# ON LOCAL RECOVERABILITY OF CIRCULAR PLANAR ELECTRICAL NETWORKS

OWEN BIESEL AND PETER MANNISTO

Abstract. Let $G$ be a circular planar graph with boundary. In [CuMo00], it is shown that the following three properties of $G$ are equivalent: (a) $G$ is recoverable, i.e., a conductivity function on the edges of $G$ is uniquely determined by the resulting Dirichlet-to-Neumann map, (b) $G$ is critical, meaning every edge removal in $G$ breaks a connection between circular sequences $P$ and $Q$ of boundary vertices, and (c) the medial graph of $G$ is lensless. In this paper we investigate the analogous properties for a single edge $e \in G$: namely, we study the relation between (a$'$) whether the conductivity of $e$ is uniquely determined by the Dirichlet-to-Neumann map, (b$'$) whether removing edge $e$ breaks a connection, and (c$'$) the location of lenses relative to $e$ in the medial graph of $G$.

Contents



## 1. Introduction

**1.1.** Let $G = (V, \partial V, E)$ be a graph with boundary with vertices $V$, boundary vertices $\partial V$, and edges $E$. A *conductivity function* on $G$ is a function $\gamma : E \to \mathbb{R}_+$ from the edges of $G$ to the positive real numbers. For any such conductivity function, we get an electrical network $\Gamma = (G, \gamma)$ with underlying graph $G$ and conductivity function $\gamma$. The conductivity function $\gamma$ has the following defining property: for any edge $e$ with endpoints $v_1$ and $v_2$, and voltage function $u$ on the vertices of $G$, the current $i(e)$ along edge $e$ from $v_1$ to $v_2$ is computed by

$$i(e) = \gamma(e) \cdot (u(v_1) - u(v_2)).$$

After ordering the vertices in $\partial V$ there is a uniquely determined matrix $\Lambda$, called the *response matrix* or *Dirichlet-to-Neumann map*, which takes a vector of voltages imposed at the boundary vertices of this electrical network and maps it to the vector of induced boundary currents into the network. A basic inverse problem of electrical network theory studied in [CuMo00] and [dVGV96] is to determine the conductivity function $\gamma$ from the matrix $\Lambda$. To state this more precisely, we recall some definitions.

**Definition 1.2.** Let $G$ be a graph with boundary, with $n$ edges and $k$ boundary vertices. Then the space of conductivity functions on $G$ is $\mathbb{R}^n_+$, and the set of response matrices $\Lambda$ for electrical networks on $G$ is contained in the set $M_{k\times k}$ of $k \times k$ matrices. We let $L : \mathbb{R}^n_+ \to M_{k\times k}$ denote the map sending a conductivity function $\gamma$ to the induced response matrix $\Lambda$. We say that $G$ is *recoverable* if the map $L$ is injective. For a given edge $e \in G$, we say that $e$ is recoverable if for every matrix $\Lambda \in M_{k\times k}$, the fiber $L^{-1}(\Lambda)$ is either empty or takes on a unique value on the edge $e$. In other words, the conductivity of the edge $e$ is uniquely determined by $\Lambda$.

**1.3.** In this paper we are interested in the question of when a specific edge $e \in G$ is recoverable. It is often the case that the conductivities of some edges are determined by boundary data, but other edge conductivities are not. A simple instance of this is given by the following graph with boundary, where all nodes are boundary nodes (in this paper, all graphs will be drawn with boundary nodes filled in and interior nodes hollow):

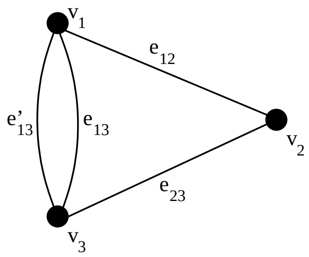


FIGURE 1. An electrical network with edges $e_{13}$ and $e'_{13}$ not recoverable

In this case the edges $e_{12}$ and $e_{23}$ are recoverable, but we can only uniquely determine the sum $\gamma_{13} + \gamma'_{13}$ of the conductivities of the two edges $e_{13}$, $e'_{13}$. So these edges are not recoverable.

Assume that $G$ is a circular planar graph with boundary. One of the main results of [CuMo00] is the following theorem (see Section 2 for definitions):

**Theorem 1.4.** *(Combining Theorem 6.1 and Proposition 9.3 in* [CuMo00]*) Let $G$ be a circular planar graph with boundary. Then the following are equivalent:*

(1) *$G$ is recoverable.*
(2) *$G$ is critical, i.e., every edge removal on $G$ breaks some connection set.*
(3) *The medial graph of $G$ is lensless.*

**Convention 1.5.** We will always assume that a circular planar graph with boundary $G$ is given a specific circular embedding, so that its medial graph is defined.

Theorem 1.4 gives a very satisfactory answer to the question of which circular planar graphs are recoverable. However, for non-recoverable graphs it gives no information on which edges are recoverable, and no information on which edge removals break a connection. These are the questions we intend to investigate in this paper.

**1.6. Overview of Results.** For this introduction we assume that the reader is familiar with the basic terminology on connection-breaking and medial graphs, as in Chs. 2 and 8 of [CuMo00]. This is briefly reviewed in Section 2. To state our main results, we make some additional definitions.

**Definition 1.7.** Let $G$ be a circular planar graph with boundary, with medial graph $M$. A *minimal lens* in $M$ is a lens which does not contain any strictly smaller lens.

A typical minimal lens is shown in Figure 2 (in dark color), together with its underlying graph (in light color). Our basic technique in this paper is to look for minimal lenses in the medial graph, and then successively remove them from the graph by a process called a pole removal which is defined below.

**Definition 1.8.** (Definition 3.4 in text) Let $G$ be a circular planar graph with medial graph $M$, and let $L$ be a minimal lens in $M$. We define the *pole removals* of $L$ by cases as follows.

(1) Suppose $L$ is a minimal 1-pole lens, corresponding to a geodesic fragment $\alpha = ...a_{-1}a_0a_0a_1...$ (i.e., a self-loop in the medial graph on vertex $a_0$). Then there is a unique pole removal of lens $L$, which replaces geodesic fragment $\alpha$ with $\alpha' = ...a_{-1}a_1...$ and leaves the rest of $M$ unchanged. See Figure 10.

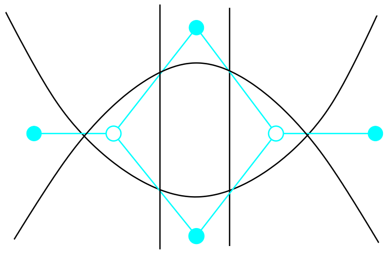

FIGURE 2. A minimal lens (dark color) and its underlying graph (light color)

(2) Suppose $L$ is a minimal 2-pole lens defined by geodesic fragments $\alpha = ...a_0a_1...a_m...$ and $\beta = ...b_0b_1...b_n...$ with $a_0 = b_0$, $a_m = b_n$, and $a_i \neq b_j$ for any other $i \neq j$. Then there are two pole removals of $L$. The first pole removal replaces geodesic fragments $\alpha$ and $\beta$ by

$$\alpha' = b_{-1}a_1a_2...a_ma_{m+1}... \quad \text{and}$$
$$\beta' = a_{-1}b_1b_2...b_nb_{m+1}...$$

(See Figure 11). The second pole removal replaces geodesic fragments $\alpha$ and $\beta$ by

$$\alpha' = a_{-1}a_0...a_{m-1}b_{m+1}... \quad \text{and}$$
$$\beta' = b_{-1}b_0...b_{m-1}a_{m+1}...$$

As defined above, the pole removals are operations on the medial graph $M$. Each pole removal, however, corresponds to an edge removal in the graph $G$, and this edge removal is also called a pole removal of the lens $L$.

**Remark 1.9.** When the minimal lens $L$ is an empty 2-pole lens, it corresponds to either a series or parallel edge in $G$. Then the pole removals correspond to contracting one of the two series edges, or deleting one of the two parallel edges. In Figure 2, the pole removals correspond to the contractions of the boundary spikes on the left and right.

The effect of a pole removal on the rest of the graph is summarized in the following proposition:

**Proposition 1.10.** *(Corollary 3.8 in text) An edge representing a pole of a minimal lens is not recoverable, and any pole removal of a minimal lens is not essential (i.e., removing the edge does not break any connection in $G$). A pole removal does not affect recoverability or essentiality of any edge not intersecting the lens.*

As a first approximation, non-recoverable edges correspond to edges which intersect minimal lenses (either on the interior or the boundary of the lens), while inessential edge removals correspond to pole removals of minimal lenses. This is complicated by the fact that when one removes a minimal lens via a pole removal, new minimal lenses may appear in the medial graph (as in Example 4.4). One can take into account these new minimal lenses by making the following definition:

**Definition 1.11.** Let $G$ be a circular planar graph with boundary, with medial graph $M$. Given an edge $e \in G$, we say that $e$ *potentially intersects a minimal lens* if there exists a sequence of graphs $G = G_0 \mapsto G_1 \mapsto ... \mapsto G_n$, with $G_{i+1}$ obtained from $G_i$ by a single pole removal of a minimal lens ($e$ cannot be the edge that is removed), such that $e$ intersects a minimal lens in the medial graph of $G_n$. Given an edge removal $r$ in $G$ (see Definition 2.5), we say that $r$ is *potentially a pole removal of a minimal lens* if there exists a sequence of graphs $G = G_0 \mapsto G_1 \mapsto ... \mapsto G_n$ as above, such that $r$ is a pole removal of a minimal lens in the medial graph of $G_n$.

Our main result is then as follows:

**Theorem 1.12.** *(combining Theorems 5.2 and 6.2 in text) Let $G$ be a circular planar graph with boundary, and $e \in G$ an edge. Then $e$ is non-recoverable if and only if $e$ potentially intersects a minimal lens. An edge removal $r$ in $G$ is inessential (does not break any connection) if and only if $r$ is potentially a pole removal of a minimal lens.*

This is our local (edge-by-edge) analogue of Theorem 1.4.

**1.13.** Some comments on this result are in order. As noted above, when one removes a minimal lens via a pole removal, new lenses may appear. Nevertheless, the process of uncrossing minimal lenses one at a time must terminate in a lensless medial graph, since each uncrossing reduces the number of crossings in the medial graph by one, and a medial graph with no crossings is obviously lensless. There are many possible ways to uncross minimal lenses to arrive at a lensless medial graph, so the following fact is somewhat surprising:

**Corollary 1.14.** *(Corollary 5.4 in text) Let $G = G_0 \mapsto G_1 \mapsto ... \mapsto G_n$ be any sequence of graphs such that each $G_{i+1}$ is obtained from $G_i$ by a single minimal lens uncrossing, and the medial graph of $G_n$ is lensless. Let $L_i$ be the minimal lens uncrossed to obtain $G_{i+1}$ from $G_i$. Then let $S$ be the set of edges in the original graph $G$ which intersect any of the minimal lenses $L_0, ..., L_{n-1}$ (this makes sense since the set of edges of each $G_i$ is a subset of the edges of $G$). This set $S$ is exactly the set of non-recoverable edges in $G$; in particular, $S$ is independent of the way in which minimal lenses are uncrossed.*

As explained in Remark 5.5, this leads to an algorithm based on the medial graph for computing the non-recoverable edges in any circular planar graph with boundary. Unfortunately, the corresponding statement for edge removals is false; the collection of edge removals which appear as pole removals of minimal lenses in a sequence $G = G_0 \mapsto ... \mapsto G_n$, with $G_n$ critical, is not independent of the sequence. Therefore to apply Theorem 1.12 to compute the inessential edge removals in a circular planar graph with boundary, one must a priori consider all possible sequences of minimal lens uncrossings yielding a critical graph. There could be many such sequences, so this does not appear to be a computationally feasible way to compute the inessential edge removals in a circular planar graph.

**1.15. Acknowledgements.** The authors would like to thank Jim Morrow and the many students of the mathematics REU at the University of Washington. Our inspiring conversations with several students, as well as with Jim Morrow, are what led us to pursue the questions studied in this paper.

## 2. Preliminaries on Medial Graphs

This section introduces the terminology on edge removals and medial graphs that we will need for this paper. Much of this appears in [CuMo00], and we will use this as a reference when appropriate. We start by introducing some definitions related to connections and connection breaking, as in Ch. 2 of [CuMo00].

**Definition 2.1.** Let $G = (V, \partial V, E)$ be a graph with boundary. $G$ is said to be *circular planar* if there exists an embedding $G \hookrightarrow \overline{D}$ into the closed unit disc $\overline{D}$ in the plane, such that the intersection $G \cap \partial\overline{D}$ is precisely the boundary vertices of $G$. Such an embedding is called a circular embedding.

**Convention 2.2.** In this paper all graphs that appear will be circular planar. Whenever we refer to a circular planar graph $G$, we will always assume that we have chosen a specific circular embedding $G \hookrightarrow \overline{D}$, along with an ordering $\partial V = (v_1, ..., v_k)$ of the boundary vertices of $G$ such that these vertices appear in clockwise order along $\partial D$.

**Definition 2.3.** Let $G$ be a circular planar graph with boundary, and let $S$ and $T$ be disjoint subsets of the boundary vertices of $G$, with $|S| = |T|$. We say that $(S; T)$ form a *circular pair* if $S$ and $T$ lie on disjoint (connected) arcs of the boundary circle. We will always assume that $S$ and $T$ are given their clockwise ordering around the boundary circle.

**Definition 2.4.** Let $G$ be a circular planar graph with boundary, and let $S$ and $T$ be a circular pair of boundary vertices, with $|S| = |T| = k$. A *connection* from $S$ to $T$ is a collection $C$ of edges of $G$ satisfying the following conditions:

- The collection $C$ forms a set of $k$ disjoint paths in $G$ containing no cycles.
- Every path has one endpoint in $S$ and one endpoint in $T$; in particular, every vertex of $S$ and $T$ is the endpoint of exactly one path.
- Any vertex used in the path except the endpoints must be an interior vertex.

**Definition 2.5.** Let $G$ be a graph with boundary, and let $e \in G$ be an edge. An *edge removal* for $e$ is one of the following two ways of producing a graph $G'$ from $G$ with one fewer edge:

(1) Take $G'$ to have the same vertex set as $G$, but with edge set $E(G') = E(G) - e$. This is called *deleting* edge $e$.
(2) Let $G'$ be the graph obtained by deleting $e$ and then identifying the two endpoints of $e$. This is called *contracting* the edge $e$. The vertex $v$ corresponding to the endpoints of $e$ in $G$ is declared to be boundary in $G'$ if and only if at least one of the endpoints in $G$ was boundary.

If $r$ is an edge removal on $G$, then we will denote the graph $G'$ obtained from applying edge removal $r$ to $G$ by $Gr$.

**Remark 2.6.** Notice that if $r$ and $s$ are edge removals of distinct edges in $G$, then we can make sense of applying edge removals $r$ and $s$ in succession. It is clear that the order in which we apply $r$ and $s$ does not matter, i.e., $(Gr)s = (Gs)r$. This simple observation will be used often in this paper.

**Remark 2.7.** Suppose that $r$ is an edge removal on $G$ which is not a boundary-boundary edge contraction. Then it is clear that the boundary vertices of $G$ and those of $Gr$ can be identified in a natural way. This is done implicitly in Definition 2.8 below.

**Definition 2.8.** Let $G$ be a graph with boundary, and $r$ an edge removal on $G$. First suppose $r$ is not a boundary-boundary edge contraction. We say that $r$ is *essential* on $G$ if there exists a circular pair $(S;T)$ of disjoint sets of boundary vertices such that there exists at least one $(S;T)$-connection on $G$, but there does not exist any $(S;T)$-connection on $Gr$. In this case we say that $r$ breaks the $(S;T)$-connection set. If $r$ is a boundary-boundary edge contraction, we say that $r$ is essential if and only if the endpoints of the edge $e$ which is to be contracted are distinct.

We say that an edge $e \in G$ is essential if both edge removals of $e$ are essential. We say that $G$ is essential (or critical, as in [CuMo00]) if every edge removal on $G$ is essential.

**2.9.** (Medial graphs) We refer to [CuMo00, Ch. 8] for the definition of a medial graph. It is convenient to subdivide the edges of the medial graph into collections called *geodesics*, as defined in [CuMo00, p. 129]. We typically specify a fragment of a geodesic in $M$ by a sequence $u_0u_1...u_m$ of vertices of $M$, such that there are edges $u_0u_1$, $u_1u_2$, ..., $u_{m-1}u_m$ such that these edges lie in consecutive order along a geodesic. By smoothing the geodesics in $M$, we can obtain that each geodesic $G$ in the medial graph $M$ is a smooth connected arc in the unit disc, either having two endpoints on the boundary of the disc or forming a closed loop in the interior of the disc.

Given a circular planar graph $G$, we will denote its medial graph by $M(G)$ or simply $M$. Then as explained in [CuMo00, Sect. 8.2], the cells of the embedding $M \hookrightarrow \overline{D}$ can be 2-colored (say, black and white) in such a way that the black cells are precisely those containing a vertex of $G$. Then $M$ together with this 2-coloring determine $G$, unless $G$ has isolated interior vertices.

**Convention 2.10.** In this paper, we will treat two graphs with boundary $G$ and $G'$ as equivalent if they only differ in some number of isolated interior vertices. Since isolated interior vertices have no effect on the electrical properties of $G$, this identification is harmless. With this convention, the medial graph $M(G)$ together with the 2-coloring induced by $G$ will uniquely determine $G$.

**Definition 2.11.** Let $v$ be a crossing of two geodesics in a medial graph $M$ (i.e., a vertex of $M$). Recall that this corresponds to an edge $e \in G$. The two edge removals of $e \in G$ then correspond to two operations on the medial graph $M$ which are called the *uncrossings* of $v$. In the typical case, these are as in Figure 3. For more on uncrossings see [CuMo00, p. 130]. An exceptional case occurs when the corresponding edge $e$ is a self-loop or (dually) a hanging interior edge, i.e., an interior vertex of degree 1. In this case (up to an isolated interior vertex) deleting and contracting $e$ are the same operation, and so the two ways of uncrossing $M$ have the same result on the medial graph, which is shown in Figure 4.

**Definition 2.12.** Let $M$ be a medial graph, viewed as a collection of smooth arcs (geodesics) in $\overline{D}$. Informally, a *lens* in $M$ is one of the three configurations shown in Figure 5. More precisely, it is one of the following three configurations in the medial graph $M$:

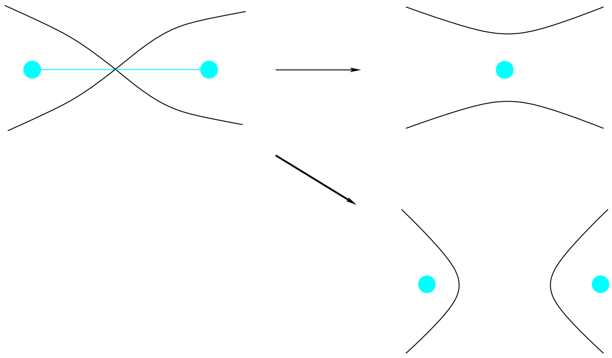

FIGURE 3. The two uncrossings of $v$ corresponding to deleting and contracting an edge $e$ in the typical case

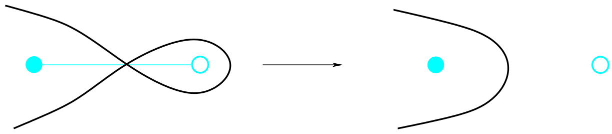

FIGURE 4. The uncrossing of $v$ corresponding to deleting an edge leading to a hanging interior vertex, and its effect on the medial graph (dark color)

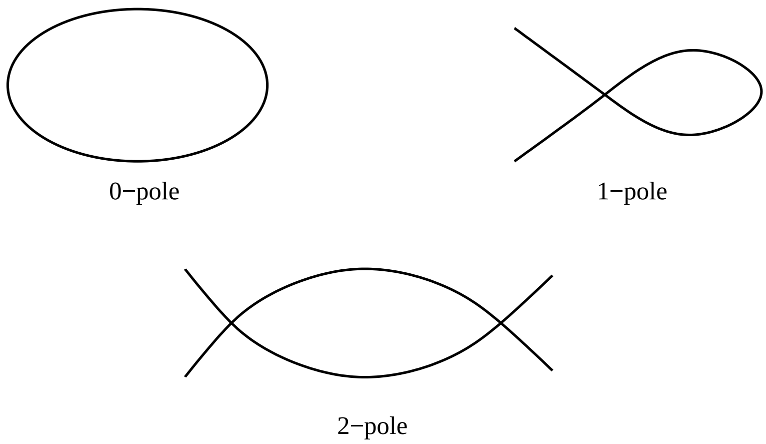


FIGURE 5. The three types of lenses in a medial graph (0-pole lens in upper left, 1-pole lens in upper right, and 2-pole lens on bottom)

(1) (0-pole) a complete geodesic $\alpha = u_0u_1...u_m$ such that $u_m = u_0$ and $u_i \neq u_j$ for any other pair $0 \leq i \neq j \leq m$.
(2) (1-pole) a geodesic fragment $\alpha = u_0u_1...u_m$, where $u_m = u_0$ and $u_i \neq u_j$ for any other pair $0 \leq i \neq j \leq m$, and such that $\alpha$ is not a full geodesic (i.e., edge $u_{m-1}u_m$ has a direct extension $u_mu_{m+1}$ which is not the edge $u_0u_1$.) The repeated vertex $u_0 = u_m$ is called the *pole* of $L$.
(3) (2-pole) two geodesic fragments $\alpha = u_0u_1...u_m$ and $\beta = v_0v_1...v_n$ such that $u_0 = v_0$, $u_m = v_n$, and for no other $i \neq j$ do we have $u_i = v_j$. The vertices $u_0 = v_0$ and $u_m = v_n$ are called the *poles* of $L$.

Note that in [CuMo00, Ch. 8], the first two configurations would have been referred to as degenerate lenses. Note that there may be other geodesics intersecting a lens in $M$. If the region contained in a lens does not intersect any geodesic, then the lens is said to be empty.

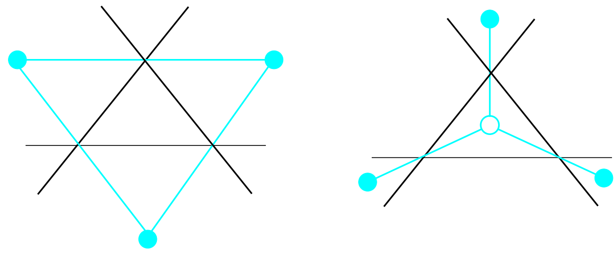

FIGURE 6. An empty triangle in the medial graph (corresponds to either a $Y$ or $\Delta$ in $G$)

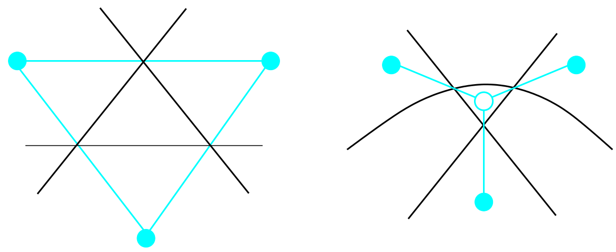

FIGURE 7. An arc switch ($Y$-$\Delta$ transformation)

**Remark 2.13.** An empty 2-pole lens in a medial graph $M$ corresponds to either a series or parallel edge in $G$. An empty 1-pole lens corresponds to a hanging interior vertex (i.e., the edge leading to an interior vertex of degree 1) or to a self-loop from a vertex to itself. An empty 0-pole lens cannot exist in the medial graph of any graph with boundary.

**Definition 2.14.** Let $M$ be the medial graph of $G$. An *empty triangle* in $M$ is a configuration topologically equivalent to that shown in Figure 6. The corresponding configuration in the graph $G$ is called a $Y$ or a $\Delta$. There is a well-known procedure for converting a graph $G$ with a $Y$ into a graph $G'$ with a $\Delta$ and vice-versa, called a $Y$-$\Delta$ transformation. The corresponding modification to the medial graph is called an arc switch (see Figure 7). See [CuMo00, Sect. 8.3] for more details.

## 3. Uncrossing minimal lenses

**3.1.** As usual, fix a circular planar graph with boundary $G$, and let $M$ be its medial graph. Our main tool for proving an edge-by-edge analogue of Theorem 1.4 will be to successively uncross lenses in $M$, attempting to reduce to the case where $M$ is lensless so that we can apply Theorem 1.4. A lens uncrossing is done via an edge removal called a *pole removal*, and this section is devoted to the definition and basic properties of pole removals of lenses. For example, if $M$ contains an empty 2-pole lens $L$ (corresponding to a series or parallel edge in $G$), then a pole removal of $L$ will correspond to contracting one of the two edges in series, or deleting one of the two edges in parallel. In general, pole removals only have good properties for a certain class of lenses, namely, the minimal lenses.

**Definition 3.2.** Let $M$ be a medial graph. A *minimal lens* in $M$ is a lens which does not contain any strictly smaller lens.

Obviously any medial graph with a lens contains a minimal lens. The following proposition shows that the only interesting minimal lenses are the minimal 2-pole lenses:

**Proposition 3.3.** *Let $L$ be a minimal lens in a medial graph $M$ which has at most one pole. Then $L$ is empty, and so must be an empty 1-pole lens.*

*Proof.* Suppose to the contrary that $L$ is not empty, and first suppose that no geodesics intersect the boundary of $L$. Then (since $L$ is non-empty) there must exist a geodesic $\alpha$ strictly contained in $L$,

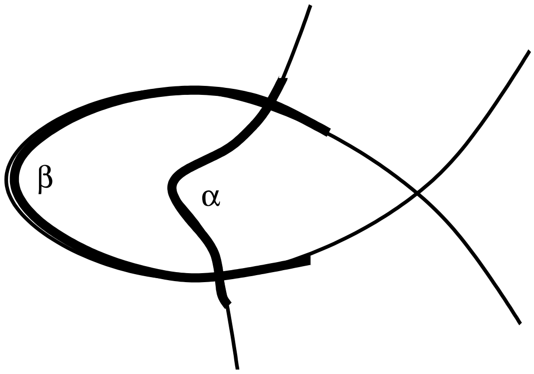


FIGURE 8. The lens produced by a geodesic passing through a 1-pole lens

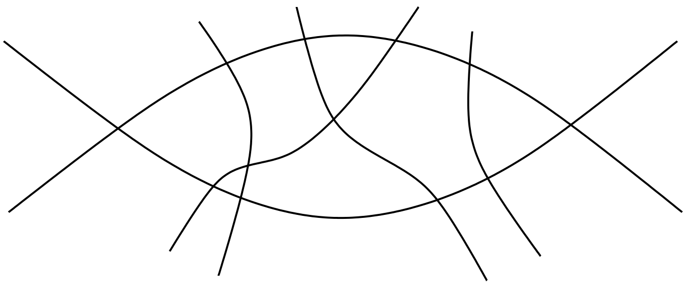

FIGURE 9. A minimal 2-pole lens

which must therefore have no endpoints. This implies that either $\alpha$ is a 0-pole lens or some sub-arc of $\alpha$ forms a 1-pole lens. In either case, this contradicts minimality of $L$.

The other possibility is that there is at least one geodesic intersecting the boundary of $L$. Let $\alpha = a_0...a_m$ be a geodesic fragment defined by crossings $a_0, ..., a_m$ such that $a_0$ and $a_m$ represent vertices of $M$ on the boundary of $L$, and all other $a_i$ are in the interior of $L$. Suppose that $a_i \neq a_j$ for $i \neq j$. Let $\beta = b_0...b_n$ be the geodesic fragment of $L$ with $b_0 = a_0$ and $b_n = a_m$. Then $\alpha$ and $\beta$ form a 2-pole lens strictly contained in $L$, contradicting the assumption that $L$ is minimal (See Figure 8). Otherwise, if $a_i = a_j$ for some $i \neq j$, then some sub-fragment of the arc fragment $\alpha$ forms a 1-pole lens strictly contained in $L$, again contradicting minimality of $L$. □

The most interesting minimal lenses are therefore the nonempty minimal 2-pole lenses. See Figure 9 for an example. The poles of such a lens can be thought of as defining two cut points along the circle defining the interior of the lens. Then the requirement for the lens to be minimal is that (1) every geodesic entering one side of the cut must exit on the other side, and (2) the medial graph in the interior of the graph must be lensless. We can summarize this by saying that the interior of the lens constitutes a permutation chord diagram from one side of the cut defined by the poles to the other side.

**Definition 3.4.** Let $G$ be a circular planar graph with medial graph $M$, and let $L$ be a minimal lens in $M$. We define the pole removals of $L$ by cases as follows.

(1) Suppose $L$ is a minimal 1-pole lens, corresponding to a geodesic fragment $\alpha = ...a_{-1}a_0a_0a_1...$ (i.e., a self-loop in the medial graph on vertex $a_0$). Then there is a unique pole removal of lens $L$, which replaces geodesic fragment $\alpha$ with $\alpha' = ...a_{-1}a_1...$ and leaves the rest of $M$ unchanged. See Figure 10.
(2) Suppose $L$ is a minimal 2-pole lens defined by geodesic fragments $\alpha = ...a_0a_1...a_m...$ and $\beta = ...b_0b_1...b_n...$ with $a_0 = b_0$, $a_m = b_n$, and $a_i \neq b_j$ for any other $i \neq j$. Then there are two

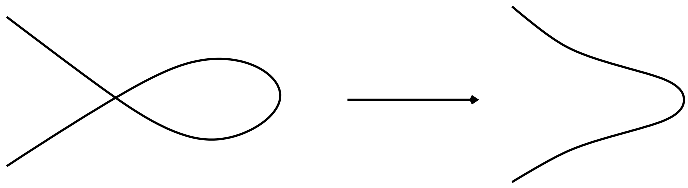

FIGURE 10. The unique pole removal of an empty 1-pole lens

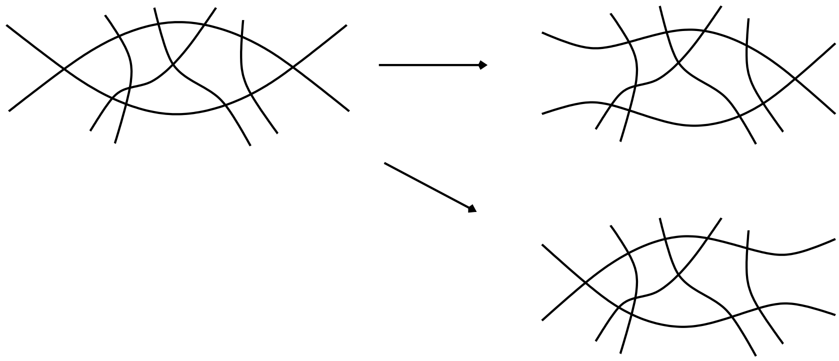

FIGURE 11. Pole removals of a minimal 2-pole lens

pole removals of $L$. The first pole removal replaces geodesic fragments $\alpha$ and $\beta$ by

$$\alpha' = b_{-1}a_1a_2...a_ma_{m+1}... \quad \text{and}$$
$$\beta' = a_{-1}b_1b_2...b_nb_{m+1}...$$

(See Figure 11). The second pole removal replaces geodesic fragments $\alpha$ and $\beta$ by

$$\alpha' = a_{-1}a_0...a_{m-1}b_{m+1}... \quad \text{and}$$
$$\beta' = b_{-1}b_0...b_{m-1}a_{m+1}...$$

As defined above, the pole removals are operations on the medial graph $M$. Each pole removal, however, corresponds to an edge removal in the graph $G$, and this edge removal is also called a pole removal of the lens $L$.

**Remark 3.5.** If $L$ is an empty 2-pole lens, then in $G$, $L$ corresponds to either a series or parallel edge. The pole removals of $L$ correspond to either contracting one half of the series edge, or deleting one half of the parallel edge. If $L$ is an empty 1-pole lens, then in $G$, $L$ corresponds to either a self-loop or a hanging interior vertex. In both of these cases, up to an isolated interior vertex, deleting and contracting the edge have the same effect, and the pole removal of $L$ corresponds to this deletion/contraction.

Our overall strategy for studying circular planar graphs will be to look for minimal lenses in the medial graph, and then successively remove one minimal lens at a time by applying a pole removal. Note that uncrossing one lens in a graph might create new lenses; nevertheless, the process of uncrossing minimal lenses must terminate in a lensless graph since there are only finitely many crossings in total, and every pole removal reduces the number of crossings in $M$ by one. The rest of this section is devoted to the key technical properties of minimal lens uncrossings which we will need in the rest of the paper.

**Lemma 3.6.** *Let $M$ be a medial graph, $L$ a minimal 2-pole lens in $M$, and $p$ one of the two poles of $L$. Then $L$ can be emptied by a sequence of arc switches never involving the pole $p$.*

*Proof.* A careful reading shows that this follows from the proof of [CuMo00, Lemma 8.2]. For the reader's convenience we include the proof here.

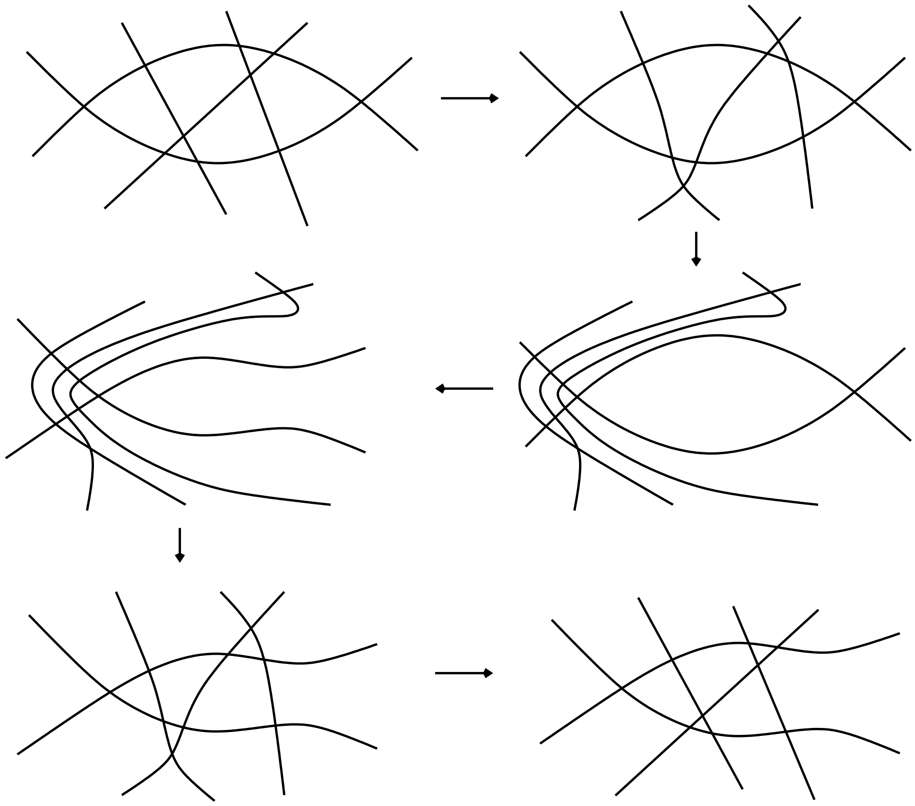

FIGURE 12. Expressing a single pole removal as a composition of $Y$-$\Delta$ transforms (arc switches) and one series/parallel edge removal

We start by showing that all crossings in the interior of $L$ can be removed by a sequence of arc switches using neither of the poles $p, q$. Suppose therefore that there is at least one crossing in the interior of $L$. Let $N$ and $S$ be the two halves of the lens $L$ defined by its poles. Since $L$ is minimal, every arc fragment which intersects $L$ must cross both $N$ and $S$. Let $\mathscr{F} = \{\alpha_i\}$ be the geodesic fragments passing through $L$; to be precise, each $\alpha_i$ is a geodesic fragment $a_0 a_1 ... a_m$ where $a_0$ is a vertex on $N$, $a_m$ is a vertex on $S$, and each $a_i$, $0 < i < m$ is a vertex in the interior of $L$. For each $i$, let $v_i$ be the point of intersection of $\alpha_i$ with $S$. Let $w_i$ be the first intersection point of $\alpha_i$ with another member of $\mathscr{F}$ after $v_i$ inside $L$ (if it exists). Let $W = \{w_i\}$ be the set of points obtained in this way. Since $L$ was assumed to have at least one interior crossing, $W$ is nonempty. For a given $w \in W$, suppose that $w$ is formed by the crossing of geodesic fragments $\alpha_i$ and $\alpha_j$. Then the geodesic fragments $\alpha_i$, $\alpha_j$, and $S$ form a triangular region $R$ in the interior of $L$. By the construction of the points $w_i \in W$, there must exist at least one $w' \in W$ such that the interior of the corresponding region $R'$ is empty (does not contain any geodesic fragments), and therefore $R'$ forms an empty triangle. Then an arc switch will move the crossing $w'$ to the exterior of $L$, and hence reduce the number of crossings in the interior of $L$ by one. Repeating this process, we will eventually remove all crossings from $L$ by a sequence of arc switches.

We have therefore reduced to assuming that the lens $L$ has no crossings in its interior. Then the arc fragment closest to $q$ forms an empty triangle with parts of $N$ and $S$, and by an arc switch we can remove this arc fragment from the interior of $L$. Repeating this process, we eventually remove all geodesic fragments from the interior of $L$ by arc swtiches via the pole $q$. □

This immediately implies the following:

**Proposition 3.7.** *Let $r$ be an edge removal in a graph $G$ which represents a pole removal of a minimal 2-pole lens in $M$. Then $r$ can be expressed as a sequence of $Y$-$\Delta$ transformations not involving the* *edge removed by $r$, then a singe series/parallel edge removal, and finally another sequence of $Y$-$\Delta$ transormations.*

*Proof.* Let $L$ be the minimal 2-pole lens with pole removal $r$. By Lemma 3.6, we can empty $L$ of all geodesics by a sequence of arc switches not involving $r$. This gives the first sequence of $Y$-$\Delta$ transformations. Then the edge removal $r$ corresponds to either a series or parallel edge removal in $G$. After applying $r$, we can apply the initial sequence of $Y$-$\Delta$ transformations in reverse order to arrive at the graph resulting from just applying edge removal $r$. See Figure 12 for an example of this process. □

**Corollary 3.8.** *An edge representing a pole of a minimal lens is not recoverable, and any pole removal of a minimal lens is not essential. A pole removal on $G$ does not affect recoverability or essentiality of any edge not intersecting the lens.*

*Proof.* The statement is obvious if the minimal lens is empty. Otherwise suppose $L$ is a 2-pole minimal lens, fix an edge $p$ representing one of the poles of the lens, and let $r$ be the pole removal of $p$. Then the proposition above shows that one can wye-delta transform $p$ to be part of a series or parallel edge, without using the edge $p$ in any of these wye-delta transformations. This corresponds to emptying the lens via arc switches only involving the other pole of the lens. Since wye-delta transformations not involving an edge $e$ do not change recoverability or essentiality of $e$, we see that $p$ is non-recoverable and $r$ is inessential. If $e$ is an edge not intersecting the lens $L$, then uncrossing $L$ at $p$ can be written as a composition of wye-delta transformations and series/parallel edge removals not involving the edge $e$. These don't change recoverability/essentiality of $e$, so the second statement is proved. □

The following two lemmas are our key lemmas for dealing with crossings intersecting a minimal 2-pole lens $L$.

**Lemma 3.9.** *Let $e$ be an edge representing a crossing on the boundary of a minimal 2-pole lens $L$, but which is not a pole of $L$. Let $\alpha$ be the geodesic fragment through $L$ containing $e$. Then $L$ can be emptied of all geodesics except $\alpha$, by a sequence of arc switches not involving $e$.*

*Proof.* Let $N$ and $S$ be the geodesic fragments defining the lens $L$, and suppose without loss of generality that $e$ is a crossing on $N$ (see Figure 13). Let $\{\beta_i\}_{i=1}^n$ be the set of geodesic fragments besides $\alpha$ passing through the interior of $L$. Assume that at least one $\beta_i$ intersects $\alpha$ in the interior of $L$ (if not, proceed to the next paragraph). Then let $\beta_j$ be the geodesic fragment which intersects $\alpha$ closest to the fragment $S$. Note that $\beta_j$ must intersect $S$ at some point. Then subfragments of the geodesics $\beta_j$, $\alpha$, and $S$ then define a region $R$ contained in $L$ as shown in Figure 13. The boundary of $R$ can be naturally divided into

$$\partial R = \beta_j' \cup \alpha' \cup S',$$

where $\beta_j'$, $\alpha'$, and $S'$ are arc fragments contained in $\beta_j$, $\alpha$, and $S$ respectively. By our construction, no geodesics intersect $\alpha'$. Therefore $R$ can be emptied of all geodesics by a sequence of arc switches: to see this, consider the region $\overline{R}$ obtained by contracting $\alpha'$; $\overline{R}$ is a minimal 2-pole lens since no geodesic intersects $\alpha'$. By Lemma 3.6, the lens $\overline{R}$ can be emptied by a sequence of arc switches not using the pole of $\overline{R}$ defined by the contracted edge $\alpha'$. The corresponding arc switches in $R$ will empty the region $R$, turning $R$ into an empty triangle. We can then apply an arc switch to remove region $R$ from the lens $L$. This reduces the number of arcs intersecting $\alpha$ by 1; we can repeat this process until no arcs intersect $\alpha$.

So we've reduced to assuming that no arcs in $L$ intersect $\alpha$ in the interior of $L$. Then let $W$ and $E$ be the two halves of $L$ defined by the arc $\alpha$. As in the previous paragraph, let $\overline{W}$ and $\overline{E}$ be the regions obtained by contracting $\alpha$. Because no geodesics intersect $\alpha$, $\overline{W}$ and $\overline{E}$ are minimal 2-pole lenses. By Lemma 3.6, $\overline{W}$ and $\overline{E}$ can be cleared by a sequence of arc switches not using the crossing defined by the contraction of $\alpha$. The corresponding arc switches in $L$ will empty $W$ and $E$ of all arcs, which completes the proof of the lemma. □

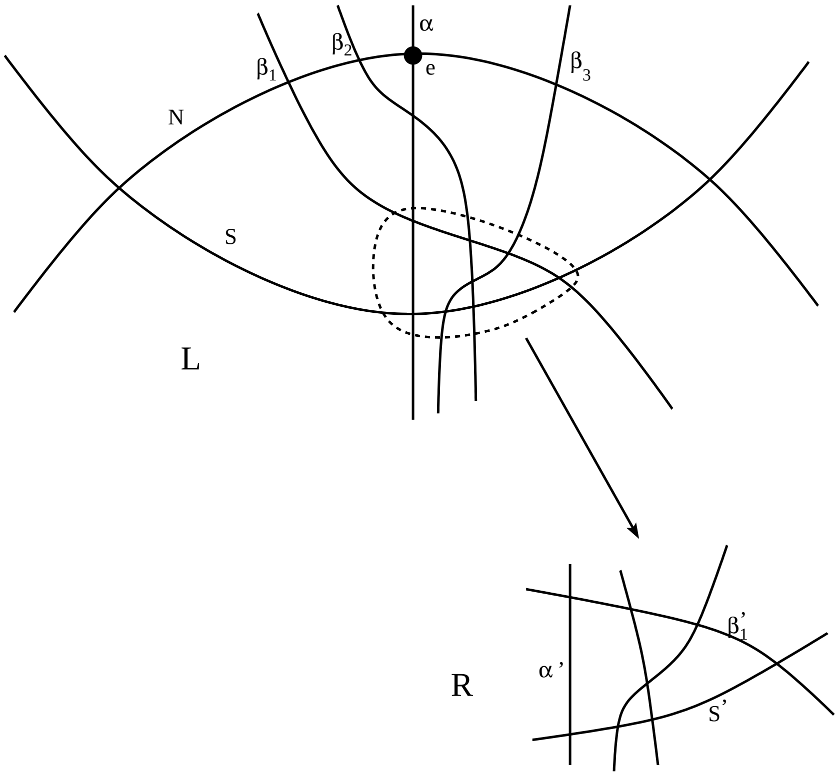


FIGURE 13. Diagram for proof of Lemma 3.9

**Lemma 3.10.** *Let $e$ be an edge representing a crossing in the interior of a minimal 2-pole lens $L$. Then $L$ can be emptied of all geodesics in the interior of $L$ except those defining $e$, by a sequence of arc switches not involving the edge $e$.*

*Proof.* Let $N$ and $S$ be the arc fragments defining the lens $L$. Let $\alpha$ and $\beta$ be the arc fragments in $L$ containing edge $e$. Then $e$ separates $\alpha$ and $\beta$ into two halves; we write $\alpha = \alpha_1 \cup \alpha_2$ and $\beta = \beta_1 \cup \beta_2$ in such a way that $\alpha_1$ and $\beta_1$ intersect $N$ (See the Figure 14). Finally, note that the geodesics $\alpha$ and $\beta$ subdivide the interior of $L$ into 4 regions $A$, $B$, $C$, $D$, as in Figure 14. Let $\{\gamma_i\}_{i=1}^n$ be the other arc fragments in $L$. If no $\gamma_i$ intersects $\alpha_1$ or $\beta_1$, proceed to the next paragraph. Otherwise, assume without loss of generality that some $\gamma_i$ intersects $\alpha_1$, and let $\gamma_j$ be the arc fragment intersecting $\alpha_1$ closest to $N$. $\gamma_j$ must intersect the arc fragment $N$ in one of regions $A$, $B$ or $C$. If $\gamma_j$ intersects $N$ in region $A$ or region $B$, then parts of geodesics $\gamma_j$, $N$ and $\alpha_1$ form a triangle $R$ with no geodesic crossing the part of $\alpha_1$ defining the boundary of $R$. Using Lemma 3.6 in the same manner as in the proof of Lemma 3.9, we can empty $R$ of all geodesics by a sequence of arc switches, turning $R$ into an empty triangle, and then remove $R$ itself from $L$ by another arc switch. The other possibility is that $\gamma_j$ intersects $N$ in region $C$, which implies that $\gamma_j$ intersects arc fragment $\beta_1$ as well. If $\gamma_j$ is the arc fragment intersecting $\beta_1$ closest to $N$, then parts of $\gamma_j$, $\beta_1$, and $N$ form a triangular region which can be emptied and then removed by arc switches, using Lemma 3.6. If not, then there is some geodesic $\gamma_k \neq \gamma_j$ which intersects $\beta_1$ closest to $N$. Since $\gamma_j$ is the geodesic intersecting $\gamma_1$ closest to $N$, $\gamma_k$ must intersect $N$ in region $B$ or region $C$, and then parts of $\gamma_k$, $N$, and $\beta_1$ form a triangular region which can be emptied and then removed by arc switches, using Lemma 3.6. This last case is illustrated in the lower half of Figure 14 (here $j = 3$) In any case, we have reduced the total number of crossings along $\alpha_1$ and $\beta_1$ by one.

By repeating the process in the above paragraph, a sequence of arc switches reduces us to a lens $L$ with no geodesics crossing $\alpha_1$ or $\beta_1$. Repeating the argument with respect to arc fragment $S$ and fragments $\alpha_2$, $\beta_2$ reduces us to a lens $L$ with no geodesics crossing $\alpha_1$, $\beta_1$, $\alpha_2$, or $\beta_2$. At this point the regions $A$ and $D$ of $L$ must be empty. Let $\overline{B}$ and $\overline{C}$ be the regions obtained by contracting $\alpha_1$

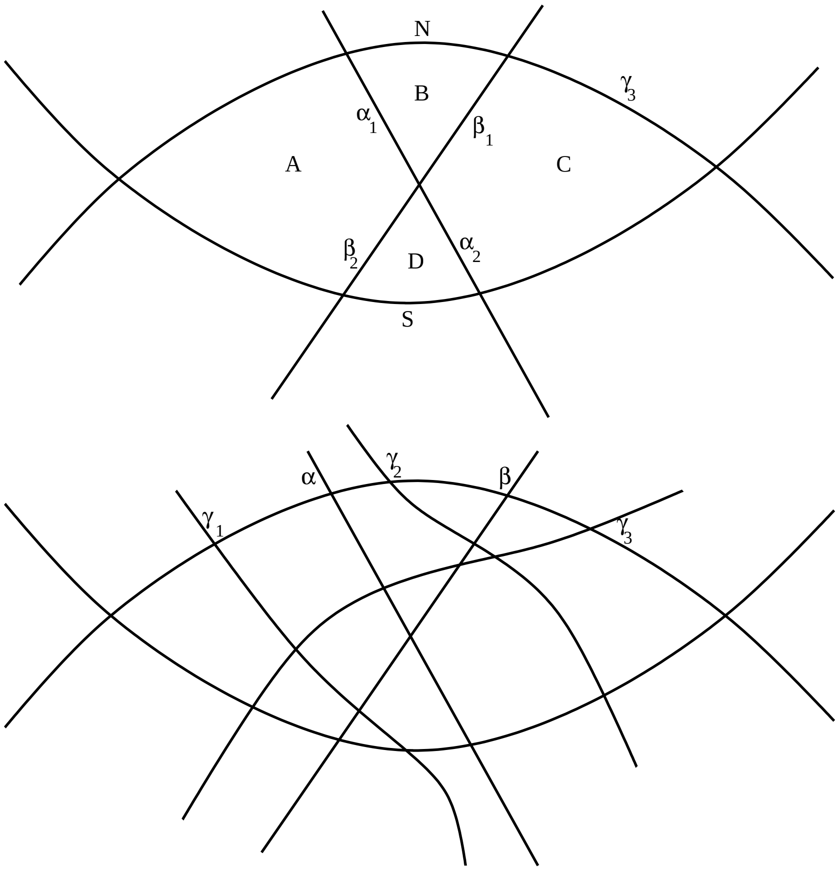


FIGURE 14. Diagram for proof of Lemma 3.10

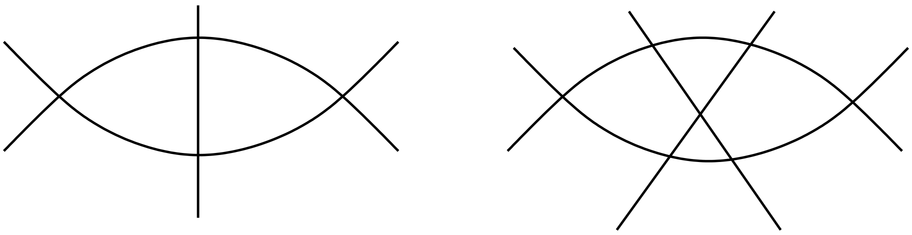

FIGURE 15. We will often be able to reduce our analysis to these two medial graphs

and $\beta_2$ to a point, and $\beta_1$ and $\alpha_2$ to a point. Then $\overline{B}$ and $\overline{C}$ must be minimal 2-pole lenses since no geodesics cross $\alpha_1, \alpha_2, \beta_1, \beta_2$. An application of Lemma 3.6 allows us to empty regions $\overline{B}$ and $\overline{C}$, and the corresponding arc switches in $L$ will empty $B$ and $C$ using the poles $p$ and $q$ respectively. $\square$

The previous two lemmas will reduce our analysis in many cases to the medial graphs shown in Figure 15. In the next section we study the graphs inducing these medial graphs in some detail.

## 4. Examples

Recall the main theorem of [CuMo00]:

**Theorem 4.1.** *Let $G$ be a circular planar graph with boundary. Then the following are equivalent:*

- *$G$ is recoverable.*

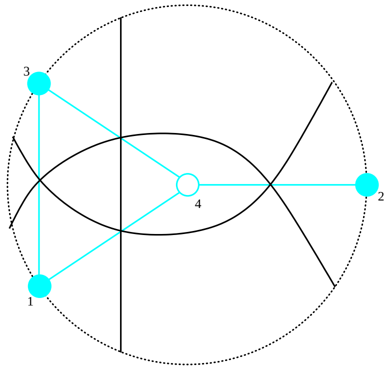


FIGURE 16. $G$ (light color) and $M$ (dark color)

- *$G$ is critical.*
- *The medial graph of $G$ is lensless.*

We will call this the main global theorem, as it relates properties of the entire graph $G$. A local theorem would be one relating the recoverability of a single edge $e$ to geometric properties of the graph. This situation turns out to be more complicated than the global situation. For example, it is not true that an edge $e$ in a circular planar graph is recoverable if and only if it is essential, and the relation of both of these properties to the location of lenses in the medial graph is a little subtle.

It is good to consider a few examples to see what is going on. Moreover, the calculations we make in these examples will be used in the proof of our main theorems, as we will reduce the proof to these special cases.

**Example 4.2.** Consider the graph $G$ in Figure 16 (in light color), with its corresponding medial graph $M$ in dark color. Notice that $G$ is self-dual, so $G$ is the only circular planar graph with boundary having $M$ as medial graph.

Let us check which edge removals are essential. It is easy to check that a $Y$ and a $\Delta$ are well-connected graphs, i.e., all possible connections between circular pairs exist. Therefore, since deleting edge 13 and contracting edge 24 leave a $Y$ and $\Delta$, respectively, they cannot break any connections. Therefore these two edge removals are not essential. All other edge removals are essential:

- 13 is critical for contraction since its endpoints are distinct.
- Deleting 14 breaks the $(1;2)$-connection; contracting it breaks the (2;3)-connection.
- Deleting 24 breaks the $(1;2)$- and $(2;3)$-connections.
- Deleting 34 breaks the $(2;3)$-connection; contracting it breaks the $(1;2)$-connection.

Notice that the two inessential edge removals correspond precisely to the 2 pole removals of the unique (minimal) lens in $M$.

Next we compute the recoverable edges of $G$. By the main global theorem, there is at least one non-recoverable edge. In fact, every edge in this graph is non-recoverable. To see this, first apply a $Y$-$\Delta$ transformation to the $Y$ to obtain the following graph shown in Figure 17.

In this figure we have parametrized the fiber over a response matrix $\Lambda$ with independent entries $a$, $b$, $c$. Here $t \in (0, a)$ is an arbitrary parameter. Now we apply the inverse $Y$-$\Delta$ transformation to obtain the parametrization of the fiber over $\Lambda$ shown in Figure 18. No edge has a constant conductivity as $t$ varies, so none of the edges are recoverable. We will eventually see that this is because every edge in $G$ intersects the lens of $M$ (i.e., every crossing in $M$ intersects the lens).

**Example 4.3.** Consider the medial graph in Figure 19, and the two graphs $G$, $G^d$ which have it as medial graph. Here it is more complicated to check which edge removals are essential and which edges are recoverable. We leave the computations to the reader; the results are as follows:

- The only inessential edge removals in $G$ are the contractions of edges 15 and 26.

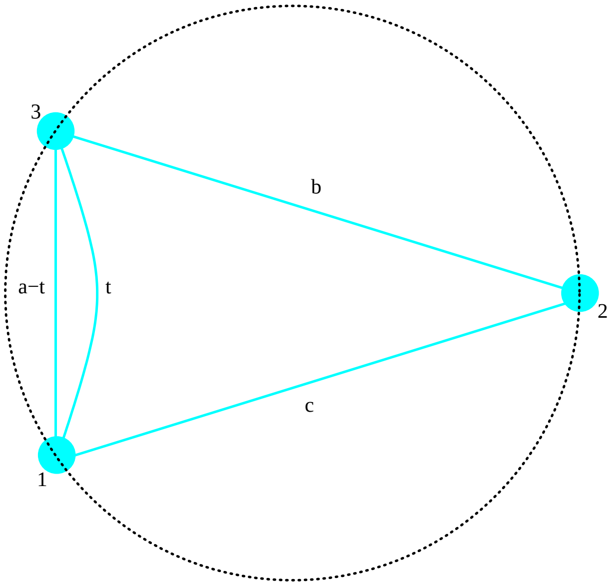


FIGURE 17. Parametrization of the fiber over a response matrix $\Lambda$ with independent entries $a, b, c$

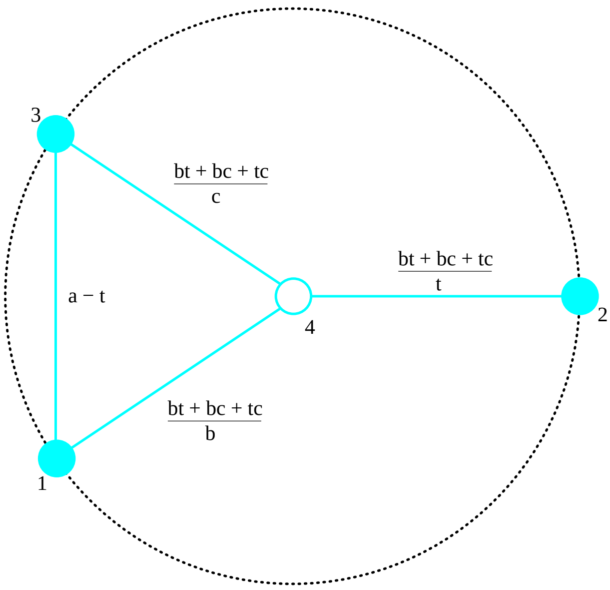


FIGURE 18. Parametrization of fiber over $\Lambda$ for original graph $G$

- The only inessential edge removals in $G^d$ are the deletions of edges 12 and 34.
- In both $G$ and $G^d$, all edges are non-recoverable. The easiest way to see this is by applying a sequence of three $Y$-$\Delta$ transformations transforming $G$ (resp. $G^d$) into a graph with a parallel edge (resp. with a series edge), parametrizing the conductivities of the resulting edges, and then applying the inverse $Y$-$\Delta$ transforms. This is the same process as in Example 1, except that one requires three $Y$-$\Delta$ transforms instead of one. Figure 20 shows the sequence of $Y$-$\Delta$ transformations used for $G$ to obtain a graph with a parallel edge. If we parametrize the the edges in the resulting graph (with parameter $t$) and then invert these $Y$-$\Delta$ transformations we get the one-parameter family in Figure 21 of conductivity functions on $G$ yielding the

same response matrix. Since the conductivity of each edge varies as $t$ varies, all edges are non-recoverable. A similar calculation on $G^d$ shows that every edge on $G^d$ is non-recoverable.

Notice that in both cases, there are exactly two inessential edge removals in $M$, corresponding to the pole removals of $L$. On the other hand, every edge in each graph is non-recoverable. One might guess from this that inessential edge removals in $M$ should correspond to pole removals of minimal lenses, while non-recoverable edges should correspond to edges which intersect minimal lenses. This is not quite right, as the following example illustrates:

**Example 4.4.** Consider the graph $G$ (light color) and medial graph $M$ (dark color) shown in the upper left half of Figure 22. Consider the edge labeled $e$. Notice that this edge does not intersect any minimal lens in $M$. Nevertheless, we leave it to the reader to check that this edge is not recoverable, and that contracting $e$ is inessential.

To understand what is happening, consider the unique minimal lens $L$. After uncrossing $L$ at pole $p$, we get the medial graph shown in the lower half of Figure 22. Notice that after uncrossing $L$, $e$ becomes

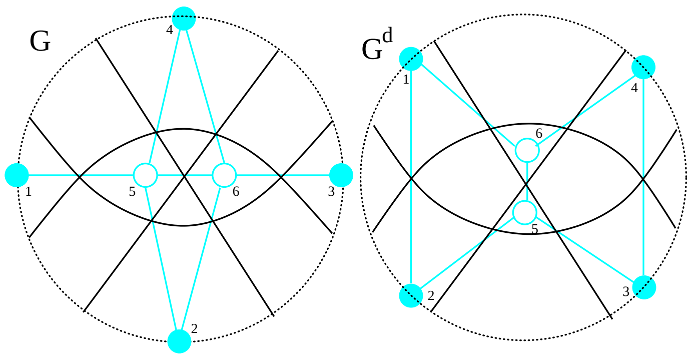


FIGURE 19. A medial graph $M$ (dark color) and the graphs $G$, $G^d$ (light color) which have it as medial graph

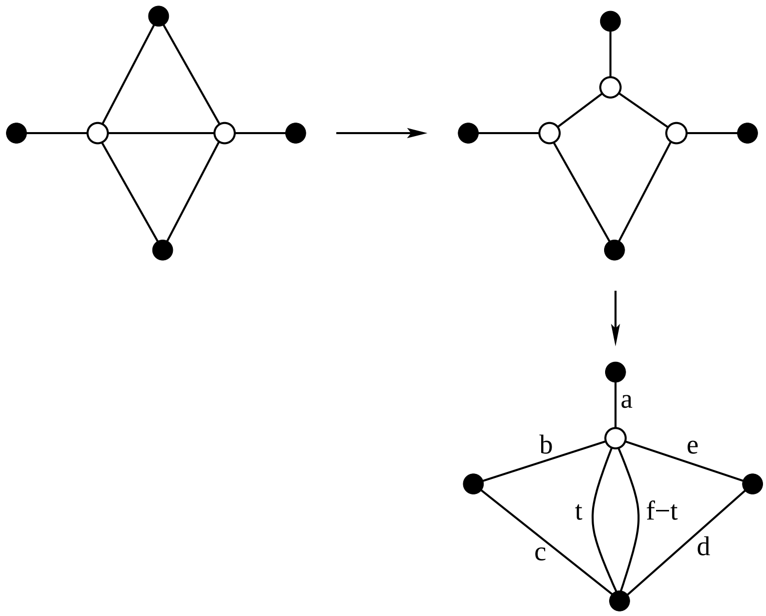


FIGURE 20. The sequence of $Y$-$\Delta$ transformations used to transform $G$ to a graph with a parallel edge, and a parametrization of the conductivities on the resulting graph yielding the same response matrix

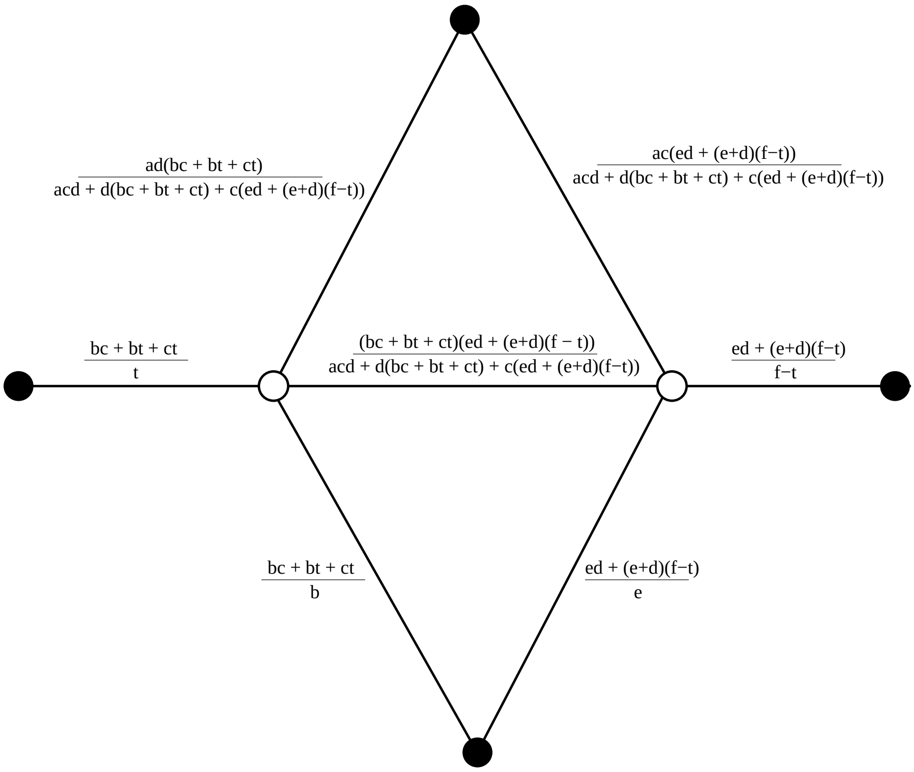


FIGURE 21. A one-parameter family of conductivity functions on $G$ (parametrized by $t$) with the same response matrix

a pole of a new minimal lens $L'$ which did not exist in the original medial graph $M$. In general, our theorems will have to keep track of the new lenses which can appear when uncrossing a minimal lens in $M$. After doing so, we will see that non-recoverable edges correspond to edges intersecting minimal lenses, and inessential edge removals correspond to pole removals of minimal lenses.

## 5. Recoverability of an edge $e$

**Definition 5.1.** Let $e$ be an edge in circular planar graph $G$, with medial graph $M$. We say that $e$ *potentially intersects a minimal lens* if these exists a sequence of graphs $G = G_0 \mapsto G_1 \mapsto G_2 \mapsto ... \mapsto G_n$ with each $G_{i+1}$ obtained from $G_i$ by a single pole removal of a minimal lens ($e$ cannot be the pole that is removed), such that $e$ intersects a minimal lens in $G_n$.

We can now state and prove our main theorem characterizing recoverability of an edge in terms of the medial graph.

**Theorem 5.2.** *An edge $e$ in a circular planar graph $G$ is non-recoverable if and only if $e$ potentially intersects a minimal lens.*

*Proof.* First suppose $e$ does not potentially intersect a minimal lens. Then fix a sequence $G = G_0 \mapsto G_1 \mapsto ... \mapsto G_n$, where $G_n$ has a lensless medial graph and each $G_i \mapsto G_{i+1}$ is obtained by a single pole removal of a minimal lens $L_i$. By assumption $e$ does not intersect $L_i$, so Corollary 3.8 tells us that the recoverability properties of $e$ are the same in $G_i$ as in $G_{i+1}$. By Theorem 1.4, $e$ is recoverable in $G_n$ since the medial graph of $G_n$ is lensless. Therefore $e$ must have been recoverable to start with.

Now suppose that $e$ does potentially intersect a minimal lens. Take a sequence $G \mapsto G_1 \mapsto ... \mapsto G_n$ of minimal lens uncrossings, such that in graph $G_n$, $e$ intersects a minimal lens, and it did not intersect a minimal lens at any earlier stage in the sequence. Then $e$ is recoverable in $G_n$ if and only if it is

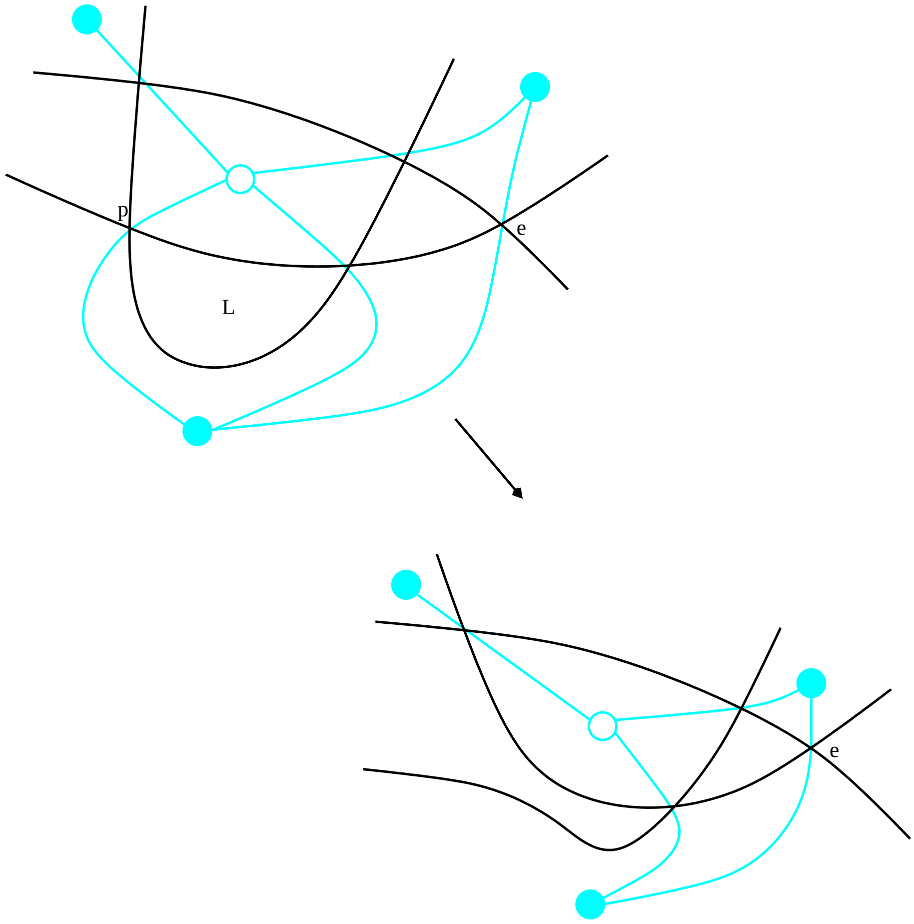


FIGURE 22. The graph and medial graph for Example 4.4. Edge $e$ becomes a pole of a minimal lens after uncrossing at $p$

recoverable in $G$. So we reduce to showing that if $e$ intersects a minimal lens, then $e$ is not recoverable. We have already shown that the poles of minimal lenses are not recoverable, so suppose $e$ is not a pole of the minimal lens. By Lemmas 3.9 and 3.10, we can empty the rest of the lens by $Y$-$\Delta$ transformations not involving $e$. These do not change recoverability of $e$, and hence we are reduced to considering the case where the lens $L$ containing $e$ looks like one of the medial graphs of Examples 4.2 and 4.3. Let $C$ be a simple closed curve in $D$ drawn sufficiently close to $L$ so as to not intersect any geodesics which do not cross $L$, as seen in Figure 23. We can then consider the graph with boundary $H$ consisting of the black cells (defining vertices of $G$) contained inside $C$, with a vertex of $H$ declared boundary if and only if it intersects the curve $C$. Examples 4.2 and 4.3 show that when $e$ is thought of as an edge of $H$, $e$ is not recoverable. The lemma below shows that since $e$ is not recoverable in $H$, it is not recoverable in $G_n$ either, so that lemma will complete the proof. □

**Lemma 5.3.** *Let $G$ be a circular planar graph with boundary, and $M$ its medial graph. Suppose that $H$ is a graph with boundary obtained as follows: draw any simple closed curve $C$ strictly contained inside the boundary circle of the medial graph $M$, and let $M_H$ be the medial graph consisting of the part of $M$ contained inside $C$. Let $H$ be the (unique) subgraph with boundary of $G$ corresponding to the cells contained in $M_H$, with a vertex of $H$ declared as boundary if and only if the corresponding cell in $M$ intersects the simple closed curve $C$. Now let $e \in H$ be any edge of $H$. If $e$ is recoverable as an edge of $G$, then it is recoverable as an edge of $H$.*

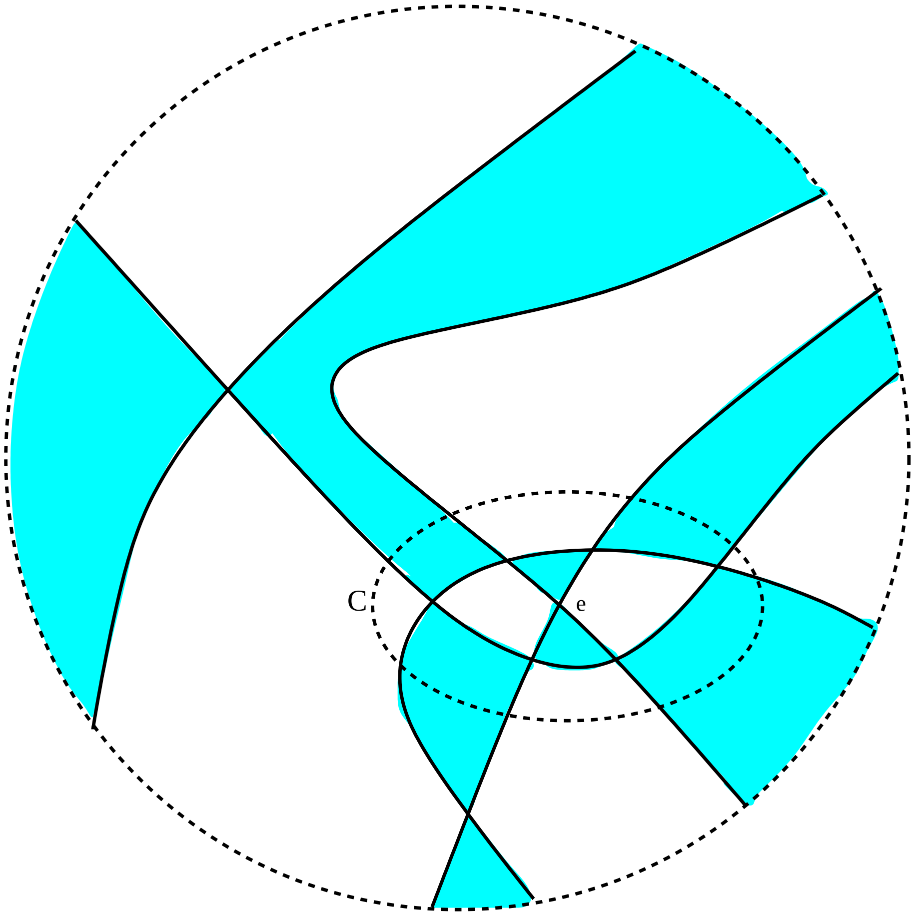


FIGURE 23. The subgraph $H$ defined by the closed loop $C$, with medial graph $M_H$

*Proof.* The key point is that $H$ can be obtained from $G$ by repeated application of the following three operations:

(1) Promoting an interior vertex of $G$ to a boundary vertex.
(2) Deleting a boundary-boundary edge.
(3) Deleting an isolated boundary vertex.

To see this, first use operation (1) promote all of the interior vertices not intersecting the medial graph $M_H$ to boundary vertices. Then all edges outside of $H$ are boundary-boundary edges, since any vertex which is interior in $H$ has the property that $H$ contains all of the edges leading out of it in $G$. Thus one can apply operation (2) to delete all of these, and then the vertices outside of $H$ are isolated boundary vertices, which we can delete by operation (3). So it suffices to prove the proposition when $H$ is a subgraph obtained by applying one of the three operations above. We handle these case-by-case:

(1) Suppose that $H$ is obtained from $G$ by promoting the vertex $v \in \text{Int } G$ to boundary. Let $L_G : \mathbb{R}_+^n \to M_{k\times k}$ and $L_H : \mathbb{R}_+^n \to M_{k+1\times k+1}$ be the Dirichlet-to-Neumann maps on $G$ and $H$, respectively. Then by standard properties of Schur complements, we have a commtutive diagram

$$\begin{array}{ccc} \mathbb{R}_+^n & \xrightarrow{L_H} & M_{k+1\times k+1} \\ \big\downarrow = & & \big\downarrow f \\ \mathbb{R}_+^n & \xrightarrow{L_G} & M_{k\times k} \end{array}$$

where the right-hand vertical map $f$ is obtained by applying a Schur complement along the row and column corresponding to vertex $v$. Then for any $\Lambda \in M_{k+1\times k+1}$, we have

$$L_H^{-1}(\Lambda) \subseteq L_G^{-1}(f(\Lambda)),$$

and in particular, if $L_G^{-1}(f(\Lambda))$ takes on a unique value on the edge $e$, then so does $L_H^{-1}(\Lambda)$.

(2) Suppose that $H$ is obtained by deleting boundary-boundary edge $pq$, for $p, q \in \partial G$. The lemma follows in this case from the fact that the response matrix $\Lambda$ depends linearly on the conductivity $\gamma_{pq}$ (because of the expression of $\Lambda$ as a Schur complement [CuMo00, 3.9]). To be more explicit, suppose that we have a pair of conductivity functions $\gamma_H, \gamma'_H$ on $H$ with equal response matrices, i.e., $L_H(\gamma_H) = L_H(\gamma'_H)$, such that $\gamma_H$ and $\gamma'_H$ take on different values on edge $e$. These can be thought of as defining 'conductivity functions' on $G$ with $\gamma_{pq} = 0$ in each case. We can define true conductivity functions $\gamma_G$, $\gamma'_G$ on $G$ by setting $\gamma_G(a) = \gamma_H(a)$ for any edge $e \in H$, and setting $\gamma_{pq} = 1$. Define $\gamma'_G$ similarly relative to $\gamma'_H$. Then, since the response matrix depends linearly on $\gamma_{pq}$, we have $L_G(\gamma_G) = L_G(\gamma'_G)$ since we have the same relation on $H$. Since $\gamma_G(e) \neq \gamma'_G(e)$, the edge $e$ is not recoverable as an edge of $G$.

(3) The recoverability of $e \in G$ is obviously not changed upon deleting an isolated boundary vertex.

□

The following corollary to Theorem 5.2 is somewhat surprising to the authors; it indicates that there is a severe restriction on where new lenses in a medial graph can appear upon uncrossing a minimal lens.

**Corollary 5.4.** *For any sequence of minimal lens uncrossings $G = G_0 \mapsto G_1 \mapsto \ldots \mapsto G_n$ with $G_n$ critical, let $L_i$ be the minimal lens uncrossed to obtain $G_{i+1}$ from $G_i$. Then let $S$ be the set of edges in the original graph which intersect any of the minimal lenses $L_0, \ldots, L_{n-1}$. This set $S$ is exactly the set of non-recoverable edges in the graph; in particular, $S$ is independent of the way in which minimal lenses are uncrossed.*

*Proof.* This follows easily from the proposition above. Fix a sequence $G \mapsto G_1 \mapsto \ldots \mapsto G_n$ of minimal lens uncrossings of $G$ resulting in a critical graph $G_n$, and take an edge $e \in G$. If $e$ is recoverable, then it does not potentially intersect a minimal lens and in particular does not intersect any of the uncrossed minimal lenses. Conversely, if $e$ does not potentially intersect any minimal lens that was uncrossed, we have a sequence of wye-delta transforms and series/parallel edge removals not involving $e$, resulting in $e$ being an edge in the critical graph $G_n$ and hence recoverable. Therefore $e$ was recoverable to begin with. □

**Remark 5.5.** Corollary 5.4 allows one to use the following procedure to determine all non-recoverable edges in a circular planar graph $G$. First, draw the medial graph $M(G)$. If $M(G)$ has no lenses, then all edges of $G$ are recoverable. Otherwise, let $L_1$ be a minimal lens of $M(G)$. Then all edges of $G$ intersecting $L_1$ are non-recoverable. Uncross $L_1$ at either pole to obtain a new graph $G_1$. If $M(G_1)$ has no lenses, then there are no further non-recoverable edges of $G$. Otherwise, let $L_2$ be a minimal lens in $M(G_1)$. Any edge of $G$ intersecting $L_2$ is then non-recoverable, so we add these to our list of non-recoverable edges of $G$. Delete $L_2$ to obtain a new graph $G_2$, with medial graph $M(G_2)$. Repeat this process until one arrives at a medial graph with no lenses. The resulting list of non-recoverable edges of $G$ is in fact the full list of non-recoverable edges, by Corollary 5.4.

**Corollary 5.6.** *For any edge $e$ in a circular planar graph $G$, if $e$ is recoverable, then $e$ is essential (for both deletion and contraction).*

*Proof.* Fix a sequence $G \mapsto G_1 \mapsto \ldots \mapsto G_n$ of minimal lens uncrossings of $G$. If $e$ is recoverable, then $e$ did not intersect any of the minimal lenses that were uncrossed in this sequence. Therefore by a sequence of wye-delta transformations and series/parallel edge removals not involving $e$, we were able to reduce $G$ to a critical graph. On this graph $e$ is essential. Since the wye-delta transformations and series/parallel edge removals did not affect essentiality of $e$, $e$ was essential to begin with. □

**Corollary 5.7.** *A boundary spike or boundary edge in a circular planar graph is recoverable if and only if it is essential.*

*Proof.* Let $e$ be a boundary spike or boundary-boundary edge. We just showed that if $e$ is recoverable, then it is essential. Conversely, if $e$ is essential we can use the boundary spike or boundary edge formulas of [CuMo00, Ch. 3] to recover $e$. □

**Remark 5.8.** Proving this corollary was the original motivation of the authors for looking at local recoverability.

**Remark 5.9.** The statements of Corollaries 5.6 and 5.7 make sense for non-planar graphs; there are several reasonable ways to generalize the notion of a connection to non-planar graphs. The statements of these corollaries are both false for any of these generalizations, seemingly for several independent reasons. Understanding the extent to which connections relate to recoverability in the non-circular planar case is an ongoing subject of research.

## 6. Essentiality of $e$ and lenses in the medial graph

Theorem 5.2 is gives a nice criterion based on the medial graph for when a given edge $e \in G$ is recoverable. There is a similar criterion for when an edge removal in $G$ is essential. To state it we make the following definition:

**Definition 6.1.** Let $G$ be a circular planar graph, and let $r$ be an edge removal on $G$. We say that $r$ is *potentially a pole removal of a minimal lens* if there exists a sequence $G \mapsto G_1 \mapsto ... \mapsto G_n$ of minimal lens uncrossings such that $r$ is a pole removal of a minimal lens in $G_n$.

We then have the following:

**Theorem 6.2.** *Let $r$ be an edge removal in the circular planar graph $G$. Then $r$ is not essential if and only if it is potentially a pole removal of a minimal lens.*

*Proof.* Our proof of this is more difficult than the corresponding statement (5.2) for local recoverability; the rest of this section is entirely devoted to its proof.

First note the following easy lemma:

**Lemma 6.3.** *Let $G$ be a circular planar graph with boundary, let $r$ be an inessential edge removal, and let $s$ be an edge removal of some other edge in $G$. If $s$ is essential in $G$, then $s$ is also essential in $Gr$ (recall that $Gr$ denotes the graph obtained by appliying edge removal $r$ to $G$).*

*Proof.* Suppose $s$ breaks the $(P;Q)$-connection set on $G$, meaning that at least one $(P;Q)$-connection exists on $G$ but no $(P;Q)$-connection exists on $Gs$. Since $r$ is inessential, at least one $(P;Q)$-connection exists on $Gr$. But no $(P;Q)$-connection exists on $(Gr)s$ since none exists on $Gs$, and $(Gr)s = (Gs)r$ is obtained by removing one edge from $Gs$. □

We next prove the 'easy' direction in the equivalence of Theorem 6.2. Suppose that $r$ is potentially a pole removal of a minimal lens. Fix a sequence of minimal lens uncrossings $G \mapsto G_1 \mapsto ... \mapsto G_n$ with $r$ the pole removal applied at stage $G_i \mapsto G_{i+1}$. Since $r$ is a pole removal of a minimal lens in $G_n$, it is inessential as an edge removal in $G_n$ (by Corollary 3.8). Then by the contrapositive of Lemma 6.3, $r$ is also inessential in $G$, since $G_n$ is obtained by a sequence of inessential edge removals.

We are left with proving:

($*$) If $r$ is inessential, then $r$ is potentially a pole removal of a minimal lens.

To get started, we need the following lemma:

**Lemma 6.4.** *Let $G$ be a circular planar graph, and let $G = G_0 \mapsto G_1 \mapsto G_2 \mapsto ... \mapsto G_n$ be a sequence of graphs where $G_n$ is critical and $G_{i+1}$ is obtained from $G_i$ by exactly one pole removal of a minimal lens. Then the number $n$ is independent of the sequence of uncrossings which was performed.*

*Proof.* (of lemma) Indeed, each $G_i \mapsto G_{i+1}$ is corresponds to one of the following operations on the graph:

(1) Deletion or contraction of an edge leading to a hanging interior vertex, or deletion of a self-loop (the possible outcomes of uncrossing a 1-pole lens).
(2) A composition of wye-delta transformations, followed by a single series/parallel edge removal, followed by more wye-delta transformations.

Thus $n$ is the number of edge removals of inessential edges used in this sequence to transform $G$ into a critical circular planar graph with the same connection set as $G$. By [CuMo00, Thm 9.5], all critical circular planar graphs with the same connection set as $G$ are $Y$-$\Delta$ equivalent, and in particular have the same number of edges. This shows that the number $n$ is independent of the sequence of minimal lens uncrossings. $\square$

For any circular planar graph $G$ with medial graph $M$, let the $n = n(G)$ be the number defined in Lemma 6.4. We call $n(G)$ the *uncrossing number* of $G$. Fix an inessential edge removal $r$, and let $e(r)$ be the edge removed by $r$. To prove $(*)$, we proceed by induction on the uncrossing number. The case $n = 0$ is vacuous, since in this case $G$ has no inessential edge removals by Theorem 1.4. For the induction we proceed by a case-by-case analysis of the location of the edge removal $r$ relative to lenses in the medial graph $M$ of $G$.

*Case 1.* Suppose that $r$ is a pole removal of a minimal lens in $M$. In this case we are done.

*Case 2.* Suppose that there exists a minimal lens $L$ such that $e(r)$ does not intersect $L$. Let $G'$ be the graph obtained by uncrossing $L$ at one of its poles. Then by Corollary 3.8, $r$ remains inessential on $G'$. By induction, we may assume that $r$ is potentially a pole removal of a minimal lens on $G'$, and hence also on $G$.

If $r$ does not fit into cases 1 or 2, then $e(r)$ must intersect some minimal 2-pole lens $L$. There are three possibilities for the location of $e(r)$ in $L$, comprising cases 3-5:

*Case 3.* Suppose that $e(r)$ is one of the poles of $L$. If $r$ is not a pole removal of $L$, then $r$ must correspond to the operation seen in the upper half of Figure 24. More precisely, if $L$ is defined by geodesic fragments $\alpha = a_0a_1...a_m$ and $\beta = b_0b_1...b_n$, then in $Gr$, $\alpha$ and $\beta$ are replaced by

$$\alpha' = ...a_0a_1...a_{m-1}b_{n-1}...b_1b_0... \quad \text{and}$$
$$\beta' = ...a_{m+2}a_{m+1}b_{n+1}b_{n+2}...$$

with all other geodesics remaining the same. Let $p = e(r)$, let $q$ be the other pole of $L$, and let $s$ be the pole removal of pole $q$ (as in the lower half of Figure 24). Recall that by Corollary 3.8, $s$ is inessential.

**Lemma 6.5.** *Let $Gr$ be the graph obtained by applying edge removal $r$ to $G$. Then $s$ remains inessential on $Gr$.*

*Proof.* (of lemma) Notice that on $Gr$, $s$ is the pole removal of a 1-pole lens $L'$. If $L'$ is empty, then $s$ is obviously inessential. If $L'$ is not empty, then it is not minimal, so we must be more careful in showing that $s$ is inessential since Corollary 3.8 does not apply. Because the original lens $L$ on $G$ is a minimal lens, it is possible to remove all interior crossings of $L$ via a series of arc switches never using either pole of $L$ (as in the proof of Lemma 3.6). The same sequence of arc switches on $L'$ will remove all interior crossings from $L'$, so we may assume $L'$ is as in Figure 25. Let $N$ be the unique minimal lens of $L'$, defined by the geodesic $\gamma$ passing through $L'$ furthest away from the pole $q$. After uncrossing $N$ at either pole, we get a new configuration $L''$ where $s$ is the pole of a minimal 2-pole lens, and is therefore inessential, as in Figure 25. This implies that $s$ is inessential on $Gr$ as well, since $Y$-$\Delta$ transformations and series/parallel edge removals not involving edge $q = e(s)$ do not change the essentiality of $s$. $\square$

We can now show that $r$ is potentially a pole of a minimal lens as follows. Since $r$ is inessential on $G$ and $s$ is inessential on $Gr$, we get that $(Gr)s$ has the same connections as $G$. Since $(Gr)s = (Gs)r$, we conclude that $r$ is inessential on the graph $Gs$. But $s$ was a pole removal of a minimal lens, so

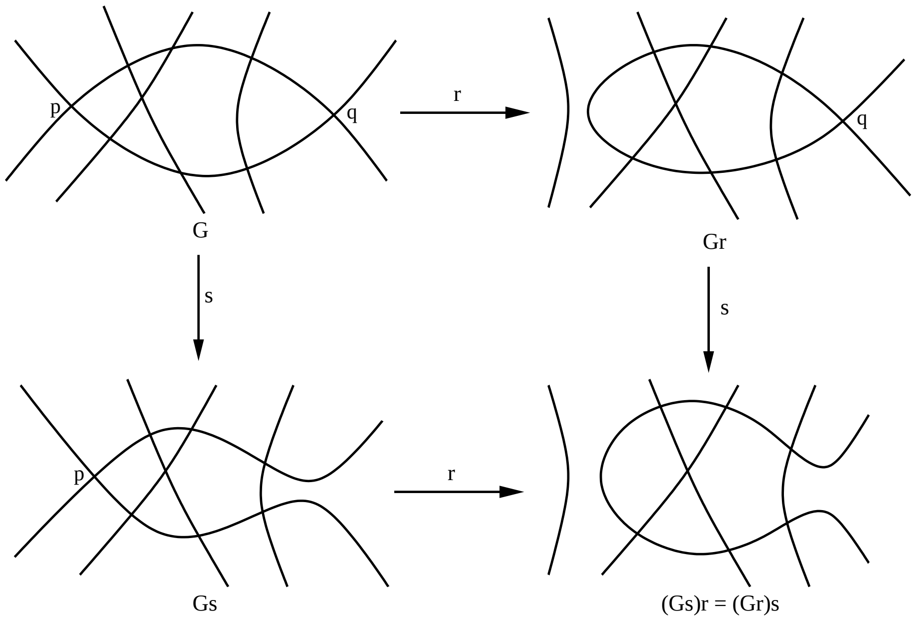


FIGURE 24. The edge removals $r$ and $s$ appearing in the proof of Case 3

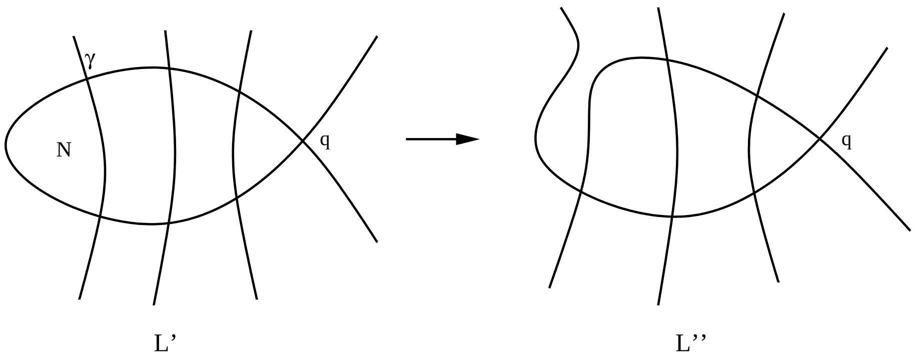


FIGURE 25. Figure referred to in Lemma 6.5

$n(Gs) = n(G) - 1$; therefore by the induction hypothesis, $r$ is potentially a pole removal of a minimal lens on $Gs$, and hence also on $G$. This completes Case 3.

*Case 4.* Suppose that $e(r)$ is a crossing on the boundary of a minimal 2-pole lens $L$. To fix notation, let $p$ and $q$ be the poles of $L$. Let $N$ and $S$ be the two arc fragments forming the lens $L$, and let $A$ be the other arc fragment defining edge $e(r)$. The edge removal $r$ is then shown in Figure 26 (together with some extra geodesics). Let $s$ be the pole removal of pole $p$. Since $L$ is minimal, $s$ is inessential on $G$. However, $p$ might not be the pole of a minimal lens in $Gr$, as happens in Figure 26. Nevertheless we claim the following:

**Lemma 6.6.** *Let $Gr$ be the graph obtained by applying edge removal $r$ to $G$. Then $s$ remains inessential on $Gr$.*

*Proof.* (of lemma) The proof is very similar to that of Lemma 6.5. Notice that on $Gr$, $s$ is the pole removal of a 2-pole lens $L'$ defined by arc fragments $S$ and the new arc fragment obtained from parts of $N$ and $A$, but this 2-pole lens is not necessarily minimal (as in Figure 26). To argue that $s$ is nevertheless inessential, consider first the original lens $L$ seen in the upper left of Figure 26. The proof of Lemma 3.9 shows that $L$ can be emptied of all interior crossings by a sequence of arc switches never

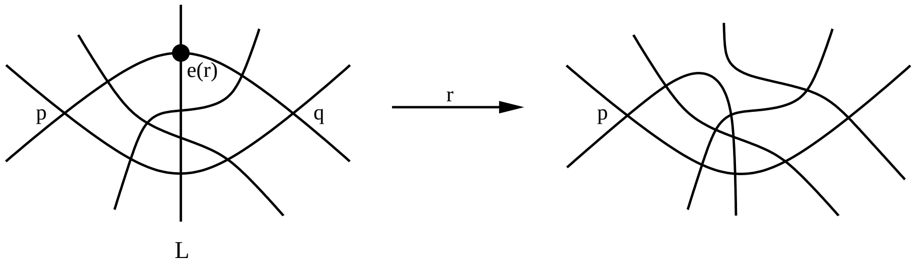


FIGURE 26. The edge removal of Case 4

using edge $e(r)$ or the poles $p$ or $q$. The same sequence of arc switches in $L'$ will transform $L'$ into a minimal 2-pole lens with pole removal $s$. This implies that $s$ is an inessential edge removal on $Gr$, since $s$ can be made a pole removal of a minimal lens by a sequence of arc switches never using $e(s)$. □

We can then show that $r$ is potentially a pole of a minimal lens on $G$ in much the same way as we handled Case 3. Namely, the above lemma shows that the graph $(Gr)s$ has the same connections as $G$, since $r$ is inessential on $G$ and $s$ is inessential on $Gr$. Since $(Gr)s = (Gs)r$, this implies that $r$ is inessential as an edge removal on $Gs$. By the induction hypothesis, $r$ is potentially a pole removal of a minimal lens on $Gs$, and therefore also on $G$.

*Case 5* The only remaining case is that $e(r)$ is in the interior of every minimal lens, and therefore in the interior of every lens. We will show that this contradicts $r$ being inessential; every edge which appears in the interior of every minimal lens in $G$ must in fact be essential for both deletion and contraction. To show this we first we need to study the configuration of the geodesics defining the crossing $e(r)$. We do this via a sequence of lemmas. All of these lemmas are under the hypothesis that $e(r)$ is in the interior of every minimal lens.

**Lemma 6.7.** *The medial graph $M(G)$ has no 0-pole or 1-pole lenses.*

*Proof.* Suppose to the contrary that $M(G)$ has at least one 0-pole or 1-pole lens. Let $L$ be such a lens which does not properly contain any 0-pole or 1-pole lens. By assumption $e(r)$ must be in the interior of $L$. Let $\alpha$ be one of the geodesics defining $e(r)$. Define two geodesics fragments $\alpha_1$, $\alpha_2$ as follows. At the crossing $e(r)$, there are two tangent directions (anti-parallel to each other) which lie on the curve $\alpha$. Let $\alpha_1$ and $\alpha_2$ be the sub-fragments of $\alpha$ obtained by following $\alpha$ along these two tangent directions until one hits the boundary of the lens $L$. Note that both $\alpha_1$ and $\alpha_2$ must eventually hit the boundary of $L$ without ever intersecting themselves: the only other possibility is that $\alpha_1$ or $\alpha_2$ forms a loop (0-pole or 1-pole lens) in the interior of $L$, and $L$ was assumed to not contain any proper 0-pole or 1-pole lens.

Now let $\widetilde{\alpha}$ be the geodesic fragment obtained by concatenating $\alpha_1$ and $\alpha_2$. Then $\widetilde{\alpha}$ and part of the boundary geodesic of $L$ form a 2-pole lens $L'$ with $e(r)$ appearing on the boundary of $L'$. This contradicts the assumption that $e(r)$ appears in the interior of every lens in $M(G)$. □

**Lemma 6.8.** *Let $\alpha$ be one of the geodesics defining the crossing $e(r)$. Then any other geodesic $\gamma$ intersects $\alpha$ at most once.*

*Proof.* Suppose to the contrary that $\alpha$ and $\gamma$ intersect at distinct points $p$ and $q$. Let $\widetilde{\alpha}$ and $\widetilde{\gamma}$ be the geodesic fragments of $\alpha$ and $\gamma$ joining $p$ to $q$. Because $\alpha$ and $\gamma$ cannot have any self-intersections by the previous lemma, $\widetilde{\alpha}$ and $\widetilde{\gamma}$ form a 2-pole lens $L'$, and $e(r)$ must be either disjoint from $L'$ or on its boundary. This contradicts the assumption on $e(r)$. □

To complete the proof of Case 5 (and hence of Theorem 6.2), we need to review the statement of the Cut-Point Lemma [CuMo00, Lemma 9.1]. Let $G$ be a circular planar graph with boundary, and let $M$ be its medial graph. Let $X$ and $Y$ be two distinct points on the boundary circle. We say that $X$ and $Y$ are cut-points for $M$ if neither point is an endpoint of any geodesic in $M$. We let $\widehat{XY}$ denote

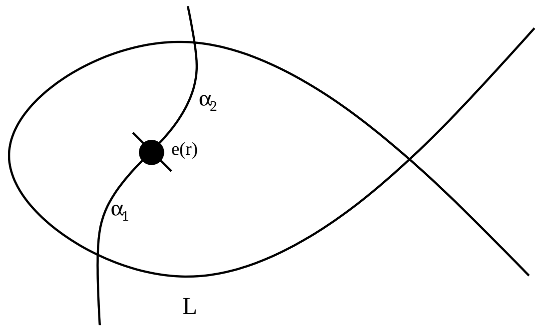


FIGURE 27. Figure for Lemma 6.7

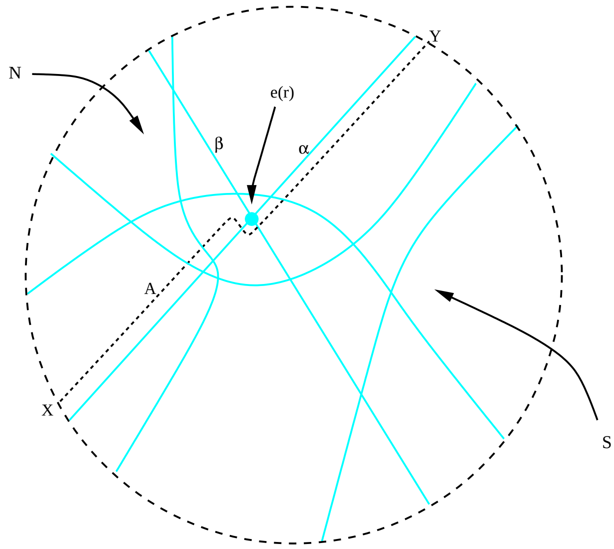


FIGURE 28. Set-up for showing that $r$ is essential via Cut-Point Lemma

the clockwise arc along the boundary circle from $X$ to $Y$, and $\hat{YX}$ the clockwise arc on the boundary circle from $Y$ to $X$. We refer to [CuMo00, p. 152] for definitions of the following terms:

- $m(X, Y) =$ the maximum integer $k$ such that there exists at $k$-connection in the medial graph $M$ which respects the cut-points $X$ and $Y$.
- $r(X, Y) =$ the number of re-entrant geodesics in the arc $\hat{XY}$.
- $n(X, Y) =$ the number of black intervals in $M$ which are entirely within the arc $\hat{XY}$.

The Cut-Point lemma the reads as follows:

**Lemma 6.9.** *(Cut-Point lemma). Let $G$ be a circular planar graph with boundary such that $M(G)$ is lensless. With $n(X, Y)$, $m(X, Y)$ and $r(X, Y)$ defined as above,*

$$m(X, Y) + r(X, Y) - n(X, Y) = 0.$$

*Proof.* See [CuMo00, Lemma 9.1]. □

We use the Cut-Point Lemma as follows (this use of the cut-point lemma is nearly identical to the use on p.166 of [CuMo00]). Let $\alpha$ and $\beta$ be the geodesics which cross to form the edge $e(r)$. Consider an arc $A$ (not a geodesic in $M$) placed as in Figure 28, so that $A$ intersects both $\beta$ and $\alpha$ exactly once, very close to $e(r)$. By Lemma 6.8, by placing $A$ close enough to geodesic $\alpha$ we may assume that no geodesic crosses $A$ more than once. Let $X$ and $Y$ be the endpoints of $A$. Let $N$ and $S$ be the two halves

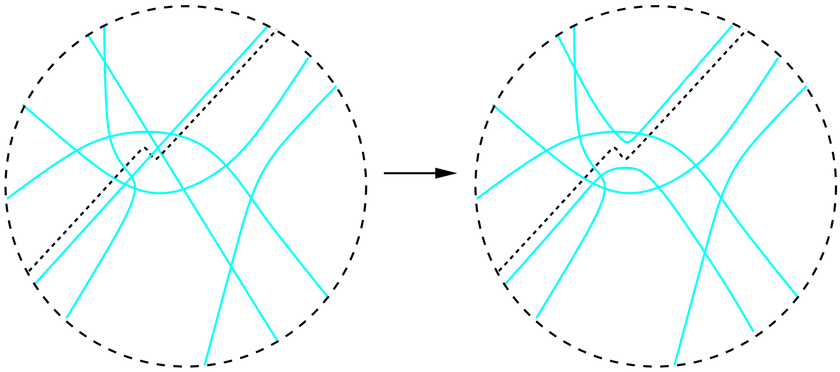

FIGURE 29. The edge removal $r$

of the unit disc $D$ defined by the arc $A$. By Lemmas 6.7 and 6.8, every lens in $M$ must be a 2-pole lens and must have one pole in each of $N$ and $S$. Therefore if we let $\mathcal{N}$ and $\mathcal{S}$ be the medial graphs obtained by restricting $M$ to $N$ and $S$, respectively, then $\mathcal{N}$ and $\mathcal{S}$ have no lenses, and no re-entrant geodesics on the boundary interval $A$. We let $m(X,Y;\mathcal{N})$ and $m(X,Y;\mathcal{S})$ be the maximum integer $k$ such that there exists a $k$-connection respecting the cutpoints $X$ and $Y$ on $\mathcal{N}$ and $\mathcal{S}$, respectively. Similarly, let $n(X,Y;\mathcal{N}) = n(Y,X;\mathcal{S})$ be the number of black intervals entirely contained in the interval $\hat{YX}$. Because no geodesic intersects $B$ twice, we have

$$m(X,Y;\mathcal{N}) = m(X,Y;\mathcal{S}) = n(X,Y;\mathcal{M}).$$

In other words, the maximum $k$-connection on both $\mathcal{N}$ and $\mathcal{S}$ respecting $X$ and $Y$ is equal to the largest possible value, namely, the number of black intervals entirely contained in $\hat{XY}$ (in $\mathcal{N}$, which equals the number of black intervals in $\hat{YX}$ on $\mathcal{S}$). Pasting these two connections together, we get a connection on the original graph $M$ of the same size respecting the cutpoints $X$ and $Y$. Now consider the edge removal shown in Figure 29. Notice that the number of black intervals on $\hat{XY}$ decreases by one on applying this edge removal. Therefore the maximum possible size of a connection respecting the cutpoints $X$ and $Y$ decreases by 1, and hence this edge removal must break a connection.

An essentially identical argument, replacing $B$ with an analogous arc placed along geodesic $\beta$ as in Figure 30, will show that the other edge removal of $e$ is essential. We have shown that if an edge appears in the interior of every lens in $M$, then that edge is essential for both deletion and contraction. Therefore Case 5 cannot occur, and we have completed the proof of Theorem 6.2. □

## REFERENCES

[CuMo00] Curtis, Edward B. and James A. Morrow. Inverse Problems for Electric Networks World Scientific Publishing Company, ©2000.

[dVGV96] Y.C.de Verdiere, I.Gitler, and D.Vertigan. Réseaux électriques planaires II. Comment. Math. Helv. 71(1), 1996, 144-167.

*E-mail address*: obiesel@math.princeton.edu

*E-mail address*: mannisto@math.berkeley.edu

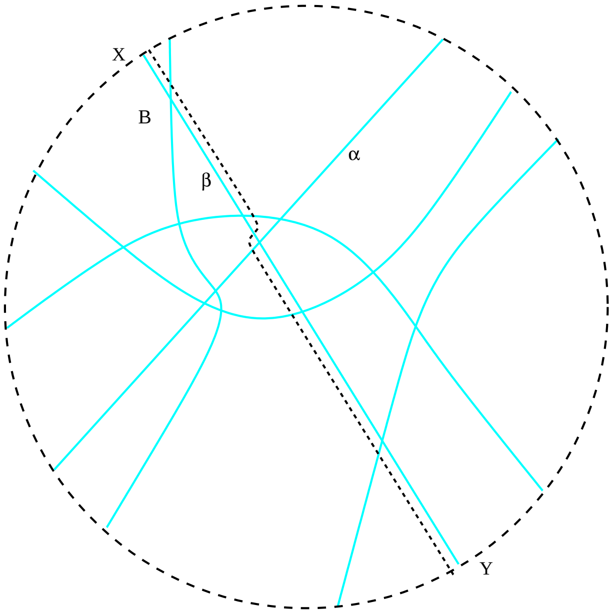


FIGURE 30. Set-up for showing that the other edge removal of $e(r)$ is essential via Cut-Point Lemma