and also causes one geodesic that previously crossed m to-be-removed edges to now cross $m+1$ to-be-removed edges. If $m = j$ a similar argument will apply. Thus,

$\Delta = 2 - [(j-1)(-1) + (m-1)(-1) + j(+1) + m(+1)]$

$= 2 - [-j + 1 - m + 1 + j + m]$

$= 2 - [2]$

$= 0$, as we want for Case 3.

Thus,

$$\begin{pmatrix}\#\\ \text{changes}\end{pmatrix}_{k+1} = 2(k+1) - \left[\sum_{i=1}^{k+1}(i-1)\begin{pmatrix}\#\\ i\times\end{pmatrix}_{k+1}\right]$$

and so, by induction,

$$\begin{pmatrix}\#\\ \text{changes}\end{pmatrix}_{n} = 2(n) - \left[\sum_{i=1}^{n}(i-1)\begin{pmatrix}\#\\ i\times\end{pmatrix}_{n}\right].\ \text{QED}$$

Remarks:

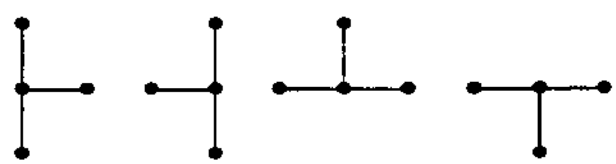
1. The definition of changed geodesic went through several stages before settling on the current definition. The first definition was connected much more closely to the

z-sequence of the medial graph after removing the edges and before removing the lenses. As it turned out, the theorem was not true with this definition. The flaw was that, according to the earlier definition, a geodesic was not changed so long as it still connected the same endpoints. However, the current, stronger, definition will not in general give the number of changes in the z-sequence.

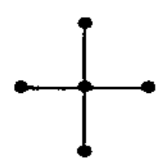
2. The z-sequence that one gets from the Λ matrix is for a critical network. However, the formula only gives the number of changes after removal of the edges, but before removal of lenses. By removing the lenses, the relationship between the number of changes, number of edges removed and the relative position of the removed edges is obscured. In addition, we would need to use a different definition of changed geodesics, because the current definition tells us nothing about geodesics that change as a result of removing lenses but that did not originally cross one of the removed edges. The definition to use would be more closely related to the z-sequence than the current definition.

   This also explains why it is much easier to find a single broken resistor. When we delete a single edge, the graph remains critical. Thus the z-sequence from the Λ matrix is the same as that from the medial graph—unlike removing multiple resistors, which in general creates lenses.

3. There is also the fact that deletion of any one of the following in a square graph:

   is indistinguishable from the deletion of:

   This is easily seen by comparing a graph with a 'T' deleted to one with a 'cross' deleted as in Figure 5.

3.2 Recovery

We begin with a few definitions.

Definition 3.2–1. Two graphs Γ and Γ' are said to be ***electrically equivalent*** if they have the same sets of Λ matrices; that is, for $\Lambda_{\Gamma,\gamma}$ and $\Lambda_{\Gamma',\gamma'}$, $\exists$ a γ and γ' such that $\Lambda_{\Gamma',\gamma'} = \Lambda_{\Gamma,\gamma}$ $\forall$ γ and γ'.

Given a graph Γ and obtaining another graph Γ' by removing edges, we can find the z-sequence Z for Γ and Z' for Γ' by using a Mathematica package written by Chris Staskewicz.

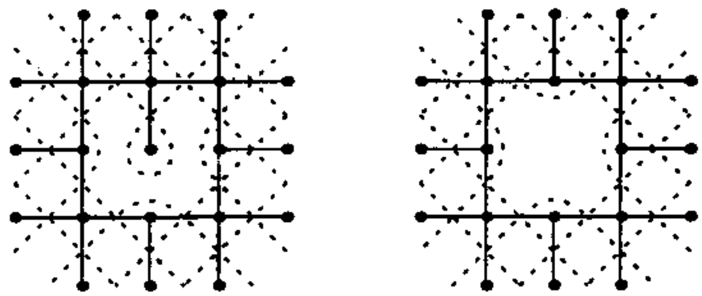


Figure 5: The removal of a 'T' produces the same z-sequence as the removal of a 'cross'

If the boundary nodes were numbered in the same way for both graphs, the numbering of the z-sequence will start at the same points on both graphs.

Definition 3.2–2. In this case, let $Z = z_1, z_2, \ldots, z_{2N}$ and $Z' = z'_1, z'_2, \ldots, z'_{2N}$. If $z_i = z_j$ but $z'_i \neq z'_j$, $1 \leq i \leq 2N$ and $1 \leq j \leq 2N$, the geodesic associated with z_i will be called a geodesic which *changes its endpoint.*

In the case of a square graph, it is possible to have a variety of graphs with different patterns of deletions which result in a critical graph.

3.2.1 Critical Graphs

In the recovery of critical square graphs, we first consider the case of a broken boundary spike.

THEOREM 3.2.1–1: *Suppose a set of edges is broken in a square resistor network (Ω, γ) to produce a new network (Ω', γ'). If Ω' is critical, then the i^{th} row and column of $\Lambda_{\Omega',\gamma'}$ will contain all zeros if and only if the boundary spike connected to the i^{th} boundary node of Ω has been deleted.*

Proof. Proof is left to the reader. □

We now will limit our search for broken resistors to any resistor which is not a boundary spike. We could not find a counter example for the following conjecture.

CONJECTURE 3.2.1–2: *Suppose a set of edges is broken in a square graph Ω to produce a new graph Ω'. Assume Ω' is critical. Then any geodesic in $\mathcal{M}(\Omega)$ which intersected a broken edge in Ω will change its endpoint.*

If Ω' is critical, the Conjecture would give a way to narrow down where the broken edges are. The geodesics which changed endpoints originally intersected at resistors which could have been broken. These are the only possible resistors which could have been broken. All intersections of these geodesics might not correspond to a broken resistor, but any resistor which is broken must lie on an intersection of these geodesics.

The Conjecture would also extend the formula for a changed geodesic

$$\binom{\#}{\text{changes}}_n = 2(n) - \left[\sum_{i=1}^{n}(i-1)\binom{\#}{i\times}_n\right].$$

since it would follow that any geodesic which changes must also change its endpoint.

Figure 6 shows an example of two out of six possible sets of removals resulting in critical Y–Δ equivalent graphs. Since the graphs are Y–Δ equivalent, they cannot be distinguished from the Λ matrix. All geodesics which intersected an edge which was removed changed their endpoints. The dashed lines represent geodesics which changed their endpoints.

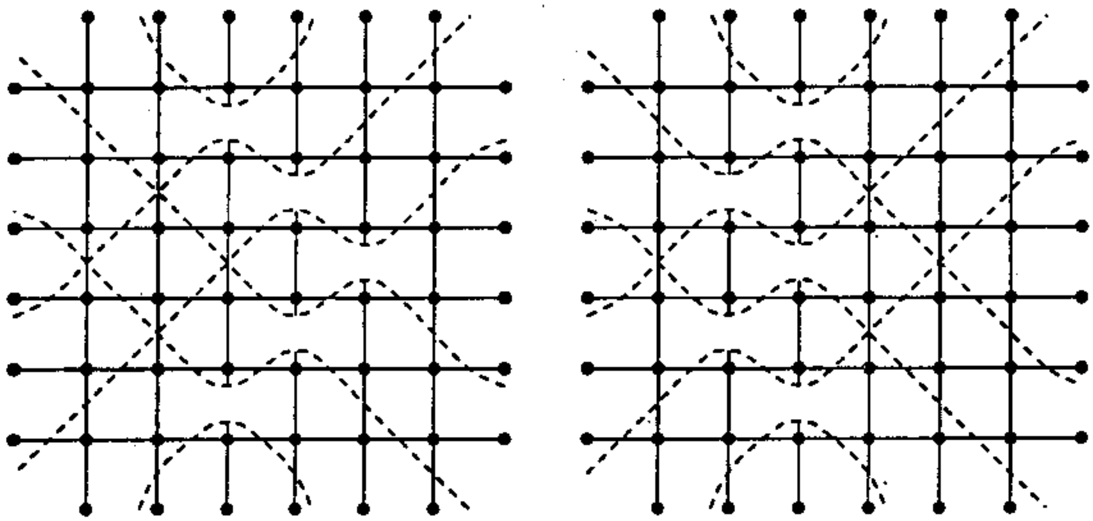


Figure 6: Y–Δ equivalent graphs

Although these graphs are Y–Δ equivalent, the conductivity function γ in general will not be the same for both graphs.

3.2.2 Non-Critical Graphs

Certain patterns of broken edges result in a Γ' which is non-critical. When the graph is non-critical, the z-sequence obtained from the Λ matrix will be the z-sequence for an electrically equivalent graph which is critical. Since we depend upon the z-sequence for the inverse problem, this becomes a problem. To overcome this problem, we must first find a way of making the graph critical and electrically equivalent. Then we can observe how the graph changes in the process. This process of making a graph critical is exactly what this section will describe.

From [1], we know a graph Γ is non-critical if $\mathcal{M}(\Gamma)$ has a lens, a geodesic with a self-intersection, or a geodesic with no endpoint on the boundary. We will now give a detailed description of the last two conditions which make a graph non-critical since the first condition, a lens, was defined in [1].

A geodesic with a self-intersection will be a subgraph of $\mathcal{M}(\Gamma)$ consisting of any geodesic $a \ldots u_0 u_1 \ldots u_k u_0 \ldots b$ where a and b are boundary nodes. Denote a sequence of nodes in a geodesic $u_0u_1 \ldots u_ku_0$ where u_0u_1 is not a direct extension of u_ku_0 by S. A *self-enclosed region* will be S together with all the nodes and edges of $\mathcal{M}(\Gamma)$ in the bounded connected component of the complement of S in the plane. Note that any geodesic with a self-intersection will form a self-enclosed region.

A geodesic with no endpoint on the boundary will be called a *closed geodesic*. A closed geodesic will be the geodesic arc $u_0u_1 \ldots u_k$ such that $u_k = u_0$ and $u_{k-1}u_k$ has u_0u_1 as a direct extension.

We now have the following for a medial graph whose graph is non-critical: the medial graph has a lens, a self-enclosed region, or a closed geodesic. These conditions are equivalent to those stated in [1]. Next, we will show how to make a non-critical graph critical in each case.

If $\mathcal{M}(\Gamma)$ has a lens, [1] states that Γ is Y–Δ equivalent to a graph Γ' with either a pair of edges in series or parallel. If $\mathcal{M}(\Gamma)$ has lenses, we may do Y–Δ or Δ–Y transformations on Γ on any one of the lenses to find an edge in series or parallel. If there is an edge in series, contract that edge, or if there is an edge in parallel, delete that edge. The resulting graph will be electrically equivalent and the resulting medial graph will have at least one less lens. This process will be called *removing a lens.*

If $\mathcal{M}(\Gamma)$ has self-enclosed regions, any geodesic intersecting the regions must form a lens. We may now use the process of removing a lens to remove any lens from this region. The process of removing lenses from the region might remove the region itself, but if it does not, now assume no other geodesics intersect the self-enclosed region. The part of the graph associated with this self-enclosed region must be one of the following in figure 7. The dashed lines are the self-enclosed region while the solid lines are the edges in the graph.

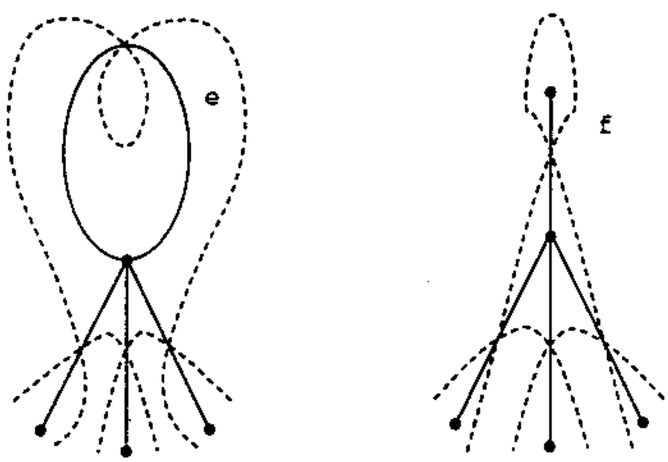


Figure 7: Self-enclosed regions

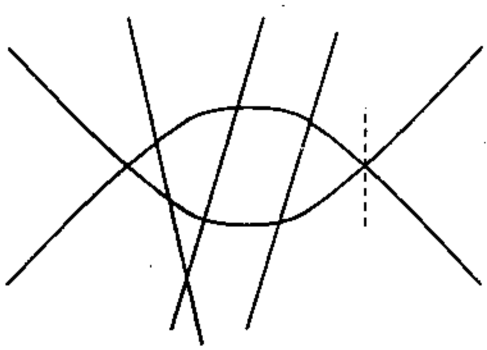
Both types of edges e and f may be removed to obtain an electrically equivalent graph.

If $\mathcal{M}(\Gamma)$ has closed geodesics, any other geodesic which intersects the closed geodesic must form a lens. These lenses may be removed by the process of removing lenses. Now assume the closed geodesic has no lenses. It may still intersect itself, in which case, must form self-enclosed regions. We know how to remove self-enclosed regions, so now assume there are no lenses or self-enclosed regions. This geodesic in $\mathcal{M}(\Gamma')$ cannot correspond to anything in Γ'; that is, if we were to color $\mathcal{M}(\Gamma')$ to find Γ', the coloring will not change if the closed geodesic is there or not. So, there will be nothing to remove in Γ' to make the graph critical; hence, all closed geodesics will be removed once lenses and self-enclosed regions have been removed.

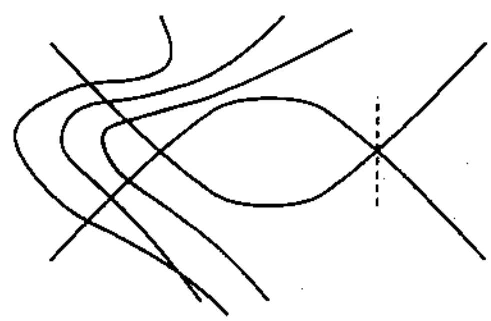
The entire process of making Γ critical depends upon removing lenses from $\mathcal{M}(\Gamma)$ since removing both self-enclosed regions and closed geodesics depend upon removing lenses. The next algorithm will describe a simple way to remove lenses.

ALGORITHM: *Given a circular planar network Γ whose medial graph $\mathcal{M}(\Gamma)$ has lenses, find in $\mathcal{M}(\Gamma)$ the lens with a least number of geodesics intersecting it. The two geodesic arcs which bound the lens intersect twice. At both of these intersections in $\mathcal{M}(\Gamma)$, there will be an edge in Γ which partial lies within the lens or is completely outside of the lens. At one of these intersections, contract the edge in Γ if it partially lies within the lens in $\mathcal{M}(\Gamma)$, or delete the edge in Γ if it does not lie within the lens in $\mathcal{M}(\Gamma)$. This process is equivalent to the process of removing lenses.*

Proof. Assume we have a lens with the least number of geodesics intersecting it. This lens cannot contain another lens, or else that lens would have the least number of geodesics intersecting it. Then by the process of clearing the lens described in [1], we can arrive at an empty lens; that is, a lens with no geodesics intersecting it. Since the process of clearing a lens may be done across either node where the geodesic arcs which bound the lens intersect, clear all geodesics across one of these nodes. If we color in this new medial graph $\mathcal{M}(\Gamma')$ to find Γ', we find that the resistor in Γ' which was at the node opposite to the one we cleared the lens on was not affected at all by the process. Figure 8 shows one case. The lines represent geodesics while the dashed line represents one resistor. Note the resistor is unaffected by the process of clearing the lens.



(a)



(b)

Figure 8: Clearing a lens

We can then remove this edge by contracting it if it lies within the lens, or deleting it if it lies outside the lens, and the resultant graph would be electrically equivalent to original graph since we are only removing an edge in series or parallel. □

In the case of removing edges to make a graph critical, the algorithm for removing lenses also gives the only way to remove edges in a given graph to make it critical and electrically equivalent. Also note that the process can be reversed—given a critical graph with edges removed, if we put edges back to try to form a lens, starting from the lens with the most geodesics intersecting it to the least, the reverse of this process will be the algorithm for removing lenses, which we know gives electrically equivalent graphs at every step. Therefore, reversing the process will give a set of non-critical graphs which could have the same Λ matrix if γ is chosen correctly. We may also use this process to find all Y–Δ equivalent graphs with the following observation.

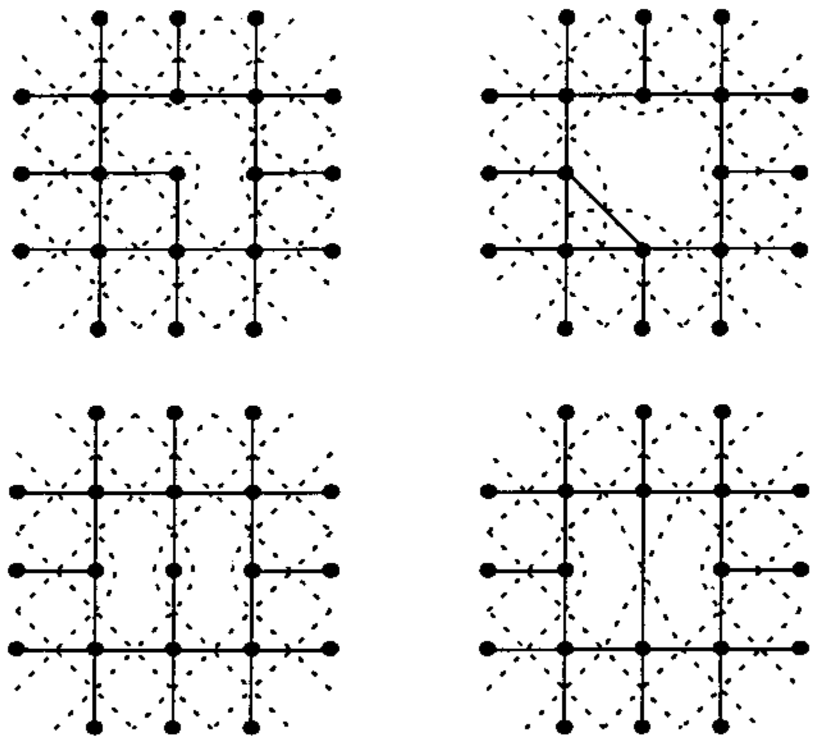
LEMMA 3.2.2–1: *Two critical circular planar graphs are electrically equivalent if and only if they are* Y–Δ *equivalent.*

Proof. It is obvious that two graphs which are Y–Δ equivalent must be electrically equivalent. To show the converse, note that we can find the z-sequence for any critical graph from the Λ matrix, and the z-sequence is independent of γ. Suppose two graphs are electrically equivalent. Then, by definition, there are conductivity functions such that both graphs have the same Λ matrices. Obviously, the same Λ matrix will give the same z-sequence. If two critical graphs have the same z-sequence, we know from [1] they must be Y–Δ equivalent. □

Since we know the algorithm gives the only way to get electrically equivalent graphs, by finding all electrically equivalent graphs which are also critical, we have found all Y–Δ equivalent graphs.

4 Open Questions

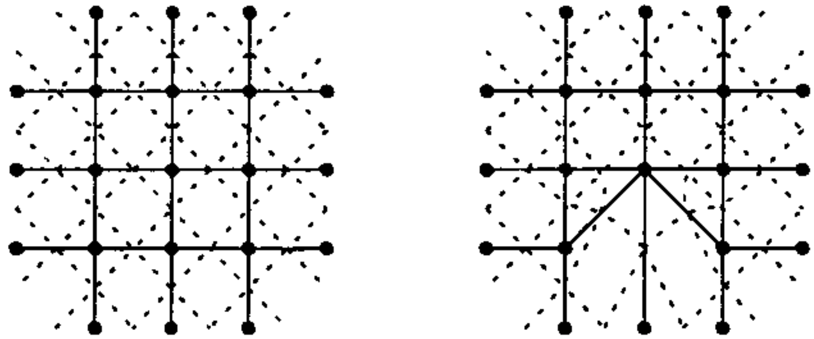
As was pointed out in Remark 2 in section 3.1, the removal of lenses obscures the relationship between the number of changes (by whatever definition), the number of edges and their relative positions. However, in the case of considering the removal of connected sets of resistors, there are two types of lenses that we encounter, and each has its own particular effect on the z-sequence:



Thus, it may be possible to reconstruct from the z-sequences of the critical networks on the right the corresponding networks to the left, by looking at the patterns of recurrences in the z-sequences suggested by the above diagrams.

By looking only at the changes in the z-sequence, we are unable to pinpoint exactly which resistors have been removed—as is possible in the case of a single removed edge. However, it may be possible to use the common patterns suggested by the diagrams to pinpoint exactly which resistors have been removed.

Another related problem that we did not examine is that of short circuits. In a short circuit, we give one of the edges resistance 0. This effectively causes the two nodes that were connected by the shorted edge to become one. As the following figure shows, it should still be possible to recover this network, much like the removal of single edge results in a recoverable network.



Is it possible to distinguish between a break and a short? Can you determine where the short was? What about the case of multiple shorts? These questions are a few questions for further research.

References

[1] E. B. CURTIS, D. INGERMAN, AND J. A. MORROW, *Circular planar graphs and resistor networks*, SIAM, (1995).

[2] E. B. CURTIS AND J. A. MORROW, *The Dirichlet to Neumann map for a resistor network*, SIAM, (1991).

[3] M. HUDELSON, *Locating faulty resistors in a network.* University of Washington REU, Oct. 17 1989.