THE TREE DIAGRAMS FORMULA

OWEN BIESEL AND PETER MANNISTO

Abstract. In this paper, we derive a generalization of Kirchhoff's matrix-tree theorem, applicable to directed graphs with boundary. As a corollary, we extend a foundational result of connected electrical networks, that every principal proper submatrix of K has positive determinant, to the case of directed electrical networks. We close with a brief discussion of connections on networks, and show that nonprincipal subdeterminants of K are nonzero polynomials in the conductivities iff the corresponding connection exists in the graph.

Contents

1. Directed Electrical Networks

The determinant of a square matrix M is perhaps its most important related quantity, and it is of perennial concern to decide when the determinant vanishes. In the study of electrical networks, one of the first questions along these lines is whether the interior-interior part of a Kirchhoff matrix is nonsingular. When M is indeed derived in this way from an (undirected) electrical network, the following result is known from the study of discrete harmonic functions:

Proposition 1.1 (Lemma 3.8 from [CuMo00]). *Let K be the Kirchhoff matrix of an undirected electrical network with conductivities in $\mathbb{R}_+$. Suppose that every interior vertex admits a path to some boundary vertex. Then the determinant of $M = K(I;I)$ is strictly positive.*

The hypothesis that every interior vertex have a path connecting it to the boundary rules out the possibility that there be an isolated interior component of the graph, in which case M would be block diagonal, and one of the blocks would have vanishing row sums and hence be singular. We will use methods unrelated to discrete harmonic analysis to prove a generalization of Proposition 1.1 to electrical networks whose Kirchhoff matrices no longer are required to be symmetric. This requires loosening the definition of electrical network to what we will refer to in this paper as a directed electrical network, which we define after the required notions of directed graph, and directed graph with boundary.

Definition 1.2.

- A *directed graph* consists of the following structure:
  - A set V of *vertices.*
  - A set E of (directed) *edges*
  - Two maps $s : E \to V$ and $t : E \to V$, associating to each edge its *source* and *target* vertex, respectively. If e is an edge with $s(e) = i$ and $t(e) = j$, we write $e : i \to j$ and say e is an edge from i to j.
- A *directed graph with boundary* consists of a directed graph (V, E, s, t) together with a subset $\partial V \subseteq V$ of the vertex set called *boundary* vertices. The remaining vertices $V \setminus \partial V$ are called *interior* vertices, and the set of interior vertices is usually denoted I.
- A *directed electrical network* consists of a directed graph with boundary $(V, E, s, t, \partial V)$ and an assignment to each edge $e \in E$ of a nonzero weight (or *conductivity*) $\gamma(e)$, chosen from a fixed commutative ring R.

We can also extend the definition of Kirchhoff matrix to the case of directed electrical networks:

Definition 1.3. The *Kirchhoff matrix* associated to a directed electrical network is a square matrix, with a row and column for each vertex. The entries are given as follows:

$$K_{ij} = -\sum_{e:i\to j} \gamma(e) \text{ if } i \neq j,$$
$$K_{ii} = \sum_{e:i\to j\neq i} \gamma(e).$$

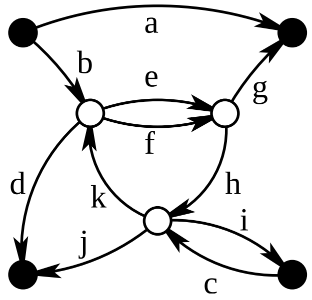
K is often presented as a block matrix, according to the partition of vertices into boundary and interior.

Remark 1.4. The above definition allows a directed graph to contain self-loops (an edge from a vertex to itself) or parallel edges (a collection of edges whose source vertices are all the same, as are their target vertices). Even if we desired to avoid them, self-loops and parallel edges can arise naturally when certain operations are applied to directed graphs without them, so it is reasonable to allow them in from the beginning. However, in the following discussion of electrical networks, it will always be safe to mentally delete self-loops, and to combine parallel edges into a single edge whose weight is the sum of the weights of the edges it replaces, since these operations do not change the network's Kirchhoff matrix. In addition, one can choose to consider the lack of any edge at all from i to j as being morally the same as a single edge $i \to j$ with weight 0, but we will stick to the convention that edges may only have nonzero weights.

The only remaining part of Proposition 1.1 to generalize is the concept of path:

Definition 1.5. If v and w are vertices of a directed graph, then a *directed path* from v to w consists of a finite sequence of distinct vertices $v = v_0, v_1, \ldots, v_n = w$ and an edge $e_k : v_k \to v_{k+1}$ for each $0 \leq k < n$.

We can now state our generalization of Proposition 1.1:



$$K = \left(\begin{array}{cccc|ccc} a+b & -a & 0 & 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & -c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -d & d+e+f & -e-f & 0 \\ 0 & -g & 0 & 0 & 0 & g+h & -h \\ 0 & 0 & -i & -j & -k & 0 & i+j+k \end{array}\right)$$

FIGURE 1. An electrical network and its Kirchhoff matrix. The vertices are ordered clockwise from the upper left: first the solid boundary vertices, then the open interior vertices.

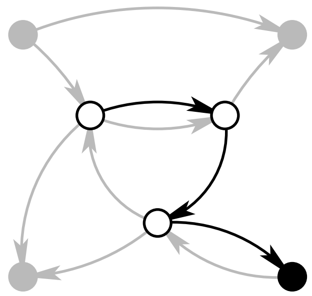


FIGURE 2. A directed path from an interior vertex to a boundary vertex.

Proposition 1.6. *Let K be the Kirchhoff matrix of a directed electrical network with conductivities in $\mathbb{R}_+$. Suppose that every interior vertex admits a directed path to some boundary vertex. Then the determinant of $M = K(I;I)$ is strictly positive.*

We will prove Proposition 1.6 using methods unrelated to discrete harmonic analysis, as a corollary of a more general result, called the tree diagram formula.

Remark 1.7. The proof in [CuMo00] of Proposition 1.1 can be extended to show that for an undirected electrical network with conductivities in the set $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, the determinant of M is nonzero. However, for directed networks with conductivities in $\mathbb{C}_+$, the determinant of M can vanish. This suggests that any proof of Proposition 1.6 cannot be too similar to the proof in [CuMo00], or else it would apply to conductivities in $\mathbb{C}_+$ as well.

2. The Tree Diagrams Formula

We begin with the notion of a tree diagram:

Definition 2.1. A *tree diagram* T on a finite directed graph with boundary G is a subset of the edges of G satisfying any of the following equivalent conditions:

- T comprises a single edge out of each interior vertex, and the subgraph consisting of edges in T has no directed cycles.
- T comprises a single edge out of each interior vertex, and every interior vertex admits a directed path to the boundary using only edges from T.
- The subgraph of G consisting of edges in T is a disjoint union of trees, each containing exactly one boundary vertex, and edges directed toward the boundary.

The set of tree diagrams on G is written $\mathbb{T}(G)$. If G has the structure of a directed electrical network, then we define the *weight* $w(T)$ of a tree diagram T as the product of the weights of all edges in T.

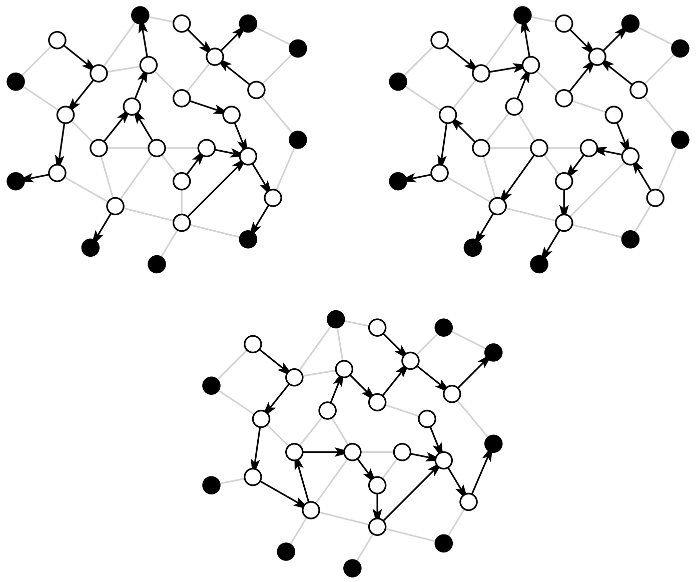


Figure 3. Three tree diagrams on the same directed graph with boundary.

Remark 2.2. It is easy to see that the three conditions defining a tree diagram are equivalent, because each is equivalent to this statement, ostensibly stronger than all three:

- T is a set of edges comprising one edge out of each interior vertex, the subgraph consisting of edges in T has no (undirected) cycles, and every interior vertex admits a unique path to the boundary using edges in T.

However, it is useful to use the weaker forms as definitions. The first definition is the one we will use in proving the tree diagrams formula below, but the second definition will be helpful later for demonstrating the existence of tree diagrams. And the third definition was the first used by the authors, and the one that gave tree diagrams their name.

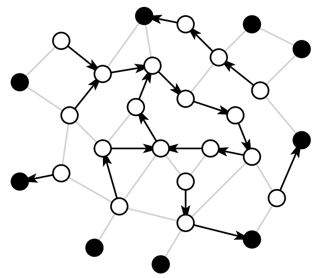


FIGURE 4. This choice of edge out of each interior vertex does not produce a tree diagram: Some edges form a cycle, leaving many interior vertices without a path to the boundary.

Theorem 2.3 (The Tree Diagrams Formula)**.** *Let G be a directed electrical network, and K its Kirchhoff matrix. Then the determinant of $K(I;I)$ is the sum of the weights of all tree diagrams:*

$$\det K(I;I) = \sum_{T\in\mathbb{T}(G)} w(T).$$

To prove Theorem 2.3, we use a method of induction based on graph operations called edge removals. There are two types of edge removals: deletion and contraction.

Definition 2.4. If $e \in E$ is an edge of a directed graph $G = (V, E, s, t)$, we can *delete* e to get a smaller directed graph $G-e$ by taking the subgraph of G consisting of all edges except e. In other words, let $E' = E\backslash\{e\}$; then $G-e = (V, E', s|_{E'}, t|_{E'})$.

If G is a directed graph with boundary ∂V, then the boundary of G is also ∂V. If G is an electrical network with weight function $\gamma : E \to R$, then $G - e$ is an electrical network with weight function $\gamma|_{E'}$.

Edge deletion is a simple form of edge removal, but there is a slightly more elaborate version called contraction:

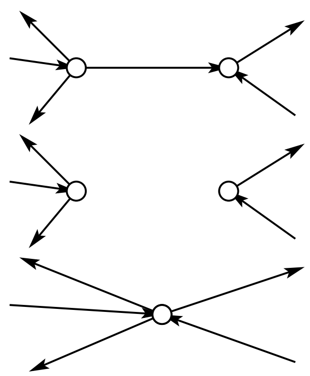


FIGURE 5. An edge, the deletion of that edge, and the contraction of that edge.

Definition 2.5. If $e : v \to w$ is an edge of a directed graph $G = (V, E, s, t)$, then we can *contract* e in two steps to form a smaller graph G/e: First identify vertices v and w, and then delete the resulting self-loop e. In other words, let $V' = V/(v \sim w)$ and $p : V \to V'$ the projection, and $E' = E \setminus \{e\}$ as before; then $G/e = (V', E', p \circ s|_{E'}, p \circ t|_{E'})$.

If G is a directed graph with boundary ∂V, then G has boundary $p(\partial V)$. If G is an electrical network with weight function $\gamma : E \to R$, then G/e is an electrical network with weight function $\gamma|_{E'}$.

Next we have a pair of lemmas describing the relationship between tree diagrams and edge removals:

Lemma 2.6. *Let T be a set of edges in a finite directed graph with boundary G, and let e be an edge of G. If $e \notin T$, then T is a tree diagram on G iff T is a tree diagram on $G - e$.*

Proof. Suppose $e \notin T$. Then the subgraph consisting of edges in T is contained in $G - e$, and is acyclic in G iff it is acyclic in $G - e$, and comprises an edge out of each interior vertex in G iff it does in $G - e$. So T is a tree diagram on G iff it is a tree diagram on $G - e$. $\square$

The next lemma is similar, but a little stricter in its hypotheses and more fiddly in its proof.

Lemma 2.7. *Let T be a set of edges in a finite directed graph with boundary G, and let T contain an edge e. If T is a tree diagram on G, then $T \setminus \{e\}$ is a tree diagram on G/e. If $T \setminus \{e\}$ is a tree diagram on G/e, and in addition e is an edge from an interior vertex to a boundary vertex, then T is a tree diagram on G.*

Proof. Let T be a tree diagram on G containing e. We must check that $T' = T \setminus \{e\}$ is a tree diagram on G/e.

First, suppose for contradiction that some edge f of T' has a boundary vertex w as its source in G/e. Then f is in T, so the source of f is an interior vertex v in G. Hence e must have been an edge from v to w. But then e and f would be two distinct edges in T with the same source. Therefore edges of T' can only lead out of interior vertices in G/e.

Now suppose for contradiction that two edges in T' have the same source in G/e. Then their sources cannot be the same in G, so those two vertices must be the source and target of e. But then two edges in T would have the same source in G, a contradiction. Hence the sources in G/e of the edges in T' are all distinct.

We have therefore established that each edge of T' leads out of distinct interior vertices of G/e - is every interior vertex of G/e the source of an edge in T'? Each interior vertex i of G/e is the image of an interior vertex j of G. There is a unique edge in T whose source in G is j, and this edge will be the desired edge out of i in G/e unless that edge is e. Supposing that vertex i is the vertex created by contracting e, we find that since i is assumed to be an interior vertex of G/e, the target of e in G must be an interior vertex. Hence it is the source of an edge $f \in T$, and the source of f in G/e is vertex i as desired. Thus every interior vertex of G/e

is the source of a unique edge of T'.

Is the subgraph of G/e given by the edges of T' acyclic? Suppose a cycle C exists among the edges of T'. Unless it used the vertex at the contracted edge e, it would lift immediately to a cycle C among the edges in T on G. So suppose C does use the contraction vertex; then C lifts to a path in G from $t(e)$ to $s(e)$. (C cannot lift to a cycle containing $s(e)$ or $t(e)$, and it cannot lift to a path from $s(e)$ to $t(e)$ or else T would contain two edges out of $s(e)$.) Then $C \cup \{e\}$ is a cycle in G among the edges of T, a contradiction. Therefore T' is a tree diagram on G/e.

Now suppose T' is a tree diagram on G/e, and that e is an edge from an interior vertex to a boundary vertex. We show that T is a tree diagram on G. The edges of T' lead out of every interior vertex of G except the source of e. Therefore $T = T' \cup \{e\}$ comprises edges out of every interior vertex of G. And T is acyclic: any cycle among edges in T cannot use e, or else T would contain an edge out of the boundary vertex $t(e)$, so that cycle would also be contained in T'. Therefore T is a tree diagram. □

Corollary 2.8. *If e is an edge of a finite directed graph with boundary G, and e leads from an interior vertex to a boundary vertex, then there is a canonical correspondence between $\mathbb{T}(G)$ and the disjoint union of $\mathbb{T}(G-e)$ and $\mathbb{T}(G/e)$.*

Corollary 2.9. *If e is an edge of a finite directed electrical network G, and e leads from an interior vertex to a boundary vertex, then*

$$\left(\sum_{T\in\mathbb{T}(G)} w(T)\right) = \left(\sum_{T\in\mathbb{T}(G-e)} w(T)\right) + \gamma(e)\left(\sum_{T'\in\mathbb{T}(G/e)} w(T')\right).$$

Proof. Each tree diagram T on G either contains e or it does not. If not, then T corresponds to a tree diagram on $G-e$ with the same weight. If T does contain e, then T corresponds to a tree diagram $T' = T \setminus \{e\}$ on G/e, and $w(T) = \gamma(e)w(T')$. □

Now we change tack and consider the effect on $K(I;I)$ when we delete or contract an interior-to-boundary edge e.

Lemma 2.10. *Let G be a directed electrical network, and let $e : i \to j$ be an edge of G with $i \in I$ and $j \in \partial V$. Let K_G be the Kirchhoff matrix of G, and let K_{G-e} be the Kirchhoff matrix of $G-e$. Then*

$$K_{G-e}(I;I) = K_G(I;I) - \gamma(e)\left(\delta_{ii}\right),$$

where (δ_{ii}) is the $|I|$-sized diagonal matrix with a 1 for the (i,i) entry and 0 everywhere else.

Proof. The only off-diagonal entry of K_G that changes is the (i,j) entry, but this is outside $K(I;I)$. However, the diagonal entry at (i,i) is the sum of weights of edges from i to vertices $j \neq i$, so when we delete e this entry decreases by $\gamma(e)$. □

Lemma 2.11. *Let G be a directed electrical network, and let $e : i \to j$ be an edge of G with $i \in I$ and $j \in \partial V$. Let K_G be the Kirchhoff matrix of G, and let $K_{G/e}$ be* *the Kirchhoff matrix of G/e, and let I' be the set of interior vertices of G/e. Then vertices in I' correspond naturally to vertices in $I \setminus \{i\}$, and*

$$K_{G/e}(I';I') = K_G(I \setminus \{i\}; I \setminus \{i\}) \subset K_G(I;I).$$

Proof. First we note that I' does biject naturally with $I \setminus \{i\}$: the only difference between the vertex sets of G and G/e is that i has become amalgamated with j and made a boundary vertex; the rest of the interior vertices have remained undisturbed. Now all that remains to show is that if k and ℓ are interior vertices of G/e, then $(K_{G/e})_{k\ell} = (K_G)_{k\ell}$. If $k \neq \ell$, this entry is the total weight of edges $k \to \ell$, which does not change when we contract e. If $k = \ell$, then this entry is the total weight of edges out of k to other vertices, which also does not change when we contract e. So $K_{G/e}(I';I')$ is indeed the principal submatrix of $K_G(I;I)$ indexed by vertices in $I \setminus \{i\}$. □

Corollary 2.12. *If e is an edge of a finite directed electrical network G, and e leads from an interior vertex to a boundary vertex, then*

$$\left(\det K_G(I;I)\right) = \left(\det K_{G-e}(I;I)\right) + \gamma(e)\left(\det K_{G/e}(I';I')\right),$$

where K_G, K_{G-e}, and $K_{G/e}$ are the Kirchhoff matrices of G, $G-e$, and G/e, respectively, and I' is the set of interior vertices of G/e.

Proof. We will expand the determinant of $K_G(I;I)$, first by using linearity of the determinant in rows. Specifically, write the ith row of $K_G(I;I)$ as a sum of two row vectors: a row vector equal to $K_G(i;I)$ except missing the $\gamma(e)$ term in the ith entry, and a row vector consisting of all zeroes except for a $\gamma(e)$ in the ith entry. Then the determinant of $K_G(I;I)$ is the sum of the determinants of the two matrices we get by substituting these two row vectors for the ith row of $K_G(I;I)$. Call these two matrices K' and K'', so that $\det K(I;I) = \det K' + \det K''$.

We can note immediately that $K' = K_{G-e}(I;I)$ by Lemma 2.10, since K' differs from $K_G(I;I)$ only in that $\gamma(e)$ has been subtracted from the (i,i) entry. Now compute the determinant of K'' with cofactor expansion along the ith row: every one of these entries is zero except the (i,i) entry, so we obtain

$$\begin{aligned}\det K'' &= \gamma(e) \det K''(I \setminus \{i\}; I \setminus \{i\}) \\ &= \gamma(e) \det K_G(I \setminus \{i\}; I \setminus \{i\}),\end{aligned}$$

since K'' and $K_G(I;I)$ only differ in the ith row,

$$= \gamma(e) \det K_{G/e}(I';I') \text{ by Lemma 2.11.}$$

Therefore

$$\det K_G(I;I) = \det K' + \det K'' = \det K_{G-e}(I;I) + \gamma(e) \det K_{G/e}(I';I').$$

□

We are now ready to prove the tree diagrams formula, Theorem 2.3. Comparing Corollaries 2.9 and 2.12, we see that most of the work in proving the theorem is already done; all that remains is to compile the corollaries into a proof by induction.

Proof of Theorem 2.3. Suppose we wish to show that all finite directed electrical networks have some property P. We know that P holds for any network with no interior-to-boundary edges. We also know that for any interior-to-boundary edge e, if P holds for $G-e$ and G/e, then P holds for G. Then P holds for all finite directed electrical networks; we prove it by induction on the number of edges of G. Suppose that P holds for all electrical networks with fewer edges than G. If G has no interior-to-boundary edges, then P holds for G. Otherwise, G contains an interior-to-boundary edge e. The networks G/e and $G-e$ both have fewer edges than G, so P holds for them, and hence holds for G as well.

Now we note that this argument applies to the property

$$\det K_G(I;I) = \sum_{T\in\mathbb{T}(G)} w(T). \tag{1}$$

This is true for graphs with no interior-to-boundary edges: if such a G has any interior vertices, then there are no tree diagrams on G, and $K_G(I;I)$ has vanishing row sums so its determinant vanishes. If G has no interior vertices at all, then there is just one tree diagram on G (namely, $T=\varnothing$ with weight $w(T)=1$), and $K(I;I)$ is the unique zero-by-zero matrix, whose determinant is 1. And if (1) holds for $G-e$ and G/e, then

$$\begin{aligned}
\det K_G(I;I) &= \Big(\det K_{G-e}(I;I)\Big) + \gamma(e)\Big(\det K_{G/e}(I';I')\Big) \quad \text{by Corollary 2.12}\\
&= \left(\sum_{T\in\mathbb{T}(G-e)} w(T)\right) + \gamma(e)\left(\sum_{T'\in\mathbb{T}(G/e)} w(T')\right) \quad \text{by assumption}\\
&= \sum_{T\in\mathbb{T}(G)} w(T) \quad \text{by Corollary 2.9,}
\end{aligned}$$

so (1) holds for G. Therefore, by the argument above, (1) holds for all finite directed electrical networks. □

We need just one more technical lemma to prove Proposition 1.6:

Lemma 2.13. *If G is a finite directed graph with boundary, and every interior vertex of G has a directed path to the boundary, then there is at least one tree diagram on G.*

Proof. We proceed by induction on the number of interior vertices of G. If G has no interior vertices, then the $T=\varnothing$ is a tree diagram on G. Now suppose G has n interior vertices. Choose one, and choose a path from it to the boundary. Then the end of this path is an interior-boundary edge e. Now G/e has $n-1$ interior vertices, so G/e has a tree diagram T by the induction hypothesis. Then $T\cup\{e\}$ is a tree diagram on G by Lemma 2.7. □

Proof of Proposition 1.6. Suppose G is an electrical network with conductivities in $\mathbb{R}_+$, and that every interior vertex has a directed path to the boundary. Then by the tree diagrams formula, we have

$$\det K(I;I) = \sum_{T\in\mathbb{T}(G)} w(T).$$

By Lemma 2.13, $\mathbb{T}(G)$ is nonempty, and since the weight of every edge is a positive real number, each term in the sum is positive. Therefore $\det K(I;I) > 0$. □

A similar result, which is useful for questions of generic recoverability (see, for example, [BiMa10]), follows:

Lemma 2.14. *Suppose G is a directed graph with boundary, for which every interior vertex has a directed path to the boundary, and make G an electrical network by weighting each edge with a distinct indeterminate conductivity. Then* $\det K(I;I)$ *is a nonzero polynomial in the conductivities. Furthermore, this result extends to the case of undirected networks.*

Proof. The key is that the weight of the tree diagram determines the edges used, if every conductivity is a distinct indeterminate, so no terms in $\sum_{T\in\mathbb{T}(G)} w(T)$ can cancel. In the undirected case, where for every edge $i \to j$ there is an edge $j \to i$ with the same conductivity, this is harder to see, because it is conceivable that distinct tree diagrams could use the same edges, but directed differently. However, this does not happen: if we know which undirected edges are used, then they form a set of trees and each edge must be directed toward the boundary. Hence every tree diagram uses a different set of conductivities, and so the weights are all distinct and cannot cancel. □

3. Another proof of the Tree Diagrams Formula

There is another proof of the tree diagrams formula based on loop partitions. In this section, we will fix an electrical network G, and attempt to compute $\det M = K(I;I)$ in terms of tree diagrams. We will also assume that G has no self-loops, since they do not contribute to K and will never be used by any tree diagram. Similarly, we will assume that G has no parallel edges, replacing any such collection by a single edge as in Remark 1.4. We have already observed that this simplification does not change K. Furthermore, if an edge in a tree diagram is replaced by an edge parallel to it, the result is still a tree diagram, so the sum of weights of all tree diagrams will not change when we make this modification. Therefore it will suffice to show that the tree diagrams formula holds for simple networks, i.e. networks whose underlying directed graphs have no parallel edges or self-loops.

First we review a different graphical interpretation of determinants; for more information, see [BiMa10].

Definition 3.1. If M is any square matrix, we can construct its *associated graph* $\mathscr{G}_M$, which is a weighted directed graph given as follows:

- The vertices of $\mathscr{G}_M$ correspond to the row (or column) indices of M.
- If M_{ij} is nonzero, there is a unique edge $e : i \to j$, with weight $w(e) = M_{ij}$. Otherwise, there is no edge $i \to j$.

Remark 3.2. If G is an electrical network, we can take $M = K(I;I)$ and consider the associated graph $\mathscr{G}_M$. This new directed graph $\mathscr{G}_M$ is not the same as the old directed graph G. However, if G is simple, we can obtain $\mathscr{G}_M$ from G as follows: First, take the subgraph of G consisting of interior vertices and edges between them. Then negate the weight of every edge. Last, at each vertex i attach a self-loop whose weight is the sum of weights of edges in G out of i. To avoid confusion, if e is an interior-interior edge of G, we will always refer to the conductivity in the electrical network as $\gamma(e)$, and the weight in the associated graph as $w(e)$, so that $w(e) = -\gamma(e)$.

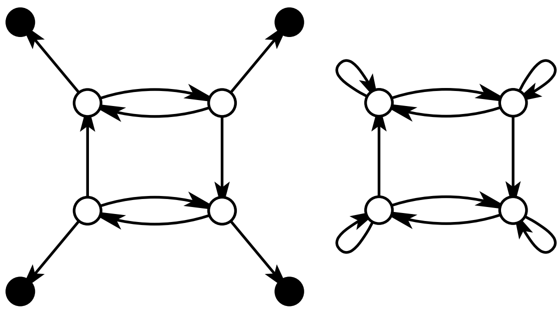


FIGURE 6. A simple electrical network G and its associated graph $\mathscr{G}_M$.

Definition 3.3. If $\mathscr{G}$ is a directed graph, a *loop partition* L on $\mathscr{G}$ is a permutation σ_L of the vertices of $\mathscr{G}$, together with edge $i \to \sigma_L(i)$ in $\mathscr{G}$ for every vertex i of $\mathscr{G}$. The set of loop partitions on $\mathscr{G}$ is denoted $\mathscr{L}(\mathscr{G})$.

We can now relate determinants to loop partitions:

Proposition 3.4 (Proposition 3.4 from [BiMa10]). *If M is a square matrix, then*

$$\det M = \sum_{L \in \mathscr{L}(\mathscr{G}_M)} \operatorname{sign}(\sigma_L) \prod_{e \in L} w(e).$$

Proof. If M is an $n \times n$ matrix, this is just a restatement of the equation

$$\det M = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{1 \leq j \leq n} M_{j\sigma(j)}. \qquad \square$$

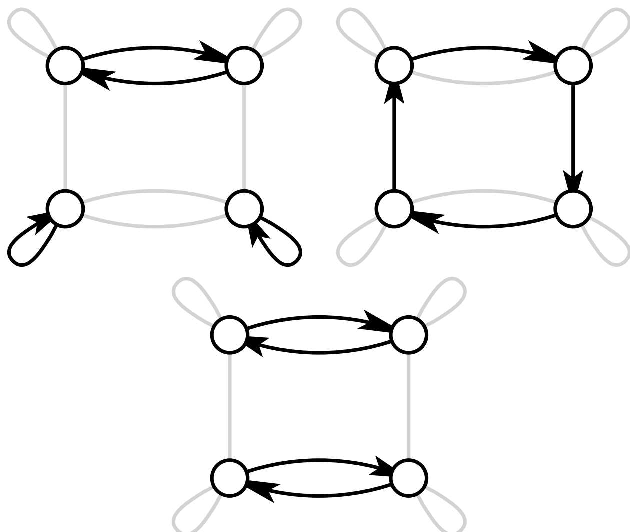


FIGURE 7. Three loop partitions on $\mathscr{G}_M$.

To use Proposition 3.4, we must relate loop partitions on $\mathscr{G}_M$ to some sort of diagrams on G. The appropriate type of diagram is called an "almost-tree diagram", so named because in this derivation, they almost appear in the tree diagrams formula.

Definition 3.5. If G is a directed graph with boundary, an *almost-tree diagram* on G is a choice, for each interior vertex i, of an edge in G out of i. The set of almost-tree diagrams on G is denoted $\mathbb{AT}(G)$. If G is an electrical network with conductivities γ, then the weight of an almost-tree diagram T is $w(T) = \prod_{e\in T} \gamma(e)$.

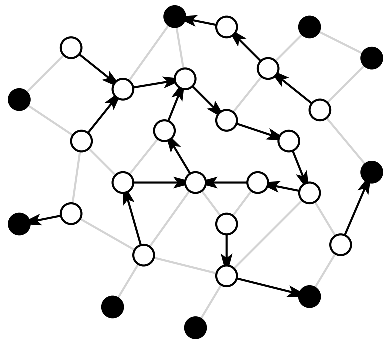


FIGURE 8. This collection of edges is not a tree diagram, but it is an almost-tree diagram that contains one cycle.

Thus the tree diagrams on G are exactly those almost-tree diagrams which contain no cycles in G. Speaking of cycles, we can make this easy observation:

Lemma 3.6. *Loop partitions on $\mathscr{G}_M$ correspond to sets of disjoint cycles among the interior vertices in G.*

Proof. Given any loop partition L on $\mathscr{G}_M$, we can decompose L into a disjoint union of cycles on $\mathscr{G}_M$. Then the *proper cycles* of L, i.e. those cycles which are not self-loops, correspond to cycles among interior vertices in G. Conversely, if we have any set of disjoint cycles among interior vertices in G, those correspond to a set of proper cycles containing some of the vertices in $\mathscr{G}_M$. We can complete this set to a loop partition on $\mathscr{G}_M$ with a self-loop at each unused vertex. $\square$

If L is a loop partition on $\mathscr{G}$, denote the set of proper cycles of L by $\mathscr{C}(L)$. Similarly, if T is an almost-tree diagram on G, denote the set of cycles in T by $\mathscr{C}(T)$. (So that in particular T is a tree diagram iff $\mathscr{C}(T) = \varnothing$.) Now we are ready to relate loop partitions and almost-tree diagrams:

Lemma 3.7. *If L is a loop partition on $\mathscr{G}_M$, then*

$$\prod_{e\in L} w(e) = (-1)^s \sum_{\substack{T\in\mathbb{AT}(G)\\ \mathscr{C}(L)\subseteq\mathscr{C}(T)}} w(T),$$

where s is the number of odd proper cycles in L.

Proof. First split the product into factors for those edges in proper cycles of L, and those edges which are self-loops:

$$\begin{aligned}\prod_{e\in L} w(e) &= \left(\prod_{C\in\mathscr{C}(L)}\prod_{e\in C} w(e)\right)\prod_{\substack{i \text{ in } \mathscr{G}_M\\ e:i\to i \text{ is in } L}} w(e)\\ &= \left(\prod_{C\in\mathscr{C}(L)}\prod_{e\in C} -\gamma(e)\right)\prod_{\substack{i \text{ in } \mathscr{G}_M\\ e:i\to i \text{ is in } L}} \left(\sum_{\substack{e'\in G\\ s(e')=i}} \gamma(e')\right),\end{aligned}$$

where we have expressed weights of edges in $\mathscr{G}_M$ in terms of conductivities of edges in G. Now note that each proper odd cycle of L contributes an odd number of factors of (-1), hence an overall sign change, while each proper even cycle of L contributes an even number of factors of (-1), hence no overall sign change. Therefore we can let s be the number of odd proper cycles of L, and so

$$= (-1)^s\left(\prod_{C\in\mathscr{C}(L)}\prod_{e\in C} \gamma(e)\right)\prod_{\substack{i \text{ in } \mathscr{G}_M\\ e:i\to i \text{ is in } L}} \left(\sum_{\substack{e'\in G\\ s(e')=i}} \gamma(e')\right).$$

Now expand the product of sums; we can interpret each resulting term as a choice of edge out of i in G for each self-loop at i in L, together with all the edges in proper cycles of L. These choices correspond to almost-tree diagrams that contain the proper cycles of L, and each term is the weight of such a diagram. Therefore

$$\prod_{e\in L} w(e) = (-1)^s \sum w(T),$$

where the sum is over almost-tree diagrams containing the proper cycles of L. □

Corollary 3.8. *If L is a loop partition on $\mathscr{G}_M$, then*

$$\operatorname{sign}(\sigma_L)\prod_{e\in L} w(e) = (-1)^{|\mathscr{C}(L)|}\sum_{\substack{T\in\mathbb{AT}(G)\\ \mathscr{C}(L)\subseteq\mathscr{C}(T)}} w(T),$$

where $|\mathscr{C}(L)|$ is the number of proper cycles in L.

Proof. If L is a loop partition, then $\operatorname{sign}(\sigma_L) = (-1)^t$, where t is the number of even cycles of L. Then

$$\begin{aligned}\operatorname{sign}(\sigma_L)\prod_{e\in L} w(e) &= (-1)^t(-1)^s\sum w(T) \text{ by Lemma 3.7}\\ &= (-1)^{s+t}\sum w(T).\end{aligned}$$

But $s+t$ is the number of cycles in L which are either proper odd cycles or even cycles, which is just the total number of proper cycles $|\mathscr{C}(L)|$. □

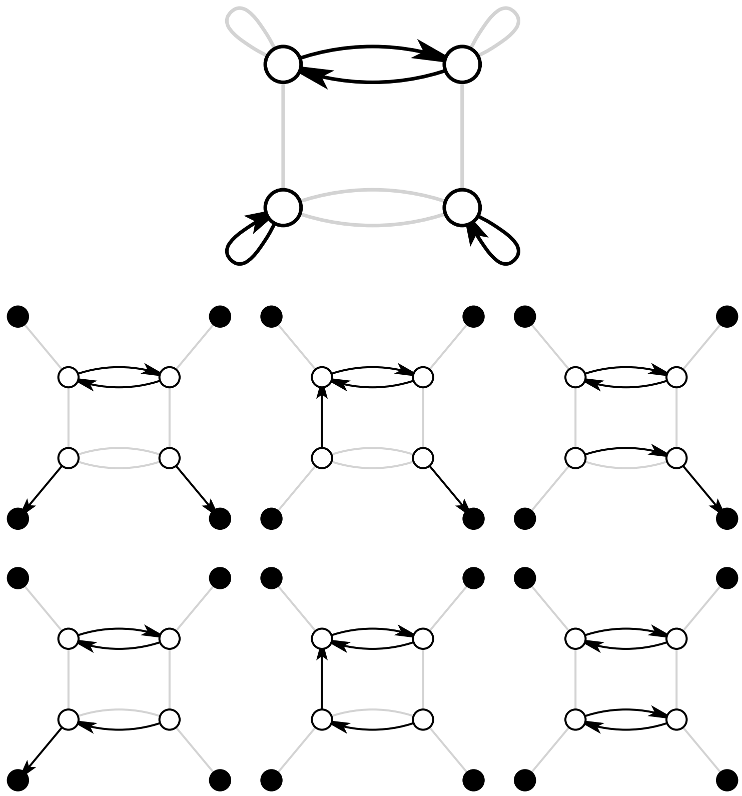


FIGURE 9. A loop partition on $\mathscr{G}_M$, and set of almost-tree diagrams corresponding to it.

Corollary 3.9. *If G is a simple directed electrical network and $M = K(I;I)$, then*

$$\det M = \sum_{\mathscr{C}} \left((-1)^{|\mathscr{C}|} \sum_{\substack{T \in \mathbb{AT}(G) \\ \mathscr{C} \subseteq \mathscr{C}(T)}} w(T) \right),$$

where $\mathscr{C}$ ranges over all sets of disjoint cycles among interior vertices of G.

Proof. According to Lemma 3.6, we may replace the sum over $\mathscr{C}$ by a sum over loop partitions on $\mathscr{G}_M$, with $\mathscr{C} = \mathscr{C}(L)$. Then the desired equation is the one obtained from Proposition 3.4 by plugging in Corollary 3.8. □

We have almost achieved our goal: we have expressed $\det M$ as a sum of weights of almost-tree diagrams, counted with signs and multiplicities. All that remains is to count, for each almost-tree diagram T, the total number of times $w(T)$ appears in Corollary 3.9.

Second proof of Theorem 2.3. Suppose T is an almost-tree diagram, and consider the contribution of $w(T)$ to

$$\det M = \sum_{\mathscr{C}} \left((-1)^{|\mathscr{C}|} \sum_{\substack{T \in \mathbb{AT}(G) \\ \mathscr{C} \subseteq \mathscr{C}(T)}} w(T) \right),$$

which we can alternatively express as

$$= \sum_{k \in \mathbb{N}} \left((-1)^k \sum_{\mathscr{C}: |\mathscr{C}| = k} \sum_{\substack{T \in \mathbb{AT}(G) \\ \mathscr{C} \subseteq \mathscr{C}(T)}} w(T) \right)$$

$$= \sum_{T \in \mathbb{AT}(G)} w(T) - \sum_{\text{cycles } C} \left(\sum_{\substack{T \in \mathbb{AT}(G) \\ C \subseteq T}} w(T) \right) + \sum_{\substack{\text{disjoint cycles} \\ C, C'}} \left(\sum_{\substack{T \in \mathbb{AT}(G) \\ C, C' \subseteq T}} w(T) \right) - \dots.$$

Suppose T contains n cycles. Then T is counted once in the ($k = 0$) sum, but n times in the ($k = 1$) sum (once for each cycle C with $C \subseteq T$). Similarly, T is counted $\binom{n}{2}$ times in the ($k = 2$) sum, once for each pair of cycles C and C' contained in T. Therefore the coefficient of $w(T)$ in the total expression is

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots = (1-1)^n = 0^n$$

$$= \begin{cases} 0 \text{ if } n > 0 \\ 1 \text{ if } n = 0 \end{cases}.$$

Therefore the coefficient of $w(T)$ in the total sum is 0 if T contains any cycles, and 1 if T is acyclic. That is, the only diagrams that remain are the tree diagrams, and they are each counted once:

$$\det M = \sum_{T \in \mathbb{T}(G)} w(T). \qquad \square$$

4. Connections and Tree Diagrams

If K is the Kirchhoff matrix of a directed electrical network G, the tree diagrams formula allows us to compute the determinant of any principal submatrix $K(J; J)$ of K, by regarding vertices J as the interior and summing the weights of the resulting tree diagrams. But what about the determinants of square submatrices of K which are not principal? We can answer through a combination of the determinant-connection formula and the tree diagrams formula.

First, we identify the square submatrix of K whose determinant we wish to compute. Such a submatrix can always be written as $K(S + J; T + J)$ where S, T, and J are disjoint sets of vertices. Since K and its subdeterminants do not depend on the choice of which vertices are boundary and which are interior, we will assume that $J = I$ is the interior of G. Then the important new diagram to consider is called a connection:

Definition 4.1. If G is a directed graph with boundary, whose interior is I, and S and T are disjoint subsets of the boundary vertices, a *connection* C from S to T (through I) is a bijection $\tau_C : S \to T$, and for every $s \in S$ a path $s \to \tau(s) \in T$, such that:

- The paths are disjoint, i.e. every vertex of G is used in at most one path.
- In each path $s \to t$, all vertices used (except s and t) belong to I.

The set of all connections from S to T is denoted $\mathscr{C}(S,T)$.

Then the determinant-connection formula follows:

Theorem 4.2 (Theorem 3.5 from [BiMa10])**.** *Let G be a directed electrical network with conductivities γ, with Kirchhoff matrix K and interior vertices I. If S and T are disjoint subsets of the boundary vertices, both of size k, then*

$$\det K(S+I;T+I) = (-1)^k \sum_{C\in\mathscr{C}(S,T)} \operatorname{sign}(\tau_C)\left(\prod_{e\in C}\gamma(e)\right)\det K(I_{\notin C};I_{\notin C}),$$

where $I_{\notin C}$ are those vertices of I not used in any of the paths in C.

Proof. See [BiMa10]. □

We can use the tree-diagrams formula to expand the determinant of $K(I_{\notin C};I_{\notin C})$, and interpret the resulting terms as weights of certain diagrams. To build such a diagram, we first choose a connection from S to T using some vertices of I. Then for the remaining vertices, we choose a single edge out of each so that each vertex has a path either to the boundary of G or to a vertex used in the connection $S \to T$. This diagram contains a tree diagram on G: Each interior vertex is either used in the connection, in which case it has a unique edge out of it (the next edge in the path), or it is not used in the connection and we have chosen an edge out of it. And each interior vertex has a path to the boundary of G: either it is used in the connection and has a path to a vertex in T, or it is not used, and then has a path directly to the boundary or by way of the vertices in the connection. The only reason the diagram is not a tree diagram is that the first edges along the paths $S \to T$ are edges out of boundary vertices. Since these diagrams are somewhat like trees and somewhat like paths, we call them “hedge diagrams”.

Definition 4.3. Let G be a directed graph with boundary, and S and T disjoint sets of boundary vertices. A *hedge diagram* H from S to T consists of the following:

- A set of edges forming a tree diagram on G.
- A bijection $\tau_H : S \to T$.
- For each vertex $s \in S$, a choice of edge out of s such that H contains a path $s \to \tau_H(s)$.

The set of hedge diagrams from S to T is denoted $\mathscr{H}(S,T)$.

Then according to the above discussion, we can apply the tree diagrams formula to Theorem 4.2 to obtain:

Corollary 4.4. *Let G be a directed electrical network with conductivities γ, with Kirchhoff matrix K. If S and T are disjoint subsets of the boundary vertices, both of size k, then*

$$\det K(S+I;T+I) = (-1)^k \sum_{H\in\mathscr{H}(S,T)} \operatorname{sign}(\tau_H)\left(\prod_{e\in H}\gamma(e)\right).$$

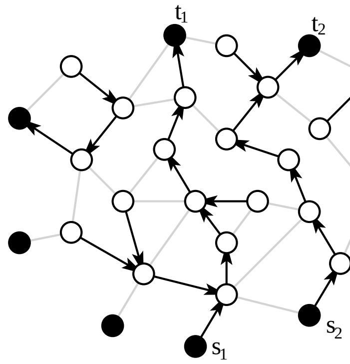


FIGURE 10. A hedge diagram from $S = \{s_1, s_2\}$ to $T = \{t_1, t_2\}$.

In addition, we have a small result analogous to Lemma 2.14.

Lemma 4.5. *Suppose G is a directed graph with boundary, for which every interior vertex has a directed path to the boundary, and make G an electrical network by weighting each edge with a distinct indeterminate conductivity. If S and T are equally-sized subsets of the boundary of G, and there is a connection $S \to T$ in G, then $\det K(S+I;T+I)$ is a nonzero polynomial in the conductivities. Furthermore, this result extends to the case of undirected networks.*

Proof. The existence of hedge diagrams is guaranteed by the assumptions: First choose any connection $C : S \to T$ and any tree diagram U on G. Then letting $U_{\notin C}$ consist of those edges of U whose sources are vertices not used in C, we have a hedge diagram $H = C \cup U_{\notin C}$. Furthermore, as in the proof of Lemma 2.14, if every conductivity is a distinct indeterminate then the weights of the hedge diagrams will be distinct monomials and cannot cancel. And again, no two hedge diagrams can differ only in the directions of the edges used: H will contain a unique connection $S \to T$, and the edges of it are all directed toward T, and the rest of the edges are directed toward the connection or the graph's boundary. So even if opposite edges have the same indeterminate conductivity, no two weights of hedge diagrams will cancel. $\square$

REFERENCES

[BiMa10] Biesel, Owen and Peter Mannisto. "The Connection-Determinant Formula, with Applications to Electrical Networks." Available at
`www.math.washington.edu/~reu/detcon/detcon1.pdf`

[CuMo00] Curtis, Edward B. and James A. Morrow. Inverse Problems for Electric Networks World Scientific Publishing Company, ©2000.