Voltage and current sources at interior nodes of an electrical network

Collin Erickson

8/13/2013

Abstract

This paper explores the recovery of networks that have voltage or current sources at interior nodes of the network. The setup assumes that we are given a network with one of these problems and are only allowed to make measurements and set voltages at boundary nodes. The response matrix can be found for every network with a single voltage source or a single current source. If the original network is critical and does not have any $Y-\Delta$ connections, then the Kirchoff matrix and the original network can be recovered using traditional methods. An attempt is made to identify the malfunctioning node once the network is recovered, but the present method does not guarantee a solution.

Contents

1 Introduction

In this paper we will consider electrical networks that are not functioning properly. The electrical network will be malfunctioning because an interior node will be stuck at some constant voltage or current, which can be thought of as the node being attached to a voltage or current source. In the cases we will consider, we will not know the graph, which node is malfunctioning, nor what the constant voltage or current is at the malfunctioning interior node, but we will have access to the network and will be able to set the boundary voltages and make current measurements at the boundary nodes.

The questions we want to answer are:

1. Can we determine whether the malfunctioning node is a voltage or current source? Can we identify the voltage or current of the malfunctioning node?

2. Can we recover the original graph if there is one malfunctioning node? If yes, can we also determine which node is the malfunctioning node?

3. Can we do these if there are multiple interior nodes that are malfunctioning?

2 Preliminaries

This is a somewhat practical question. It is not uncommon to have some circuit where some wires are touching that are not supposed to be. Many circuits require voltage sources, usually batteries, within a circuit in order for certain chips to operate, and it is easy for wires to touch when there is a circuit with a large amount of components put into a small box. However, any useful circuit would have components other than resistors, and we are only considering resistor networks.

Definition 1. *The leakage current is defined to be the net amount of current through the boundary nodes. If there are m boundary nodes, the voltage at the boundary nodes is u_B, and the current at boundary node i is ϕ_i, then the leakage current, $\mathcal{L}$, is:* $\mathcal{L}(u_B) = \sum_{i=1}^{m} \phi_i$.

A special case that will come up is when there are m boundary nodes and a single interior node, called ν_β, that does not follow Kirchoff's law. The net current flowing through the network is always zero, so in this case another formula for the leakage current is $\mathcal{L}(u_B) = -\phi$

In the study of electrical networks the Kirchoff matrix, K, and the response matrix, Λ, are often used. The Kirchoff matrix gives the conductance of wires connecting each pair of nodes in the network. The response matrix is then created by taking the Schur complement of K with respect to the interior nodes, $\Lambda = K/K(I;I)$, where $I = \nu_1, \ldots, \nu_n$ is the set of interior nodes. The response matrix can also be created by first taking the Schur complement of K with respect to some of the interior nodes, and then taking the Schur complement of the result with respect to the rest of the interior nodes. The order,the number of interior nodes, or the number of times the Schur complement is taken does not matter as long as the each interior node is Schur complemented out once in the process. The entries of the response matrix can be calculated by setting the voltage at one boundary node to be one and the rest to be zero and measuring the result. If the response matrix if found this way, then K can be found by using recovery methods.

In this paper we will use the matrix Ξ to represent one such matrix intermediary to the Kirchoff and the response matrix. Ξ will usually represent what results from taking the Schur complement of K with respect to all of the interior nodes except for one. Sometimes multiple interior nodes will be included in Ξ. It should be clear by the context how many nodes, and which nodes, are included in Ξ. Sometimes the notation Ξ^J will be used to denote that the set of interior nodes J have been included in Ξ, so $\Xi^J = K/K(I-J; I-J)$. Once Ξ is known, Λ can be found by taking the Schur complement of Ξ with respect to the remaining interior nodes. Also the values of Ξ can be measured in a similar way to the response matrix, except that the interior nodes of Ξ must also be treated and measured the same way.

3 Identifying whether a malfunctioning node is at constant current or constant voltage

Having a voltage source within a network is different than having a current source. A current source will provide a constant current regardless to what it is connected to. In contrast, the current provided by a voltage source varies depending on its connections, but its voltage will be constant. In both cases Kirchoff's law is not followed. It is simple to determine whether a malfunctioning network has an interior current source, interior voltage source, or neither. This can be done by measuring the net current at the boundary nodes after setting all boundary voltages to 0 and also after setting all boundary voltages to 1. If the net current out the boundary nodes is zero for both cases, then the network is functioning properly. If it is the same for both cases and nonzero, then there is a current source inside the network. If it is different for both cases, then there is an interior voltage source.

The same method will work if there are multiple current sources or multiple voltage sources in the network. However, we may not be able to tell the number of malfunctioning nodes in the network.

4 A single interior node is at a constant voltage

Suppose we have a graph, G, with m boundary nodes, $\mu_1, \ldots, \mu_m$, and $n+1$ interior nodes, $\nu_1, \ldots, \nu_n, \nu_\beta$. Suppose that an interior node, which we will arbitrarily designate as $\nu_\beta$is stuck at a constant voltage k. For now we will assume $k \neq 0$, although the following can easily be adjusted for that case because the voltage is always defined relative to an arbitrary constant. Our goal is to try to identify which interior node is malfunction. However, in order to do this we must recover the original graph.

4.1 Finding the voltage of the voltage source

The first thing we will do is determine the value of k.

Theorem 1. *The value of k can easily be determined by the following process. Set the voltage of all of the boundary nodes to be V. Vary the value of V until $\mathcal{L} = 0$. Then $k = V$.*

Proof. If the voltage at all of the boundary nodes are V and the voltage at $\nu_\beta$is k, then the leakage current can be calculated as follows. First we must find the current at each of the boundary nodes. Recall that the row and column sums of Ξ are zero.

$$\Xi \begin{bmatrix} V \\ \vdots \\ V \\ k \end{bmatrix} = \begin{bmatrix} V\sum_{i=1}^{m}\xi_{1i} + k_{1b} \\ \vdots \\ V\sum_{i=1}^{m}\xi_{mi} + k_{mb} \\ V\sum_{i=1}^{m}\xi_{bi} + k_{bb} \end{bmatrix} \begin{bmatrix} -V\xi_{1b} + k\xi_{1b} \\ \vdots \\ -V\xi_{mb} + k\xi_{mb} \\ -V\xi_{bb} + k\xi_{bb} \end{bmatrix} \begin{bmatrix} (k-V)\xi_{1b} \\ \vdots \\ (k-V)\xi_{mb} \\ (k-V)\xi_{bb} \end{bmatrix}$$

These first m currents are summed to find the leakage current.

$$\mathcal{L}(\begin{bmatrix} V \\ \vdots \\ V \\ k \end{bmatrix}) = \sum_{i=1}^{m}(k-V)\xi_{ib} = (k-V)\sum_{i=1}^{m}\xi_{ib} = (V-k)\xi_{bb}$$

When the leakage current is zero, than either $(V-k)$ or ξ_{bb} are zero. ξ_{bb} cannot be zero because that would mean there is no path from nubd to the boundary in the original graph. Thus, we must have $(V-k)$=0, so $k = V$ only at the voltage that causes the leakage current, which can be measured, to be zero. □

While this method gives a way to find k, it is not very efficient. The following theorem gives another way to find k.

Theorem 2. *If L is the ratio of the leakage current when all boundary nodes are set to voltage V_1 to the leakage current when all boundary nodes are set to voltage V_2, then $k = \frac{V_2 - V_1}{L-1}$.*

Proof. Let $\vec{V} = \begin{bmatrix} V \\ \vdots \\ V \\ k \end{bmatrix}$. Then we will be able to measure $\mathcal{L}(V_1)$ and $\mathcal{L}(V_2)$, where $\mathcal{L}(V_1) = (V_1 - k)\xi_{bb}$ and $\mathcal{L}(V_2) = (V_2 - k)\xi_{bb}$. If one of these is zero, then that voltage is what k is, as was explained above. Otherwise, we will define L to be the ratio of these two measured values.

$$L = \frac{\mathcal{L}(V_1)}{\mathcal{L}(V_2)} = \frac{(V_1 - k)\xi_{bb}}{(V_2 - k)\xi_{bb}} = \frac{V_1 - k}{V_2 - k}$$

Solving for k gives $k = \frac{V_2 - V_1}{L-1}$. □

4.2 Finding the response matrix

Now that we have the value of k, we can continue the process and try to reconstruct the original graph. In order to recover the graph, we want the response matrix, Λ, for the graph. However, we are not able to measure the values of the response matrix as we would in the normal case because of the voltage source at ν_β. We will proceed by creating the Ξ matrix for ν_β. If K is the Kirchoff matrix for G, then Ξ can be obtained by taking a Schur complement:

$$\Xi = K/K(\nu_1, \ldots, \nu_n; \nu_1, \ldots, \nu_n) = \begin{bmatrix} \xi_{\mu_1,\mu_1} & \xi_{\mu_1,\mu_2} & \cdots & \xi_{\mu_1,\mu_n} & \xi_{\mu_1,\nu_\beta} \\ \xi_{\mu_2,\mu_1} & \xi_{\mu_2,\mu_2} & \cdots & \xi_{\mu_2,\mu_n} & \xi_{\mu_2,\nu_\beta} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi_{\mu_n,\mu_1} & \xi_{\mu_n,\mu_2} & \cdots & \xi_{\mu_n,\mu_n} & \xi_{\mu_n,\nu_\beta} \\ \xi_{\nu_\beta,\mu_1} & \xi_{\nu_\beta,\mu_2} & \cdots & \xi_{\nu_\beta,\mu_n} & \xi_{\nu_\beta,\nu_\beta} \end{bmatrix}$$

Ξ is essentially the response matrix for G if we considered the interior node $\nu_\beta$to be a boundary node. In some ways $\nu_\beta$acts as a boundary node because it's voltage is not determined by the voltage of any other nodes. However, like other interior nodes we cannot measure the voltage or current at ν_β. The following calculation treats $\nu_\beta$as a boundary node. We will call H the graph that is the same as G except $\nu_\beta$is a boundary node. Ξ is the response matrix for the graph H. Note that Λ can be obtained from Ξ by taking another Schur complement.

We will find the values of Ξ by applying voltages to the boundary nodes and measuring the current at the boundary nodes. The voltage at $\nu_\beta$will always be k because it cannot be changed, but it is simple to work around this. We also will not be able to measure the current at ν_β. We can find most of the values of Ξ by placing a single unit voltage at a boundary node and subtracting the measured currents when the voltage at all boundary nodes is 0.

$$\Xi \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ k \end{bmatrix} - \Xi \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k \end{bmatrix} = \Xi e_i = \begin{bmatrix} \xi_{\mu_1,\mu_i} \\ \xi_{\mu_2,\mu_i} \\ \vdots \\ \xi_{\mu_n,\mu_i} \\ \xi_{\nu_\beta,\mu_i} \end{bmatrix}$$

By doing this we can find all of elements of Ξ except for the last row and column. What we are doing here is taking two current measurements at each boundary node, and their difference gives m values in Ξ. We repeat this process for every boundary node, changing with boundary node we set to one.

Since we know the value of k, we can also find the values of Ξ except for the bottom-right value, $\xi_{\nu_\beta,\nu_\beta}$, by just setting all boundary voltages to 0 and then dividing the result by k.

$$\frac{1}{k}\Xi \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{1}{k} \begin{bmatrix} k\xi_{\mu_1,\nu_\beta} \\ k\xi_{\mu_2,\nu_\beta} \\ \vdots \\ k\xi_{\mu_n,\nu_\beta} \\ k\xi_{\nu_\beta,\nu_\beta} \end{bmatrix} = \begin{bmatrix} \xi_{\mu_1,\nu_\beta} \\ \xi_{\mu_2,\nu_\beta} \\ \vdots \\ \xi_{\mu_n,\nu_\beta} \\ \xi_{\nu_\beta,\nu_\beta} \end{bmatrix}$$

We can then find the last value in Ξ because the row and columns must be zero. (Note that we have n equations for the row/column sums of the first n rows/columns that we haven't used. We could have found the values for the last row/column of Ξ using those instead.) Now that we have all of the values of Ξ, we can find the Schur complement to find the response matrix of the original graph, Λ. This response matrix can be used to recover the original network.

4.3 Recovery and identification of the voltage source node

We must realize that we may not get what we want in our recovered graph. Since recovery is only up to $Y - \Delta$ equivalences, our malfunctioning node may not even be in the graph we recovered. For now, we will assume that there are no $Y - \Delta$ equivalences in the original graph.

Our task now that we know the original network is to determine which node is malfunctioning. To do this, we will first set all of the boundary voltages to 0. Without loss of generality, we will assume $k > 0$. Now we will use the recovered graph and find the voltage at all of the interior nodes by working inward from the boundary nodes. We know all of the conductances and the voltages and currents at the boundary nodes. Using this information we will work inward from each boundary node. At each interior node we come to, we will be able to determine the voltage.

5 Multiple interior nodes are voltage sources

Now we want to consider the situation in which two interior nodes are malfunctioning and are stuck at constant voltages. We will call these nodes ν_{β_1} and ν_{β_2}, and their voltages k_1 and k_2. We will create the matrix Ξ in a similar way, so that $\Xi = \begin{bmatrix} \xi_{\mu_1,\mu_1} & \xi_{\mu_1,\mu_2} & \cdots & \xi_{\mu_1,\mu_n} & \xi_{\mu_1,\nu_{\beta_1}} & \xi_{\mu_1,\nu_{\beta_2}} \\ \xi_{\mu_2,\mu_1} & \xi_{\mu_2,\mu_2} & \cdots & \xi_{\mu_2,\mu_n} & \xi_{\mu_2,\nu_{\beta_2}} & \xi_{\mu_2,\nu_{\beta_2}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ \xi_{\mu_n,\mu_1} & \xi_{\mu_n,\mu_2} & \cdots & \xi_{\mu_n,\mu_n} & \xi_{\mu_n,\nu_{\beta_1}} & \xi_{\mu_n,\nu_{\beta_2}} \\ \xi_{\nu_{\beta_1},\mu_1} & \xi_{\nu_{\beta_1},\mu_2} & \cdots & \xi_{\nu_{\beta_1},\mu_n} & \xi_{\nu_{\beta_1},\nu_{\beta_1}} & \xi_{\nu_{\beta_1},\nu_{\beta_2}} \\ \xi_{\nu_{\beta_2},\mu_1} & \xi_{\nu_{\beta_2},\mu_2} & \cdots & \xi_{\nu_{\beta_2},\mu_n} & \xi_{\nu_{\beta_2},\nu_{\beta_1}} & \xi_{\nu_{\beta_2},\nu_{\beta_2}} \end{bmatrix}$. In order to see if we can find all unknown values and recover the original graph, we will see how many unknowns and how many equations we have available. Initially we do not know any values of Ξ (which is an $(m+2)$ by $(m+2)$ matrix), but we do know that it is symmetric. We can find all values except for the last two rows and columns by using a similar process to before. We first set all of the boundary voltages to zero except for one value we set to 1. From the resulting current at each boundary node we subtract the current that results from all of the boundary nodes having voltage 0. By doing this for each of the boundary nodes, we will get most of Ξ. For example, to get row i of Ξ, for $1 \leq i \leq m$:

$$\Xi \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ k_1 \\ k_2 \end{bmatrix} = \Xi \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_1 \\ k_2 \end{bmatrix} = \Xi e_i =$$

$$\begin{bmatrix} \xi_{\mu_1,\mu_1} & \xi_{\mu_1,\mu_2} & \cdots & \xi_{\mu_1,\mu_n} & \xi_{\mu_1,\nu_{\beta_1}} & \xi_{\mu_1,\nu_{\beta_2}} \\ \xi_{\mu_2,\mu_1} & \xi_{\mu_2,\mu_2} & \cdots & \xi_{\mu_2,\mu_n} & \xi_{\mu_2,\nu_{\beta_2}} & \xi_{\mu_2,\nu_{\beta_2}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ \xi_{\mu_n,\mu_1} & \xi_{\mu_n,\mu_2} & \cdots & \xi_{\mu_n,\mu_n} & \xi_{\mu_n,\nu_{\beta_1}} & \xi_{\mu_n,\nu_{\beta_2}} \\ \xi_{\nu_{\beta_1},\mu_1} & \xi_{\nu_{\beta_1},\mu_2} & \cdots & \xi_{\nu_{\beta_1},\mu_n} & \xi_{\nu_{\beta_1},\nu_{\beta_1}} & \xi_{\nu_{\beta_1},\nu_{\beta_2}} \\ \xi_{\nu_{\beta_2},\mu_1} & \xi_{\nu_{\beta_2},\mu_2} & \cdots & \xi_{\nu_{\beta_2},\mu_n} & \xi_{\nu_{\beta_2},\nu_{\beta_1}} & \xi_{\nu_{\beta_2},\nu_{\beta_2}} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \xi_{\mu_i,1} \\ \vdots \\ \xi_{\mu_i,m} \\ \xi_{\mu_i,\beta_1} \\ \xi_{\mu_i,\beta_2} \end{bmatrix}$$

We still do not know the last two rows or columns, so since Ξ is symmetric there are $2m+3$ unknown values in Ξ. In addition, we do not know the values of k_1 and k_2. So in total there are $2m+5$ unknown values: $k_1, k_2, \xi_{\mu_1,\beta_1}, \xi_{\mu_2,\beta_1}, \ldots, \xi_{\mu_m,\beta_1}, \xi_{\beta_1,\beta_1}, \xi_{\mu_1,\beta_2}, \xi_{\mu_2,\beta_2}, \ldots, \xi_{\mu_m,\beta_2}, \xi_{\beta_1,\beta_2}, and \xi_{\beta_2,\beta_2}$.

For equations, we have $m+2$ equations resulting from the row sums having to be zero, and we have m equations from measuring the current that result from setting all of the boundary voltages to zero. The latter equations are of the form $k_1\xi_{\mu_i,\beta_1} + k_2\xi_{\mu_i,\beta_2} =$measured value. So in total we have $2m+2$ equations.

In total, this leaves us with $2m+5$ unknown values and $2m+2$ equations, so there is no possible way for us to solve this general system. By setting certain initial conditions, there may be useful solutions to this system that can be found.

One specific case which we may want to attempt to solve is that of when there are two interior nodes stuck at some unknown voltage, but we know that they are at the same voltage k, so $k = k_1 = k_2$. We can find k by using the same method as used with a single voltage source. This knowledge gives us two more equations, which would leave us one equation short of being able to find Ξ and solve the system. However, we actually lost equations since the row sum equations and the measurements made with all boundary nodes set to voltage 0 are no longer linearly independent. Thus, although we would be able to find the voltage, we have about twice as many unknown values as we have equations.

The problem of too many unknown values and too few equations gets worse if we add more unknown interior voltage sources; the number of unknown values grows faster than the number of equations we have to find them.

6 Current Source

Now we will consider the case when an interior node is attached to a current source. This means that Kirchoff's law is not satisfied only at that one interior node, and that the net current out of the node is constant. We will assume the amount of current flowing into the network through that node, denoted by I is nonzero. Since the current there is always flowing from that node at the same rate, the voltage at the node will change depending on the voltages of the other nodes.

Suppose we are given a critical circular planar network with m boundary nodes and a single interior node, which we will call node b, that is a current source with current I, which is initially not given. We are allowed to set the voltages at the boundary nodes and measure the currents at the boundary nodes, but we cannot access the interior node. We want to answer the following questions:

1. Can we determine I?

2. Can we recover the original network?

3. Can we identify which node in the original network was malfunctioning?

6.1 Determining I

Because the amount of current through node b is always I, the amount of current going out of the boundary nodes must sum to I. Thus, finding I

is trivial and can be done by measuring the current at each boundary node (while leaving the voltages at any set of values) and summing the measurements.

6.2 Recovering the original network

We will let Ξ be the matrix that represents what the response matrix if node b is treated as a boundary node. This is equivalent to taking the Schur complement of the Kirchoff matrix with respect to all of the interior nodes except for node b. If we are able to find Ξ, then we will be able to find the response matrix, Λ for the original network by taking the Schur complement with respect to the last row and column.

$$\Xi = \begin{bmatrix} \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,m} & \xi_{1,b} \\ \xi_{2,1} & \xi_{2,2} & \cdots & \xi_{2,m} & \xi_{2.b} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi_{m,1} & \xi_{m,2} & \cdots & \xi_{m,m} & \xi_{m,b} \\ \xi_{b,1} & \xi_{b,2} & \cdots & \xi_{b,m} & \xi_{b.b} \end{bmatrix}$$

We will work toward our goal will by first determining every value of Ξ in terms of ξ_{bb}.

If we apply voltages u to the boundary nodes, then the voltage at node b will be determined as a function of u, which we will call $f(u)$. We want to determine the function f.

$$\Xi u = \Xi \begin{bmatrix} u_1 \\ \vdots \\ u_m \\ f(u) \end{bmatrix} = \begin{bmatrix} u_1\xi_{1,1} & \cdots & u_m\xi_{1,m} & f(u)\xi_{1,b} \\ \vdots & \ddots & \vdots & \vdots \\ u_1\xi_{m,1} & \cdots & u_m\xi_{m,m} & f(u)\xi_{m,b} \\ u_1\xi_{b,1} & \cdots & u_m\xi_{b,m} & f(u)\xi_{b,b} \end{bmatrix} = \begin{bmatrix} I_1 \\ \vdots \\ I_m \\ I \end{bmatrix}$$

In this equation, the values I_1,...,I_m could be measured for any u. This would allow us to use any row to determine f in terms of other values. To keep it simple we will use the last row.

$$f(u) = \frac{I - u_1\xi_{b,1} - \cdots - u_m\xi_{b,m}}{\xi_{b,b}} = \frac{I - \displaystyle\sum_{i=1}^{m} u_i\xi_{b,i}}{\xi_{b,b}} = \frac{I - \sum_{i=1}^{m} u_i\xi_{bi}}{\xi_{bb}}$$

Now we will put a voltage of zero at all of the boundary voltages. According to the equation for f, the voltage at node b will be I/ξ_{bb}.

$$\Xi \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I/\xi_{bb} \end{bmatrix} = \begin{bmatrix} I\xi_{1b}/\xi_{bb} \\ \vdots \\ I\xi_{mb}/\xi_{bb} \\ I \end{bmatrix} = \begin{bmatrix} h_1 \\ \vdots \\ h_m \\ I \end{bmatrix}$$

The values $h_1,\ldots,h_m$ can be measured. This is important because now we can find each value in the last row and column of Ξ in terms of ξ_{bb}.

$$\xi_{ib} = \xi_{bi} = \frac{h_i}{I}\xi_{bb}$$

Also, since the net current at all of the nodes must sum to zero, we have $\sum_{i=1}^{m} h_i = -I$. Often the ratios of these values are easier to use, so we will define a set of values k_{ij} for $1 \leq i, j \leq m$ (for the rest of the section assume that i, j, and l are all between 1 and m inclusive).

$$k_{ij} = \frac{h_i}{h_j} = \frac{I\xi_{ib}/\xi_{bb}}{I\xi_{jb}/\xi_{bb}} = \frac{\xi_{ib}}{\xi_{jb}} = \frac{1}{k_{ji}}$$

A useful relation is gained by summing k_{ij}'s over one of the values.

$$\sum_{i=1}^{m} k_{ij} = \sum_{i=1}^{m} \frac{h_i}{h_j} = \frac{1}{h_j}\sum_{i=1}^{m} h_i = \frac{-I}{h_j}$$

Using these ratios gives another way to express ξ_{ib} in terms of $\xi_b b$ by using the fact that row sums are zero.

$$\sum_{j=1}^{m} \xi_{jb} = -\xi_{bb} = \sum_{j=1}^{m} \frac{\xi_{ib}}{k_{ij}} = \xi_{ib}\sum_{j=1}^{m} k_{ji}$$

$$\xi_{ib} = \frac{-\xi_{bb}}{\sum_{j=1}^{m} k_{ji}}$$

Or by looking at the values of h gives another formula for the same value.

$$\xi_{ib} = \frac{h_i}{I}\xi_{bb}$$

Now that all of the values of the last row and column can be found in terms of ξ_{bb}, we must find the rest of the values of values of Ξ in terms of the entries in the last row or column. This will be done by applying specific voltages to the boundary nodes and measuring the resulting current. First we will apply a unit voltage to boundary node i (with voltage zero at the other boundary nodes) and measure the current at each of the boundary nodes. From this value we will subtract the curent resulting from applying voltage k_{ij} to boundary node j (with voltage zero at the other boundary nodes). We will call the resulting vector p^{ij}.

$$\Xi \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \frac{I-\xi_{bi}}{\xi_{bb}} \end{bmatrix} - \Xi \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_{ij} \\ 0 \\ \vdots \\ 0 \\ \frac{I-\xi_{bi}}{\xi_{bb}} \end{bmatrix} = \Xi \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -k_{ij} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} =$$

$$\Xi(e_i - k_{ij}e_j) = \begin{bmatrix} \xi_{1i} - k_{ij}\xi_{1j} \\ \vdots \\ \xi_{mi} - k_{ij}\xi_{mj} \\ I \end{bmatrix} p^{ij} = \begin{bmatrix} p_1^{ij} \\ \vdots \\ p_m^{ij} \\ I \end{bmatrix}$$

From these equations we can relate values of Ξ that are in the same row. So for any l between 1 and m:

$$\xi_{li} - k_{ij}\xi_{lj} = p_l^{ij}$$

$$\xi_{li} = p_l^{ij} + k_{ij}\xi_{lj}$$

Now we can use the fact that row sums are zero to find any value of Ξ in terms of another value of Ξ that is in the last row or column.

$$\sum_{i=1}^{m} \xi_{li} = -\xi_{lb} = \sum_{i=1}^{m} (p_l^{ij} k_{ij}\xi_{lj}) = \sum_{i=1}^{m} p_l^{ij} + \xi_{lj} \sum_{i=1}^{m} k_{ij}$$

$$\xi_{lj} = \frac{-(\sum_{i=1}^m p_l^{ij}) - xi_{lb}}{\sum_{i=1}^m k_{ij}} = \frac{-\sum_{i=1}^m p_l^{ij}}{\sum_{i=1}^m k_{ij}} + \frac{\xi_{bb}}{(\sum_{i=1}^m k_{ij})(\sum_{i=1}^m k_{il})} =$$

$$\frac{h_j}{I}\sum_{i=1}^m p_l^{ij} + \frac{h_j h_l}{I^2}\xi_{bb}$$

We now have an equation for every value of Ξ in terms of measured values (currents) and the variable ξ_{bb}.

We can use these values to find the values of the response matrix Λ by taking the Schur complement of Ξ with respect to the last row and column.

$$\Lambda = \begin{bmatrix} \xi_{1,1} & \cdots & \xi_{1,m} \\ \xi_{2,1} & \cdots & \xi_{2,m} \\ \vdots & \ddots & \vdots \\ \xi_{m,1} & \cdots & \xi_{m,m} \\ \xi_{b,1} & \cdots & \xi_{b,m} \end{bmatrix} - \begin{bmatrix} \xi_{1,b} \\ \vdots \\ \xi_{m,b} \end{bmatrix} \begin{bmatrix} \xi_{b,b} \end{bmatrix}^{-1} \begin{bmatrix} \xi_{1,b} & \vdots & \xi_{m,b} \end{bmatrix}$$

$$\Lambda_{lj} = \xi_{lj} - \xi_{lb}\frac{1}{\xi_{bb}}\xi_{jb} = \frac{h_j}{I}\sum_{i=1}^m p_l^{ij} + \frac{h_j h_l}{I^2}\xi_{bb} - \frac{h_l}{I}\xi_{bb}\frac{1}{\xi_{bb}}\frac{h_j}{I}\xi_{bb} =$$

$$\frac{h_j}{I}\sum_{i=1}^m p_l^{ij} + \frac{h_j h_l}{I^2}\xi_{bb} - \frac{h_j h_l}{I^2}\xi_{bb} = \frac{h_j}{I}\sum_{i=1}^m p_l^{ij}$$

Miraculously this shows that the response matrix does not depend on ξ_{bb}. Instead every value of Λ can be found by making various measurements at the boundary nodes of the broken graph. Although it is not obvious from the equation, it can easily be shown that, as expected, $\Lambda_{ij} = \Lambda_{ji}$. We can now take this response matrix and recover the original graph without the current sink. All that remains is to identify which interior node was the current sink.

7 Recovering the original network after finding the response matrix

In this section we will attempt to determine which interior node was malfunctioning when only a single node is malfunctioning. We will assume that we have the Ξ^i matrix for the interior node i, which allows us to get the response matrix, Λ, and then the original Kirchoff matrix.

We can immediately see that there will be some problems with recovering the original network. One problem is that the recovery gives us a critical graph. This means that if we did not know that we started with a critical graph, it will be impossible to find the node. Thus our first assumption will be to assume that the network that we start with is critical. The recovered network also only gives a network up to $Y - \Delta$ equivalence. This can be very problematic. Assume that the malfunctioning was at the center of a 3-connection. The graph we recover may change this Y into a Δ, which means that the node we are trying to locate is not even in the network. This problem can be largely be avoided by assuming that the broken node is not in a Y or Δ, or maybe just that is not at the center of a 3-connection, but we will assume that there are no $Y - \Delta$ equivalences in the entire network.

One way that we can determine what interior nodes might possibly be the malfunctioning node i, is to use the Kirchoff matrix to get the Ξ^j matrix for every interior node j. If $\Xi^i \neq \Xi^j$, then j cannot possibly be i. While this eliminates nodes that definitely are not i, it does not guarantee that we will get a unique answer.

While we cannot mathematically find a way to prove that Ξ^i is unique, it seems that it is unique in nearly all cases. This topic needs to be studied more in-depth.

8 Further topics to study

1. Is there a way to determine which node was malfunctioning that works in all cases?

2. Can partial recovery be used when not all of the response matrix can be found, or can only be found in terms of variables?

3. Can networks with two or more current sources be recovered? How about networks with a combination of voltage sources and current sources?

4. How are $Y - \Delta$ equivalences affected by having a source at the center node?