

ANOTHER LOOK AT CONNECTIONS AND DETERMINANTS

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ABSTRACT. This paper states and proves the connection-determinant formula for any subdeterminant of a square matrix. Along the way we give a definition of connection, suitably generalized for arbitrary matrices.

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1. INTRODUCTION

There are two major algorithms available for recovering electrical networks: the boundary edge removal/boundary spike contraction method in the book [1], and the K-Star method best described in Nick Addington's paper [2]. Both of these methods rely, as conditions for recoverability, on the determinants of submatrices of the Kirchhoff matrix. In this paper we prove the connection-determinant formula, which gives us a way of expressing determinants of submatrices in terms of connections on the associated graph of the matrix, which will be defined later. This allows one to use graph theory to discuss the inverse problem, which is done to great effect in this case of circular planar graphs in [1].

We next discuss the boundary edge formula, which is our basic method of recovering the conductivities on boundary edges of an electrical network. This condition admits a nice interpretation in terms of connection-breaking properties of the edge, using the connection-determinant formula. We finish by discussing Nick Addington's algorithm [2], and how its recovery process relates to the boundary edge formula.

2. A GRAPH-THEORETIC VIEW OF DETERMINANTS

NOTE: In this paper K will denote an arbitrary square matrix unless explicitly stated otherwise (i.e., it's not necessarily a Kirchhoff matrix).

The primary fact used in recovering individual entries of the Kirchhoff matrix is the boundary edge formula [1]. The conditions under which one can use the formula are a bit more general than those stated in [1], though. A fully general version, stated algebraically, reads as follows:

BOUNDARY EDGE FORMULA. Let K be a $k \times k$ matrix decomposed into $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are square, such that $\Lambda = K/D$ is well-defined (i.e., D is non-singular). Suppose that $P' = p + P$ and $Q' = q + Q$ are two sequences of indices in A , where p and q are single indices, P and Q have the same cardinality, and $p \notin P, q \notin Q$. Define a new matrix K' obtained from K by zeroing out the entry κ_{pq} . Define $\Lambda' = K'/D$. Suppose that $\det \Lambda(P'; Q') \neq 0$, but $\det \Lambda'(P'; Q') \neq 0$. Then

$$(1) \quad \kappa_{pq} = \frac{\Lambda(P'; Q')}{\Lambda(P; Q)} = \frac{\det \Lambda(P'; Q')}{\det \Lambda(P; Q)}$$

The hypotheses used in this theorem can be formulated in terms of intrinsic properties of the graph of the electrical network using the connection-determinant formula. Our first goal is to derive this formula.

We first review some facts about permutations. A **permutation** is a bijective map from a set to itself. In this paper we will only discuss permutations on finite sets of indices $\{1, \dots, n\}$, in which case the most obvious way of writing down a permutation σ is $\begin{pmatrix} 1 \\ \sigma(1) \end{pmatrix} : \dots : \begin{pmatrix} n \\ \sigma(n) \end{pmatrix}$, meaning $1 \mapsto \sigma(1)$, etc. Clearly this notation gives us all the information we need to calculate with σ , but it isn't very compact, and it is actually rather inconvenient for computation. A more useful notation (for us, at least) is cycle notation. A **cycle** is a permutation which acts on a subset $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ in a 'cyclic' manner, i.e., $i_1 \mapsto i_2 \mapsto \dots \mapsto i_k \mapsto i_1$, and fixes all other indices. For example, the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$, is a cycle $1 \mapsto 3 \mapsto 2 \mapsto 1$ which fixes the only other element 4. We write this cycle as (132) . It is a fact that any permutation can be decomposed into a product of disjoint cycles in an essentially unique way, so that each index appears in exactly one cycle. I say 'essentially' unique because it does not matter which disjoint cycle one applies first in making the composition pq of two disjoint cycles, since they act 'independently' of one another. As an example, the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ decomposes into $(15)(24)(3)$. This result on decomposition of permutations is in some ways similar to the unique factorization of an integer into primes.

Next we must discuss the sign of a permutation, which is defined as follows: given a permutation $\sigma \in S_n$, write σ as a product of transpositions (2-cycles). Note that these cycles are not necessarily either disjoint or unique. In fact, one decomposition may have more cycles than another. However, if the number of transpositions in one decomposition is even, then so is the number of transpositions in every other decomposition. In this case the sign of the permutation is 1. If the number of transpositions is odd, the sign of the permutation is -1. An important fact about this function is that it is a **homomorphism**, i.e., $\text{sgn}(\sigma\tau) = \text{sgn}(\sigma)\text{sgn}(\tau)$ for all permutations σ, τ . This implies that the sign of a permutation is equal to the product of the signs of its component cycles. A given cycle $(i_1 \dots i_k)$ can be written

as the product $(i_1 i_2)(i_2 i_3) \dots (i_{k-1} i_k)$ of even transpositions, and there are $k - 1$ transpositions in this decomposition. Thus the sign of a cycle $\mathcal{C}$ is 1 if the cycle has an odd number of elements, and -1 otherwise, or, $\text{sgn}(\mathcal{C}) = (-1)^{|\mathcal{C}|-1}$, where $|\mathcal{C}|$ is the number of elements in the cycle $\mathcal{C}$. Multiplying the signs of all the cycles in a permutation $\sigma \in S_n$, we find $\text{sgn}(\sigma) = (-1)^{|\sigma|_c - n}$, where $|\sigma|_c$ is the number of cycles in σ (This number is well-defined, by unique decomposition).

We now move on to determinants. Given an $n \times n$ matrix K , the determinant of K can be defined as

$$(2) \quad \det K = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n \kappa_{i, \sigma(i)}$$

Note that for each σ , exactly one entry from each row and column goes into the product $\prod \kappa_{i, \sigma(i)}$.

There is a graph which is naturally related to K , called the **associated graph** $\mathcal{G}_K$. This graph is produced as follows: given an $n \times n$ matrix K , create a graph with n vertices, and for each non-zero entry κ_{ij} , create a directed edge from vertex i to vertex j with weight κ_{ij} . The result is the associated graph $\mathcal{G}_K$ associated to K .

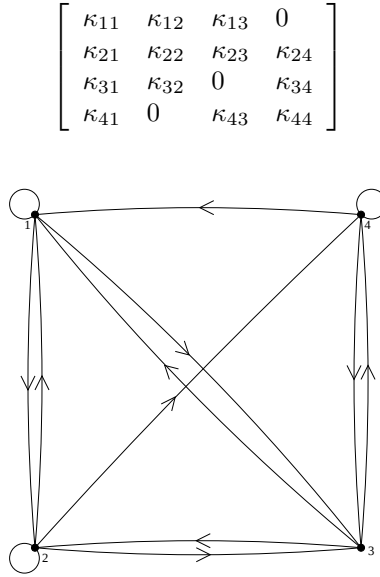


FIGURE 1. A matrix K and its associated graph $\mathcal{G}_K$

We can express $\det K$ in terms of certain sets of edges on $\mathcal{G}_K$ called **loop partitions**. The easiest way to define a loop partition is as a set of directed edges on $\mathcal{G}_K$ such that every vertex has exactly one directed edge leading into it, and one directed edge leading out of it, i.e., every vertex is part of exactly one loop. In the case that an edge goes from a vertex to itself, we count that as both an edge leading into the vertex and an edge leading out of that vertex. Loop partitions are intimately related to permutations. In fact, by decomposing permutations (uniquely)

into products of disjoint cycles, we can form a natural bijective correspondence between permutations on n elements and loop partitions on a graph with n elements, by sending the cycle $(i_1 i_2 \dots i_k)$ to the loop with directed edges $(i_1 i_2), (i_2 i_3)$, etc. It should be obvious that cycles in the permutation correspond to loops in the loop partition, and that distinct cycles map to distinct loops.

We can then easily translate expression (2) for $\det K$ into a sum over loop partitions, just by changing our terminology a bit. For every particular term $\prod_i^n \kappa_{i, \sigma(i)}$ in the sum defining $\det K$, we associate the collection of directed edges $(i, \sigma(i))$, $1 \leq i \leq n$, with weights $\kappa_{i, \sigma(i)}$. This collection of edges forms a loop partition of $\mathcal{G}_K$ because σ is a permutation.

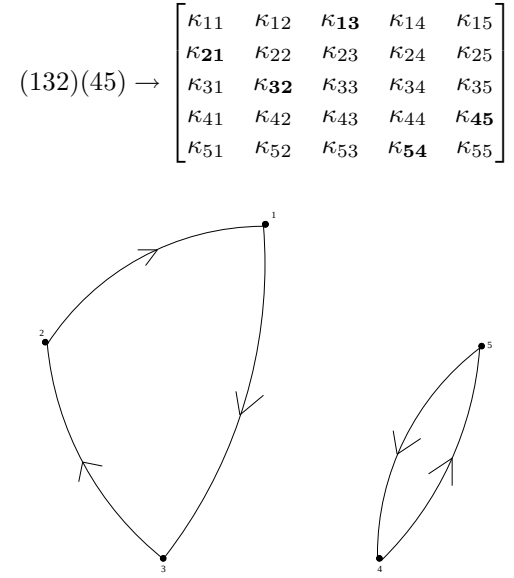


FIGURE 2. A permutation and its corresponding loop partition

Hence we can express $\det K$ as a sum over all loop partitions on $\mathcal{G}_K$. We define the sign $\text{sgn}(L)$ of a loop partition L to be the sign of the corresponding permutation σ . If we define $|L|_c$ to be the number of disjoint loops in L (which is equal to $|\sigma|_c$), then

$$(3) \quad \det K = \sum_{L \in \mathcal{L}(\mathcal{G}_K)} (-1)^{k-|L|_c} \omega(L)$$

where $\mathcal{L}(\mathcal{G}_K)$ is the set of all loop partitions on $\mathcal{G}_K$ and $\omega(L)$ is the product of the weights of all directed edges used in L .

This was all fairly straightforward. We next extend this analogy we are creating between K and $\mathcal{G}_K$ to submatrices of K . We will find that the natural structure on $\mathcal{G}_K$ used to express this relationship is the connection, in a generalized sense from that used in [1]. First we define a submatrix of K as follows. Partition the matrix K into $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are square, say $c \times c$ and $m \times m$, so that $c + m = n$. The vertices on $\mathcal{G}_K$ corresponding to rows in A are called **boundary vertices**, with those corresponding to rows in D called **interior vertices** (following the terminology of [1]). We then choose k (> 0) rows in A and k columns in A . The

set of row indices we denote by P , and the set of column indices we denote by Q . We then form the submatrix $K(P+D; Q+D)$; i.e., the submatrix of K consisting of the intersection of rows with indices in P or D and columns with indices in Q or D . Define the vertices on $\mathcal{G}_K$ corresponding to rows $P+D$ to be **starting vertices**, and the vertices corresponding to columns $Q+D$ to be **ending vertices**. Thus every interior vertex is both a starting vertex and an ending vertex, whereas a boundary vertex may be either a starting vertex, an ending vertex, both, or neither.

We need just one more bit of terminology to write down $\det K(P+D; Q+D)$. Given some integer i which is a possible index for $K(P+D; Q+D)$ (i.e., between 1 and the size of $K(P+D; Q+D)$), let $s(i)$ denote the corresponding row of the original matrix K , and let $e(i)$ denote the corresponding column of K . s and e stand for ‘starting vertex’ and ‘ending vertex’, respectively. We have

$$(4) \quad \det K(P+D; Q+D) = \sum_{\sigma \in S_\eta} \operatorname{sgn}(\sigma) \prod_{i=1}^{\eta} \kappa_{s(i), e(\sigma(i))}$$

where $\eta = k + m$ is the size of $K(P+D; Q+D)$. Take note of how we chose the corresponding matrix entry in K : given some index i , we form the pair $(i, \sigma(i))$, and then take the matrix entry corresponding to the starting vertex of the former, and the ending vertex of the latter.

How can we express this formula on $\mathcal{G}_K$? To do so, we must introduce connections.

Definition 1. A **path** from boundary vertex p to boundary vertex q is defined as a sequence of directed edges $(pi_1), (i_1i_2), \dots, (i_nq)$ such that $i_1, i_2, \dots, i_n$ are distinct interior nodes on $\mathcal{G}_K$. A **connection** $(P; Q)$ from boundary vertices $P = (p_1, \dots, p_k)$ to $Q = (q_1, \dots, q_k)$, called a $(P; Q)$ -connection, is a set of disjoint, directed paths from p_j to q_j , for each $1 \leq j \leq k$, such that any interior node on $\mathcal{G}_K$ is traversed at most once. The set of all $(P; Q)$ -connections is called the $(P; Q)$ -connection set.

Notice that this is almost exactly the same definition as in [1]; the only changes are that edges are now thought of as directed, and a connection (or even a path) from a boundary vertex to itself is allowed. In the case of Kirchhoff matrices, the former requirement that the edges be directed is usually ignorable, since both edges have the same weight. So this definition reduces to the old definition whenever the old definition applies.

Let us now see that the set of edges corresponding to $\{\kappa_{s(i), e(\sigma(i))}\}$ (from (4)) induces a unique connection $(P; Q)$ on $\mathcal{G}_K$. Given some starting boundary vertex $p \in P$, we start by following the (unique) directed edge leading out of this vertex. If we arrive at a boundary vertex, we have a path from starting vertex p to ending vertex q . If not, then we are at an interior node, which has a unique directed edge leading out of it, by definition. So we may continue following this path (in a unique way) until we arrive at some boundary vertex q . We then choose another starting boundary vertex p_2 , and follow the directed edge leading out of this vertex until we arrive at some ending boundary vertex q_2 . We repeat this process for each starting vertex, to arrive at a set of paths from boundary vertices P to boundary vertices Q . These paths are disjoint because there is always only one directed edge leading into and out of a vertex, and if a path goes through an interior node, it must use up both of these on its traversal of the node. Then no other path can lead into this node, since no edge is used twice.

So every permutation induces a set of edges on $\mathcal{G}_K$, which then induces a unique connection on $\mathcal{G}_K$. However, this correspondence is NOT bijective. In general there will be a set of interior nodes which do not figure into the connection $(P; Q)$. Two permutations could induce the same connection but then differ on what paths they induce between these unused interior nodes. This is something we will have to deal with in deriving the connection-determinant formula.

$$(15)(24)(3) \rightarrow \begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \\ 9 \end{array} \begin{bmatrix} 1 & 3 & 7 & 8 & 9 \\ \kappa_{11} & 0 & \kappa_{17} & 0 & \kappa_{19} \\ 0 & 0 & \kappa_{27} & \kappa_{28} & 0 \\ \kappa_{71} & 0 & \kappa_{77} & 0 & 0 \\ 0 & \kappa_{83} & 0 & \kappa_{88} & 0 \\ \kappa_{91} & \kappa_{93} & 0 & 0 & \kappa_{99} \end{bmatrix}$$

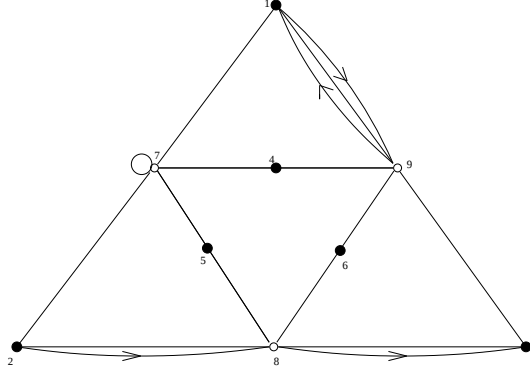


FIGURE 3. A permutation and its corresponding $(1,2;1,3)$ connection on the triangle-in-triangle graph

3. THE CONNECTION-DETERMINANT FORMULA

Recall first our original formula for $\det K(P + D; Q + D)$:

$$(5) \quad \det K(P + D; Q + D) = \sum_{\sigma \in S_\eta} \text{sgn}(\sigma) \prod_{i=1}^{\eta} \kappa_{s(i), e(\sigma(i))}$$

We wish to write this formula as a sum over connections on $\mathcal{G}_K$. Remember that every permutation $\sigma \in S_\eta$ induces a connection $(P; Q)$. Decompose σ into $\sigma = \phi\mu$, where ϕ is the product of all cycles including at least one boundary vertex, and μ is the product of all remaining cycles, if any (if there are none, then μ is the identity permutation). Then the edges represented by the cycles in ϕ are exactly the edges used in the connection $(P; Q)$: clearly any edge used in the $(P; Q)$ -connection must be in one of the cycles in ϕ . Conversely, any edge which is apart of an interior-interior loop must be in one of the cycles represented by μ , since a loop on interior nodes corresponds to a cycle consisting entirely of interior nodes, in our correspondence discussed earlier. So writing $\sigma = \phi\mu$ is tantamount to dividing σ into its $(P; Q)$ -connection edges and its edges between interior nodes unused in the connection. As before, $\text{sgn}(\sigma) = \text{sgn}(\phi)\text{sgn}(\mu)$. Now consider the permutation $\tau \in S_k$ which

takes starting vertices P to ending vertices Q as induced by σ . Since σ and τ map starting to ending vertices in the same way, $|\phi|_c = |\tau|_c$. Then

$$\begin{aligned}\operatorname{sgn}(\phi) &= (-1)^{t-|\phi|_c} \\ &= (-1)^{t+k-k-|\tau|_c} \\ &= (-1)^{t-k}\operatorname{sgn}(\tau) = (-1)^{t+k}\operatorname{sgn}(\tau)\end{aligned}$$

where t is the number of elements in the permutation ϕ .

The connection-determinant formula is constructed to be first a sum over all permutations σ which induce the same $(P; Q)$ -connection, then a sum over all permutations σ which send P to Q in the same way (i.e., τ for each permutation is the same). Then we sum over all possible τ . The most difficult part is finding an expression for the sum over all permutations σ which induce the same $(P; Q)$ -connection.

Let $S(\phi)$ denote the set of permutations $\sigma \in S_\eta$ with the same boundary cycles ϕ , or equivalently, the set of permutations σ which induce the same $(P; Q)$ -connection. The set of edges used in the connection (denoted by E_ϕ) is the same for each σ , and so $\operatorname{sgn}(\sigma) \prod_{(i,j) \in E_\phi} \kappa_{i,j}$ can be taken out of the summation.

$$\begin{aligned}& \sum_{\sigma \in S(\phi)} \operatorname{sgn}(\sigma) \prod_{i=1}^{\eta} \kappa_{s(i), e(\sigma(i))} \\ &= \sum_{\sigma \in S(\phi)} (-1)^{k+t} \operatorname{sgn}(\tau) \prod_{(i,j) \in E_\phi} \kappa_{i,j} \cdot \operatorname{sgn}(\mu) \cdot \prod_{i \in J_\mu} \kappa_{i, \mu(i)} \\ &= (-1)^k \operatorname{sgn}(\tau) \cdot \left(\prod_{(i,j) \in E_\phi} -\kappa_{i,j} \right) \cdot \left(\sum_{\sigma \in S(\phi)} \operatorname{sgn}(\mu_\phi) \prod_{i,j \in J_{\mu_\phi}} \kappa_{i, \mu_\phi(i)} \right)\end{aligned}$$

Since the permutations σ in $S(\phi)$ all have equal ϕ s, they differ only in μ , so the sum over $\sigma \in S(\phi)$ is really a sum over all μ ; hence the expression

$$\sum_{\sigma \in S(\phi)} \operatorname{sgn}(\mu_\phi) \prod_{i,j \in J_{\mu_\phi}} \kappa_{i, \mu_\phi(i)}$$

is a sum over all permutations μ , and hence is equal to the determinant of the principal proper submatrix of K determined by the interior vertices unused by the connection ϕ . We denote this determinant by U_ϕ .

Finally, we sum over all possible connections ϕ inducing a particular permutation $\tau \in S_k$, and then we sum over all $\tau \in S_k$, to obtain the following:

Theorem 1. CONNECTION-DETERMINANT FORMULA. *Suppose K is a square matrix decomposed into $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and suppose $K(P+D; Q+D)$ is a submatrix of K where P is a set of row indices in A and Q is a set of column indices in A , and $|P| = |Q| = k$. Then*

$$(6) \quad \det K(P+D; Q+D) = (-1)^k \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \left(\sum_{\phi, \tau_\phi = \tau} \left(\prod_{(i,j) \in E_\phi} -\kappa_{i,j} \right) U_\phi \right)$$

where

- τ is a permutation from starting indices P to ending indices Q
- ϕ is a permutation corresponding to a $(P; Q)$ -connection
- E_ϕ is the set of edges in $\mathcal{G}_K$ used in ϕ

- U_ϕ is the determinant of the principal submatrix of D corresponding to all interior nodes unused by the connection represented by ϕ

Using the connection-determinant formula, we obtain the following ‘dictionary’ for determining when a submatrix $K(P \cup D; Q \cup D)$ is singular or non-singular:

Corollary 1. *Let K be a square matrix, and let $K(P + D; Q + D)$ be a submatrix of K .*

- *If no $(P; Q)$ -connection exists on $\mathcal{G}_K$, then $\det K(P + D; Q + D) = 0$.*

In addition, $\det K(P + D; Q + D) \neq 0$ if at least one of the following conditions holds:

- *Exactly one $(P; Q)$ -connection exists on $\mathcal{G}_K$ and it uses every interior node.*
- *K is a Kirchhoff matrix and exactly one $(P; Q)$ -connection exists on $\mathcal{G}_M$.*
- *K is a Kirchhoff matrix, $P \cap Q = \emptyset$ and every $(P; Q)$ -connection induces a permutation τ from P to Q of the same sign.*
- *K is a Kirchhoff matrix, $P \cap Q = \emptyset$, $\mathcal{G}_K$ is circular planar, and at least one $(P; Q)$ -connection exists.*

Proof: These are all straightforward consequences of the connection-determinant formula. If no $(P; Q)$ -connection exists on $\mathcal{G}_K$, then the sum over connections is obviously empty, so $\det K(P + D; Q + D) = 0$. If exactly one $(P; Q)$ -connection exists and every interior node is used, then the summation collapses to a single non-zero term. If K is a Kirchhoff matrix, and exactly one $(P; Q)$ -connection exists, then again the summation collapses to a single term, for which $U_\phi > 0$ since a principal submatrix of a Kirchhoff matrix has positive determinant. If in addition P and Q are disjoint, then all of the terms in $\prod_{(i,j) \in E_\phi} -\kappa_{i,j}$ are off-diagonal entries of the Kirchhoff matrix, which are negative. Hence this product is positive, and since U_ϕ is positive, we are summing over a collection of strictly positive terms. Finally, if K represents a circular planar electrical network, then we know there exists only one permutation τ of the boundary indices, so by the previous sentence $\det K(P + D; Q + D) \neq 0$ as long as at least one $(P; Q)$ -connection exists. $\square$

We are most interested in $(P; Q)$ -connection sets for which $\det K(P + D; Q + D) \neq 0$ for all valid choices of matrix entries κ_{ij} . It is natural to call such a connection set **well-behaved**. The above four conditions give us our main criteria for identifying well-behaved connection sets. Note that the bottom condition implies that every non-empty $(P; Q)$ -connection set between disjoint sets of boundary indices on a circular planar graph is well-behaved.

4. THE BOUNDARY EDGE FORMULA

The boundary edge formula is our main tool for recovering entries in the Kirchhoff matrix. Its proof is very simple.

Theorem 2. BOUNDARY EDGE FORMULA. *Let K be an $n \times n$ matrix decomposed into $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with A and D square and D invertible, so that $\Lambda = K/D$ can be defined. Suppose that $P' = p + P$ and $Q' = q + Q$ are two sequences of indices in A , where p and q represent single boundary vertices, and P and Q are arbitrary sequences of boundary vertices, and $p \notin P, q \notin Q$. Now create a new matrix K' obtained from K by zeroing out the entry κ_{pq} . Define $\Lambda' = K'/D$. Suppose that $\det \Lambda(P'; Q') \neq 0$,*

but $\det \Lambda'(P'; Q') = 0$. Then

$$(7) \quad \kappa_{pq} = \frac{\det \Lambda(P'; Q')}{\det \Lambda(P; Q)} = \frac{\Lambda(P'; Q')}{\Lambda(P; Q)}$$

Proof. Since $\Lambda = A - BD^{-1}C$, if κ_{pq} is subtracted from an entry of A , the only change in Λ is that κ_{pq} is subtracted from λ_{pq} . Hence we have

$$\det \Lambda'(P'; Q') = \det \begin{pmatrix} \lambda_{pq} - \kappa_{pq} & \Lambda(p; Q) \\ \Lambda(P; q) & \Lambda(P; Q) \end{pmatrix} = 0$$

Since the determinant function is linear in the columns of a matrix,

$$\begin{aligned} \det \begin{pmatrix} \lambda_{pq} & \Lambda(p; Q) \\ \Lambda(P; q) & \Lambda(P; Q) \end{pmatrix} &= \det \begin{pmatrix} \kappa_{pq} & \Lambda(p; Q) \\ 0 & \Lambda(P; Q) \end{pmatrix} \\ \det \Lambda(P'; Q') &= \kappa_{pq} \cdot \det \Lambda(P; Q). \end{aligned}$$

Since the left-hand side is non-zero by hypothesis, so must $\det \Lambda(P; Q)$ be non-zero as well. Hence we may divide by it to obtain

$$(8) \quad \frac{\det \Lambda(P'; Q')}{\det \Lambda(P; Q)} = \kappa_{pq}.$$

Our basic condition for recovering an entry κ_{pq} is that $\det \Lambda(P'; Q')$ is non-zero, but zeroing out the entry κ_{pq} of K zeroes out $\det \Lambda(P'; Q')$. We can translate this condition into a connection-breaking property on $\mathcal{G}_K$ using the connection-determinant formula. First note that since Λ is obtained by taking the Schur complement of K , we have (see [1])

$$(9) \quad \det \Lambda(P; Q) \cdot \det K(D; D) = \det K(P + D; Q + D)$$

In taking the Schur complement, we implicitly assumed that $\det K(D; D) \neq 0$, so $\Lambda(P; Q)$ is non-singular if and only if $K(P + D; Q + D)$ is non-singular. Hence we can use the ‘dictionary’ (Corollary 1) to determine the singularity of submatrices of Λ . Using our definition of well-behaved connection, we see that to recover an entry κ_{pq} , it is sufficient to find a well-behaved $(p + P; q + Q)$ -connection set for which deleting the edge pq breaks every connection in this set. This is important enough to be stated as a corollary:

Corollary 2. *Let p and q be two (not necessarily distinct) boundary vertices of an electrical network connected by a boundary edge. If there is a well-behaved $(p + P; q + Q)$ -connection set for some sequences of boundary indices P and Q such that after deleting the boundary edge (pq) , the $(p + P; q + Q)$ -connection set disappears, then the entry κ_{pq} of the response matrix is recoverable.*

It is important to note that the converse of this statement is not true: it is often the case that a boundary edge on an electrical network is recoverable, even though deleting the edge does not break any well-behaved connection set. This point will be discussed further later in the paper.

4.1. Boundary spikes. In [1], a formula is given which will recover the conductivity γ_{pr} of a boundary spike under certain conditions (here p denotes the boundary node and r the interior node). This formula has no analogue in the recovery method presented by Nick Addington in [2], which purports to be a general recovery algorithm. In Addington’s method, to recover a boundary spike conductivity γ_{pr} , one must first recover the diagonal entry κ_{pp} of the Kirchhoff matrix, and then use the

relation $\kappa_{pp} = \gamma_{pr}$ to recover the boundary spike. In fact, we will see that this is essentially what the boundary spike formula in [1] does.

In the proof of the boundary spike formula, it is first noted that if the edge pr is a boundary spike, then within the Kirchhoff matrix, there is a submatrix of the form

$$K(p, r; p, r) = \begin{bmatrix} \kappa_{pp} & -\kappa_{pp} \\ -\kappa_{pp} & \sigma \end{bmatrix}$$

where the remaining entries in row p and column p are all 0. The next step is to expand $K(P + p + I; Q + p + I)$ (corresponding to the connection $(P + p; Q + p)$ on the graph) along the row corresponding to node p , which results in the formula

$$\det K(P + p + I; Q + p + I) = \kappa_{pp} \det K(P + I; Q + I) - \kappa_{pp}^2 \det K(P + I - r; Q + I - r).$$

The conditions required to use the formula then imply that $\det K(P + I - r; Q + I - r) = 0$, so we can use the same Schur complement identity (Proposition 1) to show that

$$\det \Lambda(P + p; Q + p) = \kappa_{pp} \det \Lambda(P; Q)$$

which recovers the diagonal entry κ_{pp} and therefore γ_{pr} .

Since the boundary spike formula is essentially recovering the diagonal entry of the Kirchhoff matrix rather than the boundary spike conductivity (they just happen to be equal), it is not surprising to find that whenever one can apply the boundary spike formula, one could have applied the boundary edge formula instead to find κ_{pp} . To do so, we directly consider the connection $(P + p; Q + p)$ (which the boundary spike formula does implicitly). Since the connection $P; Q$ exists and $\kappa_{pp} \neq 0$, the extended connection also exists using the loop edge from p to itself. Furthermore, p must loop to itself; any other connection would necessarily use interior node r twice. Therefore, each possible $(P; Q)$ connection's contribution in the connection-determinant formula is simply multiplied by a non-zero factor κ_{pp} , and so if $(P; Q)$ is well-behaved, so is $(P + p; Q + p)$. To use the boundary edge formula, we must know that zeroing out κ_{pp} breaks the connection $(P + p; Q + p)$. Zeroing out the entry κ_{pp} means that any possible connection $(P + p; Q + p)$ with non-zero weight must have the set of paths from P to Q not use interior node r , since the connection from p to p must now use r . This is exactly the same restriction on $(P; Q)$ which results from contracting pr . Therefore, if contracting pr breaks a connection $(P; Q)$ (allowing us to use the boundary spike formula), then zeroing out the entry κ_{pp} will break the connection $(P + p; Q + p)$, allowing us to use the boundary edge formula. The boundary spike formula, as it turns out, can be subsumed by the boundary edge formula.

5. THE ADDINGTON RECOVERY METHOD

The Addington recovery method, presented in [2], is currently the most versatile recovery algorithm we have. Central to the algorithm is examining a set of residue (R) matrices to recover entries in the upper left corner of the Kirchhoff matrix. In this section we will examine the R matrix method of recovering information and compare this to using the boundary edge formula. To begin with, we will define the four matrices we will be considering, denoted K, Λ, Z, R :

Definition 2. Given a $k \times k$ Kirchhoff matrix K decomposed into $\begin{pmatrix} A & B \\ B^\top & D \end{pmatrix}$ where A and D are square, let Λ denote the Schur complement K/D . The Z matrix is obtained from K by replacing A with a matrix of zeroes; that is, $Z = \begin{pmatrix} 0 & B \\ B^\top & D \end{pmatrix}$. The R matrix is then defined as the Schur complement Z/D .

(Note that it is not necessary to restrict K to be a Kirchhoff matrix. To avoid confusion, however, K will denote a Kirchhoff matrix in this section and the rest of the paper.)

When tackling the inverse problem, we are given Λ and asked to construct K . Nick Addington's method actually attempts to recover R , and then uses the fact that $A = \Lambda - R$ to recover A . To recover an entry of R , we must know all but one of the entries in a particular submatrix of R , we must know that the particular submatrix is singular, and we must know that the cofactor of the unknown entry is non-singular. We can then solve for the unknown entry. Clearly, we must have some method of determining whether a determinant of R will be zero or non-zero. To do this, we note that since $R = Z/D$, for any submatrix $R(P; Q)$ of R ,

$$\det R(P; Q) \cdot \det D = \det Z(P + D; Q + D)$$

Since D is non-singular, $\det R(P; Q) = 0$ if and only if $\det Z(P + D; Q + D) = 0$. The connection-determinant formula tells us that we can inspect determinants of submatrices of Z by looking at Z 's associated graph $\mathcal{G}_Z$. We can easily obtain $\mathcal{G}_Z$: it is the graph which results from removing all boundary edges from the associated graph of K (which itself is the electrical network we are studying with loops representing diagonal entries added in). If a connection on $\mathcal{G}_Z$ cannot be made, then the corresponding submatrix of R is singular. Conversely, if a connection can be made in precisely one way on $\mathcal{G}_Z$, then the corresponding submatrix of R is non-singular.

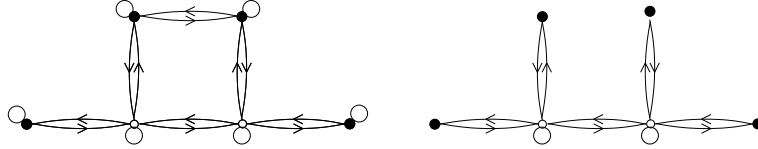


FIGURE 4. $\mathcal{G}_K$ and $\mathcal{G}_Z$ for the well-connected graph on 4 nodes

As we will now see, looking at the residue matrices will not give us any new information. All of the information we recover by looking at R matrices could have been recovered directly using the boundary edge formula, if we remove each boundary edge from the graph as we recover it. We state this as a theorem:

Theorem 3. *If an entry in R is recoverable, then the corresponding entry in A , the upper left corner of K is recoverable directly by using the boundary edge formula, if each boundary edge is removed from the graph as soon as it is recovered.*

Proof. Assume that we are recovering an entry r_{pq} of R . To recover r_{pq} , there must be some submatrix of R in which r_{pq} is the only unknown entry. Then the corresponding entries in the Kirchhoff matrix are known and so are 0, by hypothesis. Let the submatrix of R we are inspecting correspond to rows $P' = p + P$ and $Q' = q + Q$. Then $R(P; Q)$ (the cofactor of r_{pq}) is non-singular by hypothesis, and since $R = Z/D$, $Z(P + D; Q + D)$ and $K(P + D; Q + D)$ are non-singular as

well ($K(P + D; Q + D) = Z(P + D; Q + D)$ since all of $K(P; Q)$ has been zeroed out). We now consider the submatrix $K(P' + D; Q' + D)$. Assume that κ_{pq} is the upper left entry of this submatrix. We consider $K(P' + D; Q' + D)$ as a linear function $F(z)$ of its first column. This first column can be represented as $z = x + y$, where $x = \begin{bmatrix} \kappa_{pq} \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ a \end{bmatrix}$. Since κ_{pq} is the only non-zero entry in $K(P'; Q')$, $F(y) = \det Z(P' + D; Q' + D) = 0$. Then

$$\begin{aligned} \det K(P' + D; Q' + D) &= F(x) + F(y) \\ &= F(x) \\ &= \kappa_{pq} \det K(P + D; Q + D) \end{aligned}$$

We have already shown that $K(P + D; Q + D)$ is non-singular, so we can proceed to solve for κ_{pq} by the boundary edge formula. (Author's note: To finalize the proof, a discussion of the square root trick should be given, since this is a way of recovering an entry in R . Perhaps in a later version this case will be handled).

It is not known at present whether the other direction of this theorem holds; that is, it is not known that one can always recover everything from the R matrix that one could recover from the boundary edge formula. Neither a proof nor a counter-example seems to be in sight.

6. CONNECTIONS AND RECOVERABILITY

6.1. Circular Planar Graphs. For circular planar graphs it has been shown that connections are intimately related to the recoverability of a network. In particular, a circular planar graph is recoverable if and only if removing or contracting any edge in the graph breaks a connection between disjoint sets of boundary nodes (for more information see [1]). A natural question to ask is whether this characteristic of circular planar graphs extends to the new loops representing diagonal entries. That is, is it true for critical circular planar graphs that zeroing out a diagonal entry breaks some connection in the graph? Unfortunately, the answer is no. The simplest recoverable graph which violates this property is the kite graph (the well-connected graph on 4 nodes) shown below on the left:

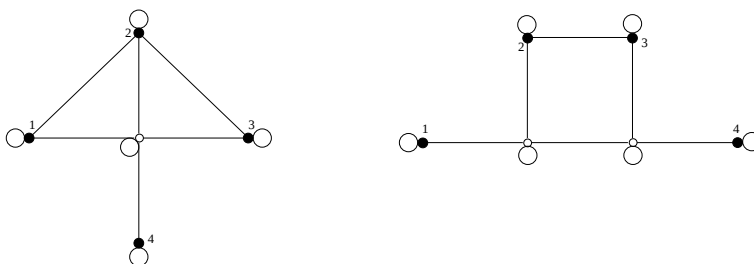


FIGURE 5. The kite graph and its wye-delta equivalent, the top hat graph

In fact, this graph is wye-delta equivalent to the “top hat” graph (depicted above on the right), which does have the property that zeroing out any entry in its Kirchhoff matrix breaks a connection on its associated graph. This shows that not even wye-delta transformations preserve the connection-breaking property (for

edges on the associated graph) which was used to characterize the recoverability of circular planar graphs. While diagonal entries can sometimes be used to expedite the recovery of circular planar graphs, they certainly do not make the recovery process conceptually simpler.

6.2. Arbitrary Graphs. The situation is exacerbated when we look at non-circular planar graphs. Here even edges between disjoint vertices (non-loops) do not require the connection-breaking property to be recoverable. For example:

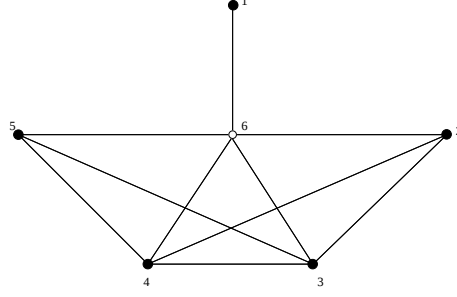


FIGURE 6. A graph without the connection-breaking property which characterizes circular planar recoverability

This graph is clearly recoverable, as we can use the $(1,2;1,5)$ connection to recover κ_{11} and thereby the boundary spike, which we can then contract to complete the recovery process. However, deleting the $(3,4)$ boundary edge does not break any connection in the graph. Every 2-connection involving the boundary edge $(3,4)$ exists in 2 different permutations. Breaking the $(3,4)$ edge only breaks one of these permutations. But if we recover, for example, the $(2,3)$ edge (deleting this edge breaks the $(1,2;3,5)$ connection) and then delete it, then we create a new electrical network in which deleting the $(3,4)$ edge breaks the $(1,3;2,4)$ connection, and so we can use the boundary edge formula to recover this edge. Therefore we can see that in the general case, there are situations in which a boundary edge is not immediately recoverable (by the boundary edge formula), but becomes so after recovering one or more boundary edges which do break connections in the graph.

This does not happen for “normal” boundary edges in the circular planar case, as it appears to require that connections $(P;Q)$ with two or more different permutations τ from starting vertices to ending vertices must exist in the graph, one of which is broken by deleting the boundary edge in question, and the others by deleting other boundary edges which are recoverable by looking at entirely different connections. By the Jordan Curve theorem, in the circular planar case there can only exist one permutation from starting vertices to ending vertices when examining any possible connection $(P;Q)$, if P and Q are disjoint. If P and Q are not disjoint, then this is not necessarily true, as we can see in the “kite” graph above. In fact, the five boundary node graph depicted above and the kite graph are quite similar. Zeroing out the diagonal entry κ_{11} in the kite graph will not break a connection for much the same reason as deleting the $(3,4)$ boundary edge did not: every 2-connection on the kite graph involving the $(1,1)$ edge exists in 2 permutations, only one of which is broken by deleting the $(1,1)$ edge, and the other by deleting some (recoverable) boundary edge in the graph.

This makes it much more difficult to say if a particular boundary edge is recoverable just by looking at the topology of the graph that results by deleting that edge. It seems conceivable that some such description exists, at least for boundary edges recoverable by the boundary edge formula. It seems that for a boundary edge to be recoverable, deleting it must break one permutation τ of a particular $(P; Q)$ connection. The difficulty is in determining whether the other permutations will be broken by deleting other boundary edges. It may be that one must consider some property of all of the boundary edges in the graph as a whole, rather than inspecting them one by one. Clearly, there is much left to do in this area.

7. FUTURE RESEARCH

- (1) The boundary edge formula is a very general formula which works for recovering entries in the upper left corner of arbitrary square matrices. Is there a formula for entries outside of this corner, that is, boundary-interior or interior-interior connections? The formula would be much more complicated than the boundary edge formula, since the variation of Λ with respect to entries in A ($\Lambda = A - BC^{-1}B^T$) is simpler than the variation with respect to the rest of K . On the other hand, Λ is still a linear function of entries in B or B^T , so it seems like there could be some formula which allows us to recover entries in these submatrices without resorting to special properties of Kirchhoff matrices (specifically, that rows and columns add to 0).
- (2) When we use the boundary edge formula, we are actually taking the Schur complement of $\Lambda(P'; Q')$ mod $\Lambda(P; Q)$, where $P' = P + p$ and $Q' = Q + q$. It is interesting that we calculated Λ by taking a Schur complement in K , and then got back an entry κ_{pq} in K by taking a Schur complement in Λ . In a sense, the Schur complement seems to be its own inverse function (under very special conditions). Can we explain the boundary edge formula in a different way using this view of the Schur complement? What happens if we take the Schur complement of some rows in Λ when the conditions of the boundary edge formula do not apply? Is it purely a coincidence that we used the Schur complement twice to get back an entry in K , or is something else going on here?
- (3) The question brought up at the end of the previous section: is there a way to characterize (by edge-breaking properties) whether or not a boundary edge will be recoverable by the boundary edge formula?
- (4) The other direction of the theorem in Section 4: Will inspecting R matrices work to recover a boundary edge whenever that edge is recoverable by the boundary edge formula? This would certainly be very convenient, since it is much easier to inspect the Z matrix and its associated graph for connections that do/do not exist rather than K itself.

8. REFERENCES

- [1] E. B. Curtis and J. A. Morrow, *Inverse problems for electrical networks*, World Scientific, (2000).
- [2] N. Addington, *A Method for Recovering Arbitrary Graphs*, (2005).
- [3] M. Goff, *Recovering Networks with Signed Conductivities*, (2003).