THE CONNECTION-DETERMINANT FORMULA, WITH APPLICATIONS TO ELECTRICAL NETWORKS

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ABSTRACT. The connection-determinant formula is a way of graphically interpreting the monomials in the determinant of an $n \times n$ matrix. The heart of this short paper is a precise formulation and proof of this formula. We also give a physical application to electrical networks, to motivate the statement of the formula and to indicate how one might use it.

CONTENTS

1. THE INVERSE PROBLEM FOR ELECTRICAL NETWORKS

Before developing the connection-determinant formula, which is a purely mathematical fact about arbitrary matrices, we introduce a physical situation which motivated the original development of the connection-determinant formula in [CuMo00]. Consider an electrical network, which we model mathematically as a graph with boundary $(V, \partial V, E)$, and a conductivity function $\gamma : E \to \mathbb{R}_+$ on the edges of the graph. We assume that the graph is simple, meaning that no loops or parallel edges exist. Thus if e is an edge of the graph with endpoints p and q, then e is uniquely defined by the label pq (or qp), and we can consider the conductivity function γ instead as a function $\gamma : V \times V \to \mathbb{R}$, where $\gamma(i, j)$ (also written γ_{ij}) is the conductivity of the edge ij if it exists, and 0 otherwise.

Suppose we choose an ordering of the vertices V, such that the boundary vertices ∂V come first in the ordering. Then we can assign a matrix to this electrical network, known as its Kirchhoff matrix K. K has entries as follows:

$$K_{ij} = \begin{cases} -\gamma_{ij}, \text{ if } i \neq j \\ \displaystyle\sum_{\substack{1 \leq j \leq n \\ j \neq i}}^{n} \gamma_{ij}, \text{ if } i = j. \end{cases}$$

K comes with a canonical partition

$$K = \begin{array}{c} \\ \partial \\ I \end{array} \begin{array}{cc} \partial & I \\ \left[\begin{array}{cc} A & B \\ B^\top & C \end{array} \right] \end{array},$$

where the off-diagonal entries in A correspond to boundary-boundary edges, entries in B correspond to boundary-interior edges, and off-diagonal entries in C correspond to interior-interior edges.

The forward problem for this electrical network asks the following: given an arbitrary vector ϕ of boundary voltages, what is the vector ψ of induced currents into the network at the boundary nodes? It is not hard to show (See [CuMo00]) that there exists a unique matrix Λ with the property that $\Lambda\phi = \psi$ for any boundary voltages ϕ and induced boundary currents ψ. We consider Λ to be the solution to the forward problem. In the case that the underlying graph G of the electrical network is connected (or more generally that every connected component includes a boundary node), the submatrix C in the above partition is invertible and we can write $\Lambda = A - BC^{-1}B^\top$.

To state the version of the inverse problem we are interested in, we need some definitions:

Definition 1.1. *Let G be a graph with boundary, with N edges and k boundary nodes. The space of conductivity functions on G is then $\mathbb{R}_+^N$, and the space of all possible response matrices for conductivity functions on G is contained in $M_{k \times k}$. We let $L : \mathbb{R}_+^N \to M_{k \times k}$ denote the map which sends a conductivity function γ on G to the resulting response matrix Λ. We say that G is recoverable if the map L is injective. We say that an edge $e \in G$ is recoverable if each fiber $L^{-1}(\Lambda)$ is constant on the edge e.*

Our main inverse problem is then: given a graph G, is G recoverable? More generally, is a given edge $e \in G$ recoverable? We will always assume that every component of G contains a boundary node. Therefore for any Kirchhoff matrix $K = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$ defining an electrical network on G, $\Lambda = A - BC^{-1}B^\top$. So our question is: given $A - BC^{-1}B^\top$ and the signs $(+, -,$ or $0)$ of entries in K, can we solve for K? The form $A - BC^{-1}B^\top$ is known as the Schur complement of K with respect to C, and is often denoted by K/C. More generally, submatrices of Λ are given as Schur complements of submatrices of K with respect to C. To state this precisely we first need a convention on how to write down submatrices:

Notation. Given a matrix M and subsequences S of the row indices and T of the column indices, we write $M(S;T)$ to mean the submatrix of M with rows S and columns T. If we have two subsequences S and I, we denote by $S + I$ the concatenation of the two sequences.

With this notation, we have $\Lambda(S;T) = K(S+I;T+I)/K(I;I)$ for any choice of row indices S and column indices T of Λ. Moreover, if $\#(S) = \#(T)$ so that $\Lambda(S;T)$ is square, we have the following relation among determinants:

$$\det \Lambda(S;T) = \det K(S+I;T+I)/\det K(I;I).$$

Again, this relation is not hard to show. But it has the following consequence, which gives us our first strategy for determining the values of entries in K (more specifically, in the upper-left corner A of K) from Λ.

Proposition 1.2 (Boundary edge formula). *Suppose that $K = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is an arbitrary partitioned square matrix with D square and invertible, and $\Lambda = K/D$. Suppose that $\Lambda(p + S; q + T)$ is a square submatrix of Λ consisting of row p then rows S, column q then columns T, such that $\det \Lambda(p + S; q + T) \neq 0$. Next construct a new matrix K' from K by zeroing out K_{pq}, and let $\Lambda' = K'/D$. Suppose that $\det \Lambda'(p + S; q + T) = 0$. Then $\det \Lambda(S; T) \neq 0$, and*

$$K_{pq} = \det \Lambda(p + S; q + T) / \det \Lambda(S; T).$$

Proof. We have the following simple computation:

$$0 = \det \Lambda'(p + S; q + T) = \det \begin{bmatrix} \lambda_{pq} - K_{pq} & \Lambda(p; T) \\ \Lambda(S; q) & \Lambda(S; T) \end{bmatrix}$$

$$= \det \begin{bmatrix} \lambda_{pq} & \Lambda(p; T) \\ \Lambda(S; q) & \Lambda(S; T) \end{bmatrix} - \det \begin{bmatrix} K_{pq} & \Lambda(p; T) \\ 0 & \Lambda(S; T) \end{bmatrix}$$
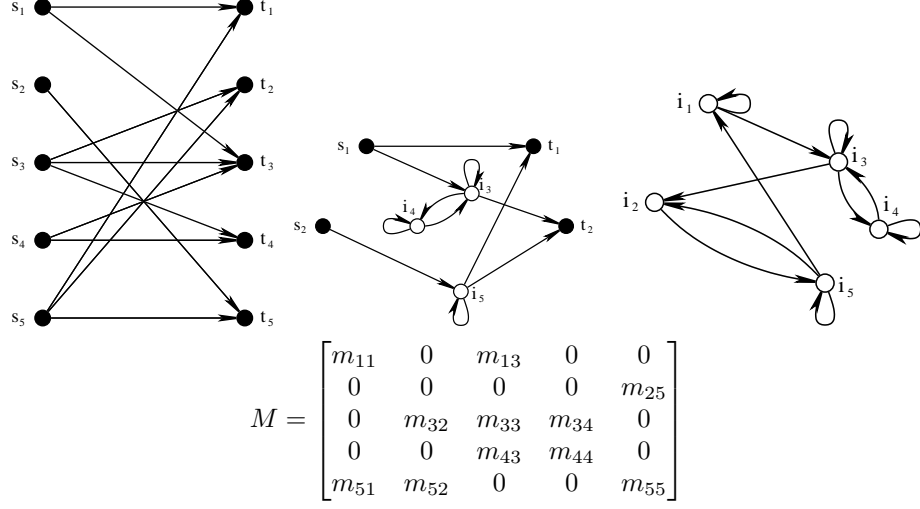
$$= \det \Lambda(p + S; q + T) - K_{pq} \det \Lambda(S; T).$$

Since $\det \Lambda(p + S; q + T) \neq 0$, so too must $\det \Lambda(S; T) \neq 0$, and so we can divide by it to solve for K_{pq}. □

It turns out that the determinantal conditions in the hypotheses of this proposition are nicely interpreted in terms of connections (which will be defined later) on the underlying graph G of the electrical network. This is a consequence of the connection-determinant formula, which is the main subject of this paper. Using these conditions, we will show how the above proposition can actually be used to determine entries in K from Λ.

Remark on Previous Work. The connection-determinant formula appeared first (to our knowledge) in the book [CuMo00] precisely to solve the above question about interpreting the conditions in the boundary edge formula. However, it was not clear from the discussion there that the connection-determinant formula is in fact an extremely general statement about how to graphically interpret the determinant of any square matrix in terms of graph-theoretic quantities. It applies in particular to the study of random walk networks as explained in [Lis11].

2. Graph Representations of Matrices

Let K be the Kirchhoff matrix of an electrical network $\Gamma = (G, \gamma)$, and let Λ be the resulting response matrix. For our application to inverse problems, we would like an answer to the following question: given sequences S and T of row and column indices, determine the sign of $\det \Lambda(S; T)$ (or at least whether it is non-zero) just by looking at the underlying graph G. Since $\det \Lambda(S, T)$ has the same sign as $\det K(S + I; T + I)$ ($\det C$ is a positive number since K is positive semi-definite), we can look instead at $\det K(S + I; T + I)$. So we would like to determine the sign of a sub-determinant of the Kirchhoff matrix from the graph of the electrical network (independently of the actual values of the conductivities). This will not be possible in all cases, but quite often it does work, especially if G is circular planar (see [CuMo00] for a definition of this term). We will show this by first proving a more general statement: for any square matrix M of size $n \times n$,

$$M = \begin{bmatrix} m_{11} & 0 & m_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{25} \\ 0 & m_{32} & m_{33} & m_{34} & 0 \\ 0 & 0 & m_{43} & m_{44} & 0 \\ m_{51} & m_{52} & 0 & 0 & m_{55} \end{bmatrix}$$

FIGURE 1. A matrix M and graph representations $\mathscr{G}_M(0), \mathscr{G}_M(3), \mathscr{G}_M(5)$

there exist associated graphs $\mathscr{G}_M(\ell)$ for each M between 0 and n, such that $\det M$ has a natural interpretation as a signed sum over certain collections of paths on $\mathscr{G}_M(\ell)$. Which associated graph $\mathscr{G}_M(\ell)$ is appropriate is typically determined by the geometry of the situation, as we will explain in the case of a submatrix of the Kirchhoff matrix K of an electrical network. But for now, we work in full generality.

Let M be an $n \times n$ matrix. We can interpret the entries of M as edges of a weighted bipartite graph, as follows: we create a graph G with $2n$ vertices labeled $s_1, ..., s_n, t_1, ..., t_n$, where the vertices s_j correspond to the row indices, and the vertices t_j correspond to the column indices. (Here, 's' and 't' stand for 'source' and 'target', respectively.) For every non-zero entry M_{jk} of M, G has a weighted, directed edge from vertex s_j to vertex t_k of weight M_{jk}. This is the graph $\mathscr{G}_M(0)$. The 0 refers to the fact that there are no interior nodes in this graph, only source and target nodes. We denote the sequence $(s_1, \ldots, s_n)$ of source vertices by S, and the sequence $(t_1, \ldots, t_n)$ by T.

Now take any ℓ between 0 and n. We form the associated graph $\mathscr{G}_M(\ell)$ by taking $\mathscr{G}_M(0)$ and identifying the last ℓ source vertices and target vertices pairwise. In other words, we identify s_j and t_j for each j between $n - \ell + 1$ and n. The resulting graph has three types of vertices: source vertex s_j and target vertex t_j for each $j \in \{1, \ldots, n - \ell\}$, and intermediate vertex i_j for each $j \in \{n - \ell + 1, ..., n\}$. These three types of vertices are grouped into sets S, T, and I. Note that a source vertex can only have an edge leading out of it, while a target vertex can only have an edge leading into it. In the next section we will interpret the determinant of M as a sum over a type of collection of disjoint paths from the source vertices S to the target vertices T, moving through the intermediate vertices I. This explains the terminology. Note that the sets of vertices S, T, and I are ordered since they come from rows/columns of the matrix M, and we will often label them by their corresponding row/column index.
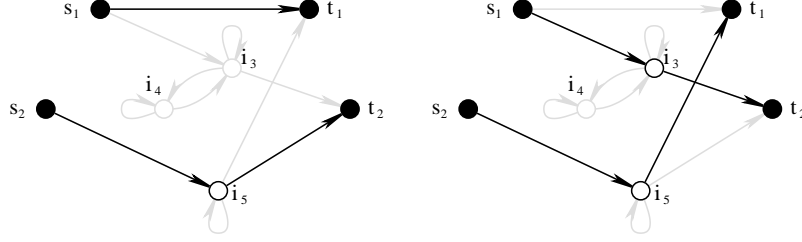
FIGURE 2. Two connections from (s_1, s_2) to (t_1, t_2) through (i_3, i_4, i_5) on a fixed underlying graph. The first has $\tau(C) = (1)(2)$ and the second has $\tau(C) = (12)$

3. THE CONNECTION-DETERMINANT FORMULA

Let M be a square matrix with n rows and columns. We will start with the following formula for the determinant of M:

$$(1) \qquad \det M = \sum_{\sigma \in \mathfrak{S}_n} \text{sign}(\sigma) \prod_{j=1}^{n} M_{j,\sigma(j)},$$

where $\mathfrak{S}_n$ is the symmetric group on n letters and $\text{sign} : \mathfrak{S}_n \to \{\pm 1\}$ is the sign homomorphism. Let $\mathscr{G}_M(\ell)$ be one of the graph representations of M. We will interpret each monomial $\prod_{j=1}^{n} M_{j,\sigma(j)}$ in terms of structures definable on $\mathcal{G}_M(\ell)$. We first make some definitions.

Definition 3.1. *Let G be a digraph, with S, T, and I subsets of the vertices of G, where $|S| = |T| = k$. S and T are assumed to be disjoint from I, but S could intersect T. A k-connection from S to T through I is a collection of k directed paths in G, each beginning at a vertex in S and ending at a vertex in T, and passing only through vertices in I at intermediate steps. In addition, any vertex of I is used in at most one path. Given a connection C from S to T through I, let $I_{\in C}$ denote the set of intermediate vertices of I used in C, and let $I_{\notin C}$ denote the set of intermediate vertices not used in C. The set of all connections from S to T through I is denoted by $\mathscr{C}(S, T; I)$.*

Now suppose furthermore that $S = (s_1, ..., s_k)$ and $T = (t_1, ..., t_k)$ are ordered sequences. Then for each connection $C \in \mathscr{C}(S, T; I)$ we define the induced permutation $\tau(C) \in \mathfrak{S}_k$ to be the unique permutation in $\mathfrak{S}_k$ such that for each j, $t_{\tau(j)}$ is the endpoint of the path through I starting at vertex s_j. In other words, τ encodes the order in which S connects to T via C.

Definition 3.2. *Let G be a digraph, and I a subset of the vertices of G, where $|I| = n$. A loop partition L of I is a collection of n edges of G forming disjoint cycles containing all and only the vertices of I. Equivalently, every edge in L has its source and target in I, and for every vertex v of I there is exactly one edge of L with source v, and one edge of L with target v. The set of all loop partitions of I is denoted by $\mathscr{L}(I)$. If $I = (i_1, ..., i_n)$ is ordered, then for any loop partition L we define $\mu(L) \in \mathfrak{S}_n$ to be the permutation induced by sending vertex $\imath_j$ to the target of the unique edge leading out of i_j.*

Now we return to the expression (1) for the determinant of M, and consider the monomial $\prod_{j=1}^{n} M_{j,\sigma(j)}$ for any permutation $\sigma \in \mathfrak{S}_n$ such that the entries
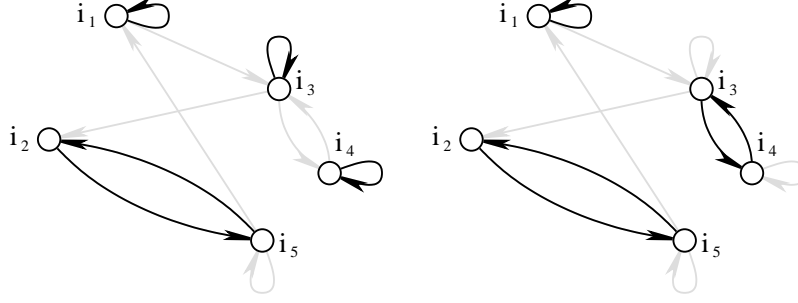
FIGURE 3. Two loop partitions on all the vertices of a fixed underlying graph. For the first, $\mu(L) = (1)(25)(3)(4)$, and for the second, $\mu(L) = (1)(25)(34)$.

$M_{j,\sigma(j)}$ are all non-zero. This determines a set of edges on $\mathscr{G}_M(\ell)$ for any ℓ, by the construction of $\mathscr{G}_M(\ell)$. First suppose $\ell = 0$, so that $\mathscr{G}_M(\ell)$ is bipartite. In this case the edges corresponding to the entries $M_{j,\sigma(j)}$ induce an n-connection from the source vertices to the target vertices, where each path is a single edge from some source vertex s_j to target vertex $t_{\sigma(j)}$. The induced permutation of this connection is just σ, so the nonzero terms (including sign) in $\det M$ correspond exactly to connections from S to T without any intermediate nodes. To summarize:

Proposition 3.3. *Let M be a square matrix, with associated bipartite graph $\mathscr{G}_M(0)$ with source vertices S and target vertices T. Then*

$$\det M = \sum_{C \in \mathscr{C}(S,T;\varnothing)} \text{sign}(\tau(C)) \left(\prod_{e \in C} w(e) \right),$$

where $w(e)$ is the weight of the edge e, i.e. the corresponding entry of M.

Now suppose $I = \{1, \ldots, n\}$, so we are considering the graph $\mathscr{G}_M(n)$ where all the vertices are intermediate vertices. In this case the monomial $\prod_{j=1}^n M_{j,\sigma(j)}$ corresponds to a loop partition on all n intermediate vertices. To see this, given a permutation $\sigma \in \mathfrak{S}_n$ corresponding to a non-zero monomial in the determinant (meaning that the entries $M_{j,\sigma(j)}$ are all non-zero), the corresponding edges $e_{j,\sigma(j)}$ all exist on $\mathscr{G}_M(n)$, and every vertex $\imath_j$ of $\mathscr{G}_M(n)$ is the source of edge $e_{j,\sigma(j)}$ and the target of edge $e_{\sigma^{-1}(j),j}$, so these edges form a loop partition. Conversely, a loop partition gives an induced permutation $\mu(L)$, as in the definition above, and the entries $M_{i,\mu(i)}$ correspond to the edges used in the loop partition. Thus we have a proposition analogous to Proposition 3.3:

Proposition 3.4. *Let M be a square matrix with n rows and columns, and consider its graph representation $\mathscr{G}_M(n)$, where all vertices are intermediate vertices in I. Then*

$$\det M = \sum_{L \in \mathscr{L}(I)} \text{sign}(\mu(L)) \left(\prod_{e \in L} w(e) \right).$$

Furthermore, if one decomposes the permutation σ into disjoint cycles, the cycles in the loop partition correspond to the cycles in the cycle decomposition.

For a general ℓ between 0 and n, a nonzero monomial in $\det M$ corresponds to a combination of a connection and a loop partition. The precise way in which this happens is expressed by the following formula, which is the main theorem of this section:

Main Theorem 1 (Connection-Determinant Formula). *Let M be an $n \times n$ matrix, and consider a graph interpretation $\mathscr{G}_M(\ell)$, with k source vertices S and target vertices T, along with ℓ intermediate vertices I, so $k + \ell = n$. Then we can express the determinant of M as*

$$(2) \quad \det M = (-1)^k \sum_{C \in \mathscr{C}(S,T;I)} \text{sign}(\tau(C)) \left(\prod_{e \in C} (-w(e)) \right) \det M(I_{\notin C}, I_{\notin C}),$$

where

- *C is a connection from S to T through I,*
- *for any edge e used in C, $w(e)$ is the* weight *of e, i.e. the corresponding nonzero entry of M,*
- *$M(I_{\notin C}, I_{\notin C})$ is the principal submatrix of M with rows and columns corresponding to the vertices in $I_{\notin C}$ (i.e., those not used in the connection C).*

In words, the determinant of M is a signed sum over connections from S to T through I which is weighted by the edges used in the connection and the determinant of the submatrix of M corresponding to unused intermediate nodes. The sign associated to each connection is just the sign of its induced permutation.

Proof. We first show that nonzero monomials in $\det M$ correspond bijectively to connections from S to T through I, along with a choice of loop partition of the unused intermediate vertices. Let $\sigma \in \mathfrak{S}_n$ be a permutation inducing a non-zero monomial in the expression for $\det M$. Then the terms in this monomial correspond to a collection of n edges on $\mathscr{G}_M(\ell)$, with the property that for each vertex in $S \cup I$, there is a unique edge in this collection leading out of the vertex, and its target is in $T \cup I$. Since σ is a permutation, the map $S \cup I \to T \cup I$ is a bijection. If we take some vertex s_j in S, we can follow the unique edge leading out of s_j to a new vertex v_1. If v_1 is in T, then we have a path from s_j to v_1. If not, we are at an intermediate node, which has a unique edge leading out of it, to some vertex v_2. If it is in T, we have a path from s_j to v_2. If not, it is an intermediate vertex, and we can follow along its unique edge leading to a new vertex. Proceeding in this way, we eventually end up with a path from s_j to some vertex in T. We repeat this process for each vertex in S, and we arrive at a collection C of edges which forms a set of paths from S to T through I. Moreover, these paths are disjoint, since no intermediate node can be traversed more than once (this would correspond to some column index having two preimages under the supposed bijection σ). So C is a connection induced by the given non-zero monomial in $\det M$.

Aside. The process above is actually a solution to a well-known puzzle: suppose one has finite disjoint sets A, B, and C, and a bijection $f : A \cup C \to B \cup C$. Thus A has the same cardinality as B. But can one use the bijection f to produce an explicit bijection $\widetilde{f} : A \to B$? The method above is one nice solution: for any element $a \in A$, repeatedly apply f until one arrives at some $f(f(...(f(a)..))$ which

$$M = \begin{array}{c} \\ s_1 \\ s_2 \\ i_3 \\ i_4 \\ i_5 \end{array} \begin{array}{ccccc} t_1 & t_2 & i_3 & i_4 & i_5 \\ \begin{bmatrix} \mathbf{m_{11}} & 0 & m_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{m_{25}} \\ 0 & m_{32} & m_{33} & \mathbf{m_{34}} & 0 \\ 0 & 0 & \mathbf{m_{43}} & m_{44} & 0 \\ m_{51} & \mathbf{m_{52}} & 0 & 0 & m_{55} \end{bmatrix} \end{array} \quad \begin{aligned} \sigma &= (1)(34)(25) \\ \phi &= (1)(25) \\ \mu &= (34) \\ \tau &= (1)(2) \end{aligned}$$
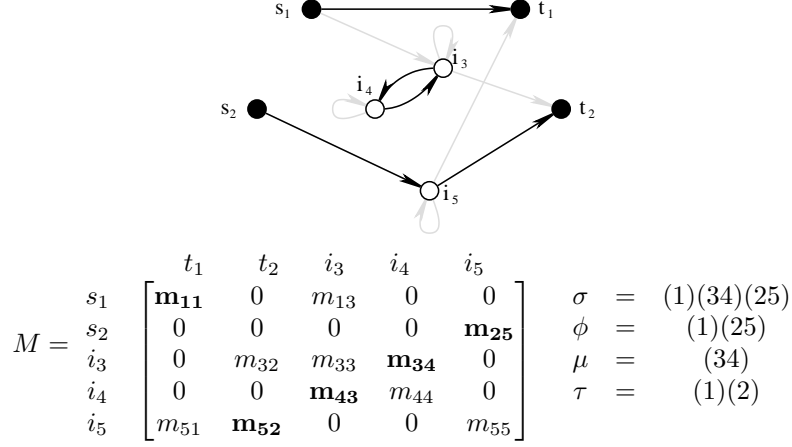
FIGURE 4. A matrix M, a non-zero monomial in its determinant expression, the corresponding edges in $\mathscr{G}_M(3)$, and the permutations associated to this monomial

is an element of B.

What is left over from the bijection $S \cup I \to T \cup I$ after we have accounted for the induced connection? It is a permutation on $I_{\notin C}$, the intermediate indices whose corresponding vertices were not used in the induced connection C. As we have already explained above, such a permutation corresponds to a loop partition of the vertices of $I_{\notin C}$.

Conversely, suppose we are given a connection C from S to T through I and a loop partition L on $I_{\notin C}$, all of whose edges are in $\mathscr{G}_M(\ell)$. Every vertex in $S \cup I_{\in C}$ is the source of a unique edge of C, whose target is then a vertex in $T \cup I_{\in C}$, and this induces a bijection between $S \cup I_C$ and $T \cup I_C$. We have already seen that any loop partition on $I_{\notin C}$ induces a bijection $I_{\notin C} \to I_{\notin C}$. Together, these two bijections give us a bijection $S \cup I \to T \cup I$. This bijection induces a permutation on the n indices of M by ordering $S \cup I$ and $T \cup I$ according to their appearance among the row and column indices, respectively, of M. This permutation in turn corresponds to a monomial in $\det M$, which is nonzero because each entry of M used corresponds to an edge of either C or L, all of which are in $\mathscr{G}_M(\ell)$. This sets up a one-to-one correspondence between the nonzero terms of $\det M$ and pairs (C, L) where $C \in \mathscr{C}(S, T; I)$ and $L \in \mathscr{L}(I_{\notin C})$.

We have thus shown that we may express $\det M$ as follows:

$$(3) \qquad \det M = \sum_{\substack{C \in \mathscr{C}(S,T;I) \\ L \in \mathscr{L}(I_{\notin C})}} \text{sign}(C, L) \prod_{e \in C \cup L} w(e),$$

where $\text{sign}(C, L) = \text{sign}(\sigma)$ for $\sigma \in \mathfrak{S}_n$ the permutation corresponding to the pair (C, L) as outlined above. We would like to interpret this sign directly in terms of properties of C and L. First, factor σ into its disjoint cycles, and then write $\sigma = \phi\mu$ where ϕ consists of those cycles including at least one index in $\{1, \ldots, n\} \setminus I$ (i.e. an index in S and T) and μ is the rest of the cycles, which use only indices in I. This corresponds to the decomposition of edges in the pair (C, L) into edges used in the

connection C and edges used in the loop partition L. Thus $|\phi| = |C|$ and $|\mu| = |L|$, where the absolute value sign means "number of elements of". Next, consider the paths in C, and let $\tau = \tau(C)$ be the permutation in $\mathfrak{S}_k$ induced by C. Then τ, which encodes only the way in which C connects S to T and the actual paths used, is obtained from ϕ by dropping the indices in ϕ corresponding to intermediate vertices. In particular, the number of cycles in ϕ equals the number of cycles in τ, since no cycle consisting entirely of intermediate vertices is included in ϕ. We can use this to relate $\text{sign}(\phi)$ to $\text{sign}(\tau)$ as follows (here $|\phi|_o$ denotes the number of cycles of ϕ, and similarly for $|\tau|_0$):

$$\begin{aligned}
\text{sign}(\phi) &= (-1)^{|\phi|-|\phi|_o} \\
&= (-1)^{|C|-|\tau|_o} \\
&= (-1)^{|C|-k}(-1)^{|\tau|-|\tau|_o} \text{ (since } |\tau| = k) \\
&= (-1)^k(-1)^{|C|}\text{sign}(\tau).
\end{aligned}$$

So $\text{sign}(\sigma) = (-1)^k(-1)^{|C|}\text{sign}(\tau)\text{sign}(\mu)$. We will use this expression for $\text{sign}(\sigma)$ and rewrite (3):

$$\begin{aligned}
\det M &= \sum_{C\in\mathscr{C}(S,T;I)}\sum_{L\in\mathscr{L}(I_{\notin C})}(-1)^k(-1)^{|C|}\text{sign}(\tau)\text{sign}(\mu)\left(\prod_{e\in C}w(e)\right)\left(\prod_{e'\in L}w(e')\right) \\
&= (-1)^k\sum_{C\in\mathscr{C}(S,T;I)}\text{sign}(\tau)(-1)^{|C|}\left(\prod_{e\in C}w(e)\right)\left(\sum_{L\in\mathscr{L}(I_{\notin C})}\text{sign}(\mu)\prod_{e'\in L}w(e')\right) \\
&= (-1)^k\sum_{C\in\mathscr{C}(S,T;I)}\text{sign}(\tau(C))\left(\prod_{e\in C}(-w(e))\right)\det M(I_{\notin C}, I_{\notin C}).
\end{aligned}$$

In the last line, we have collapsed the sum over loop partitions of $I_{\notin C}$ into the determinant of $M(I_{\notin C}, I_{\notin C})$. □

4. Electrical Networks revisited

Let K be the Kirchhoff matrix of an electrical network with underlying graph G, and let $K(P;Q)$ be an arbitrary square submatrix of K. We posed the following question at the beginning of the previous section: can one determine the sign of $\det K(P;Q)$ based entirely on the graph G? This is not possible in general; for example if G is a complete graph there is very little one can say about signs of determinants. But if G is sparse, there is typically quite a lot of information about the signs of determinants that can be obtained without knowing the actual conductivities. We assume now that G is a connected graph. Then one has the following two facts about K:

- Principal proper submatrices of K have strictly positive determinant.
- Off-diagonal entries of K are ≤ 0.

We will not use any other properties of K besides these two; in particular, we will not assume that K is symmetric.

Let $M = K(P;Q)$ be a submatrix of K formed from rows P and columsn Q. Assume that $P \neq Q$ since we already know that the sign of any principal proper submatrix of K is positive. Let $S = P \setminus Q$, $T = Q \setminus P$, and $I = P \cap Q$. By
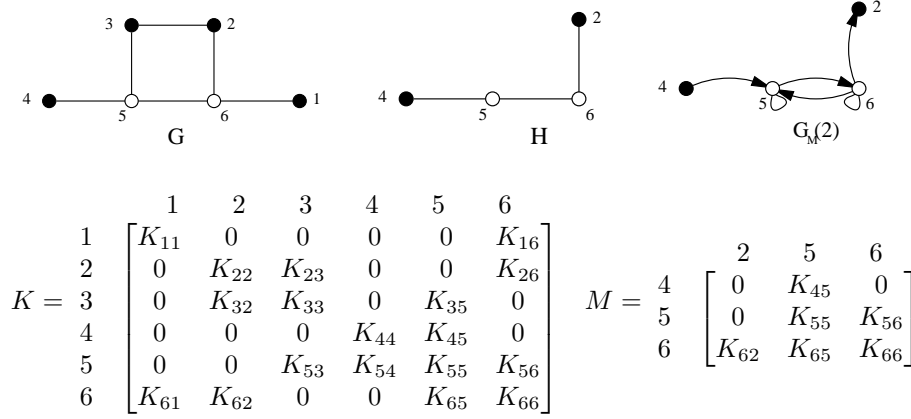
$$K = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \left[\begin{array}{cccccc} K_{11} & 0 & 0 & 0 & 0 & K_{16} \\ 0 & K_{22} & K_{23} & 0 & 0 & K_{26} \\ 0 & K_{32} & K_{33} & 0 & K_{35} & 0 \\ 0 & 0 & 0 & K_{44} & K_{45} & 0 \\ 0 & 0 & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & 0 & 0 & K_{65} & K_{66} \end{array}\right] \end{array} \qquad M = \begin{array}{c} \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{ccc} 2 & 5 & 6 \\ \left[\begin{array}{ccc} 0 & K_{45} & 0 \\ 0 & K_{55} & K_{56} \\ K_{62} & K_{65} & K_{66} \end{array}\right] \end{array}$$

FIGURE 5. A Kirchhoff matrix K, its submatrix $M = K(4,5,6; 2,5,6)$, the underlying graph G of the electrical network, the subgraph H induced by the vertices 2, 4, 5, 6, and the appropriate associated graph

re-ordering P and Q and keeping track of the resulting sign changes in M, we can assume that $P = S + I$, $Q = T + I$ so that the intersection comes last in the ordering. Let $k = |S| = |T|$ and $\ell = |I|$. Then it is natural to consider the connection-determinant formula for the associated graph $\mathscr{G}_M(\ell)$. In fact, $\mathscr{G}_M(\ell)$ corresponds naturally to the subgraph of the underlying graph G of the electrical network defined by the vertices corresponding to indices S, T, and I. This is shown in a specific case in the figure above. The connection-determinant formula for $M = K(P; Q)$ reads

$$(4) \qquad \det M = (-1)^k \sum_{C \in \mathscr{C}(S,T;I)} \operatorname{sign}(\tau(C)) \left(\prod_{e \in C} (\gamma(e)) \right) \det M(I_{\notin C}, I_{\notin C}),$$

where we have used the fact that $-w(e)$ is just the conductivity $\gamma(e)$ of the corresponding edge of the electrical network. This is a sum over all connections from S to T through I on $\mathscr{G}_M(\ell)$. These connections are the same as the connections from S to T through I on the original electrical network G, so we could just as well look at G instead of $\mathscr{G}_M(\ell)$. Notice that the other associated graphs of M do not correspond to submatrices of the original electrical network, which explains why we chose $\mathscr{G}_M(\ell)$ rather than some other associated graph.

The basic criteria for determining the sign of $\det K(S+I; T+I)$ are summarized in the following proposition:

Proposition 4.1. *Let $M = K(S+I; T+I)$ be an arbitrary submatrix of an $n \times n$ Kirchhoff matrix K, with S, T, and I disjoint subsequences of $\{1, ..., n\}$.*

- *If no connection from S to T through I exists on the graph G of the electrical network, then $\det M = 0$.*
- *If at least one connection from S to T through I exists, and every such connection induces the same permutation $\tau(C)$, then $\det M \neq 0$, and the sign of the determinant is $(-1)^k \operatorname{sign} \tau(C)$.*

Proof.

- If no connection from S to T through I exists, this corresponds to no non-zero monomial existing in the expansion of det M as a sum over permutations. Therefore it must be 0.
- The sign of any term in the connection-determinant formula is entirely determined by sign $\tau(C)$, since $\gamma(e)$ is always positive and so is det $M(I_{\notin C}, I_{\notin C})$. So if these are all the same, then det M is the sum of non-zero monomials of the same sign.

$\square$

We can say something even stronger when G satisfies a planarity condition, known as circular planarity:

Definition 4.2. *Let G be a graph with boundary. Then G is called circular planar if there exists an embedding of G in the unit disc D such that $G \cap \partial D$ equals the boundary vertices of G. G is called circularly embedded if it is given such an embedding. Given G a circularly embedded graph with boundary, let S and T be disjoint subsets of boundary vertices. S and T are said to form a circular pair if they lie on disjoint arcs of the unit circle. We always give S and T the counterclockwise ordering around the circle.*

Proposition 4.3. *Let K be the Kirchhoff matrix of a circularly embedded electrical network, with the boundary vertices given in counterclockwise order. Let S and T be disjoint subsets of the boundary vertices forming a circular pair. Let I be the set of interior vertices. Then $\det K(S+I; T+I)$ is non-zero if and only if there exists a connection from S to T through I, and the sign of this determinant is $(-1)^{k(k+1)/2}$, where $k = |S| = |T|$.*

Proof. We will show that every connection induces the same permutation $\tau(C)$ and compute its sign. Let $S = (s_1, ..., s_k)$ and $T = (t_1, ..., t_k)$. We will show that s_i must map to t_{k-i} for each i. One can reduce immediately to showing that s_1 cannot map to t_j for $j < k$. Suppose to the contrary that s_1 mapped to such a t_j. Then by the Jordan curve theorem, the path from s_1 to t_j separates the disk into two connected components, one containing t_k and the other containing s_i for all $i > 1$. Hence there cannot be any path from a vertex of S to t_k which does not cross the path from s_1 to t_j, so no such connection can exist (see figure on next page). Hence in any connection, the path beginning at s_1 must end at t_k. Proceeding inductively, we find that $\tau(j) = k - j$ for each $1 \le j \le k$. Thus all the monomials have the same sign. To compute this sign, we must compute the sign of the permutation in $\mathfrak{S}_k$ sending j to $k - j$. This one easily computes to be $(-1)^{k(k-1)/2}$. Hence the overall sign of the determinant is $(-1)^k (-1)^{k(k-1)/2} = (-1)^{k(k+1)/2}$. $\square$

4.1. **The Inverse Problem on Electrical Networks.** Recall the proposition we proved earlier for expressing an entry in the Kirchhoff matrix as a function of entries in the response matrix:

Proposition 4.4 (Boundary edge formula)**.** *Suppose that $K = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a partitioned square matrix with C square and invertible, and let $\Lambda = K/D$. Suppose that $\Lambda(p + S; q + T)$ is a square submatrix of Λ consisting of row p then rows S, column q then columns T, such that $\det \Lambda(p + S; q + T) \ne 0$. Next construct a*
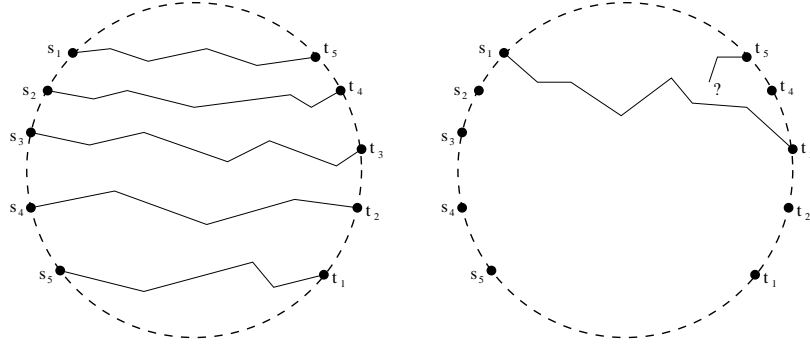
FIGURE 6. Every connection of a circular pair $(S;T)$ must map s_i to t_{k-i}. If s_1 did not map to t_k, then nothing could map to t_k.

new matrix K' from K by zeroing out K_{pq}, and let $\Lambda' = K'/D$. Suppose that $\det \Lambda'(p+S; q+T) = 0$. Then $\det \Lambda(S;T) \neq 0$, and

$$K_{pq} = \det \Lambda(p+S; q+T)/\det \Lambda(S;T).$$

We would like simpler conditions under which we can apply this formula in the case of electrical networks. First note that $\det \Lambda(P;Q)$ has the same sign as $\det K(P+I; Q+I)$ for any P and Q so the determinantal conditions in the proposition are really conditions on subdeterminants of K. When applying the connection-determinant formula, it is easiest to work generically:

Definition 4.5. *Let G be a graph with boundary, with N edges and k boundary nodes. Recall that $L : \mathbb{R}_+^N \to M_{k \times k}$ denotes the map sending a conductivity function on G to the resulting response matrix. We say that G is generically recoverable if there exists an open dense subset $U \subseteq \mathbb{R}_+^n$ such that $L|_U$ is injective. We say that an edge $e \in G$ is generically recoverable if there exists an open dense subset $U \subseteq \mathbb{R}_+^n$ such that the fibers of $L|_U : U \to M_{k \times k}$ are constant on the edge e.*

The simplest geometric version of the boundary edge formula is the following:

Proposition 4.6. *Let G be a graph with N edges and k boundary nodes, and let pq be a boundary edge of G. Suppose that there are sequences S and T of boundary vertices (not necessarily disjoint) such that there exists a connection from $p + S$ to $q + T$ on G, but that on the graph G' obtained from deleting boundary edge pq, no $(p+S; q+T)$ connection exists. Then edge pq is generically recoverable, and in fact*

$$\gamma_{pq} = -\frac{det\Lambda(p+S; q+T)}{\det \Lambda(S;T)}$$

on the open dense subset where $\det K(p+S+I; q+T+I) \neq 0$. Here $\gamma : V \times V \to \mathbb{R}_+$ is a conductivity function on G and Λ is the resulting response matrix.

If $(p+S; q+T)$ is a circular pair and also satisfies the conditions above, then the edge is globally recoverable, not just generically.

Proof. Existence of a connection from $p+S$ to $q+T$ is equivalent to there existing at least one non-zero monomial in the expression $\det K(p+S+I; q+T+I)$. Therefore the condition $\det K(p+S+I; q+T+I) = 0$ defines a proper Zariski-closed subset of $\mathbb{R}_+^N$; we defer the proof until the end of this proposition. The fact that there

is no connection from $p + S$ to $q + T$ when one removes the edge pq then implies that $\det K'(p + S + I; q + T + I) = 0$ for all choices K' of a Kirchhoff matrix on the graph G' obtained from deleting edge pq. Thus the conditions in the statement of this proposition are a direct translation of the conditions in the boundary edge formula.

If $(p + S; q + T)$ is a circular pair on a circular planar graph, then the existense of a connection implies that $\det K(p + S + I; q + T + I) \neq 0$ for all conductivity functions on G, not just on a Zariski-open subset of them. This was proved in (4.1). $\square$

Now we return to the unproved lemma cited in the proof:

Lemma 4.7. *Let G be a graph with N edges and k boundary nodes, and S, T subsets of the boundary vertices such that there exists a connection from S to T. Then there exists an open dense subset $U \subset \mathbb{R}_+^N$ of the space of all conductivity functions such that for all Kirchhoff matrices K constructed from conductivity functions in U, $\det K(S + I; T + I) \neq 0$.*

Proof. We may assume S and T are disjoint by relabeling the intersection $S \cap T$ as part of the interior. The fact that there exists a connection from S to T means that if we consider each entry in $K(S + I; T + I)$ as an independent variable, then $\det K(S + I; T + I)$ is a non-zero polynomial in these entries. This is because the conection-determinant formula tells us that the non-zero monomials in $\det K(S + I; T + I)$ correspond bijectively to a choice of connection from S to T, plus a loop partition on the unused interior vertices. However, the entries in $K(S + I; T + I)$ are not all independent, since the original matrix K was symmetric with diagonal entries determined as functions of the off-diagonal entries. What we must show is that $\det K(S + I; T + I)$ is not the zero function of the conductivities γ_{ij}. To do this, we need an alternate connection-determinant formula, describing the non-zero monomials in $\det K(S + I; T + I)$ when this expression is explicitly written as a function of the conductivities of the network. This alternate formula is given by the Tree Diagram formula, proved in [BiMa10]. The tree diagram formula states that the non-zero monomials in $\det K(S + I; T + I)$ correspond bijectively to the choice of a connection from S to T through I, together with a tree diagram (see [BiMa10]) connecting the unused interior vertices to vertices in the rest of the electrical network. It is shown there that if there is a connection from S to T through I, then there exists at least one tree diagram on the unused interior vertices, and hence at least one non-zero monomial in the expression $\det K(S + I; T + I)$ when this is considered as a function of the independent variables γ_{ij}. Hence this determinant does not vanish identically, and so the locus on which it does not vanish is open and dense in $\mathbb{R}_+^N$ (since it is non-empty and Zariski open). $\square$

One can also give a geometric version of the boundary edge formula for recovering diagonal entries K_{pp} of the Kirchhoff matrix. However, it is very inconvenient to do this by referring to connections on the underlying graph G of the electrical network. Instead, we look at a directed graph $\mathcal{G}$, called the secondary graph of the electrical network, constructed as follows. For every edge pq of G, there exist directed edges pq and qp on $\mathcal{G}$. In addition, there exists a self-loop on every vertex of $\mathcal{G}$. Another way of stating this is that $\mathcal{G}$ is the associated graph $\mathscr{G}_K(n)$ for any valid Kirchhoff matrix on the electrical network G, where n is the total number of vertices of G.

Proposition 4.8. *Let G be a graph with N edges and k boundary nodes, and n total nodes. Let p be a boundary vertex of G. Suppose that there are sequences S and T of boundary vertices such that at least one $(p+S; p+T)$-connection exists on the secondary graph $\mathcal{G}$ of the electrical network, but that every such connection must connect p to itself via its self-loop. Then the sum of the conductivities leading out of vertex p is generically recoverable, and given by the following formula when $\det \Lambda(p+S; p+T) \neq 0$:*

$$K_{pp} = \frac{\det \Lambda(p+S; p+T)}{\det \Lambda(S; T)}.$$

Proof. This is is similar to the previous proposition. The fact that there is at least one $(p + S; p + T)$ connection means that there is at least one non-zero monomial in the expansion of $\det K(p + S + I; q + T + I)$, and hence the locus on which $\det K(p + S + I; q + T + I) = 0$ is a proper Zariski-closed subset of $\mathbb{R}_+^N$ (using the lemma above). The fact that every such connection must connect p to itself via its self-loop means that when one zeroes out K_{pp}, corresponding to deleting the self-loop in the secondary graph, then $\det K'(p + S + I; q + T + I) = 0$. So the conditions of the boundary edge formula apply. $\square$

Now we give an example. Consider the electrical network with graph and secondary graph shown below:

Graph   Secondary Graph
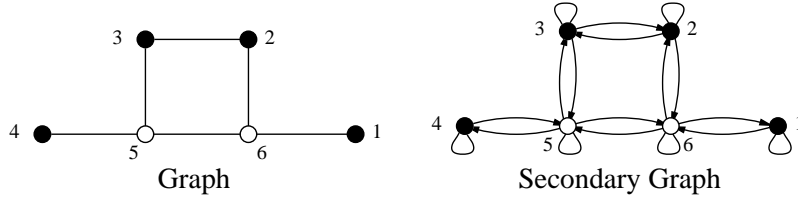
We can pose the recovery problem for electrical networks on this graph. It turns out that this graph is recoverable. We do not quite have the tools to prove this (see [CuMo00] for the remaining necessary tools), but we can recover all of the conductivities except γ_{56}. First consider the boundary edge 23. This edge can be recovered by considering the connection $(1, 2; 3, 4)$, which is broken when one deletes edge 23. Therefore

$$\gamma_{23} = -\frac{\det \Lambda(2, 1; 3, 4)}{\lambda_{14}} = -\frac{\lambda_{23}\lambda_{14} - \lambda_{13}\lambda24}{\lambda_{14}}.$$

Next consider the self-loop on boundary vertex 1 in $\mathcal{G}$. This equals the sum of the conductivities out of vertex 1, which is just γ_{16}. A connection on $\mathcal{G}$ which must use the self-loop on 1 is $(1, 2; 1, 4)$, so

$$\gamma_{15} = \frac{\lambda_{11}\lambda_{24} - \lambda_{21}\lambda_{14}}{\lambda_{24}}.$$

Finally, consider the self-loop on boundary vertex 2. This represents the sum $\gamma_{23} + \gamma_{26}$. We already obtained a formula for γ_{23}, but we can obtain an independent formula for $\gamma_{23} + \gamma_{26}$ from the boundary edge formula and hence compute γ_{26}. Inspecting the secondary graph $\mathcal{G}$, we see that every connection from $(2, 1)$ to $(2, 4)$ has to use the self-loop on 2. Hence

$$\gamma_{23} + \gamma_{26} = \frac{\lambda_{22}\lambda_{14} - \lambda_{12}\lambda_{24}}{\lambda_{14}}.$$
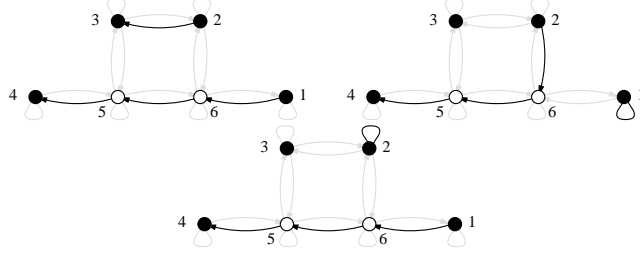
FIGURE 7. These three connections allow us to recover conductivities γ_{23}, γ_{16}, and $\gamma_{23} + \gamma_{26}$

Using symmetric arguments, we can compute γ_{45} and $\gamma_{35} + \gamma_{23}$. This gives us all of the conductivities except γ_{56} as functions of entries in the response matrix, so we have proved that every edge in the graph except γ_{56} is recoverable. In fact, the whole graph is recoverable as stated above, but we need more tools to prove that.

The boundary edge formula and the connection-determinant formula are basic tools for recovering electrical networks, but more tools are needed to give strong results. The strongest general theorem proved to date is requires one more definition to state:

Definition 4.9. *Let G be a graph with boundary, and e an edge in G. A valid edge removal for e is one of the following two operations:*

- *Deleting e from the graph, leaving the vertex set the same.*
- *Contracting e, meaning that one identifies the endpoints of e, and then deleting e (which is now a self-loop) from the resulting graph. One can only contract an edge if both its neighbors are interior nodes, or if exactly one is boundary and in the resulting graph G' the vertex obtained by contracting the endpoints is designated as boundary.*

Theorem 4.10. *Let G be a circular planar electrical network. Then the following are equivalent:*

- *G is generically recoverable.*
- *G is recoverable.*
- *For any edge e of G and valid edge removal of e, there exists a pair $(S;T)$ of sequences of boundary vertices such that there exists an $(S;T)$-connection on G, but after removing e there does not exist an $(S;T)$-connection.*

The proof is long, and is one of the principal results of [CuMo00].

References

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[BiMa10] Biesel, Owen and Mannisto, Peter. The Tree Diagram Formula.
[Lis11] Lister, Eddie. Input-Output Random Walk Networks.

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