Medial Graphs and Y-Delta Equivalences (Preliminary Version)

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August 6, 2013

Abstract

We use certain tools related to chord diagrams to give a discussion of critical circular planar electrical networks with no *Y-Delta* equivalents. As a corollary we have that these medial graphs can be given a characterization which enables us to uniquely determine the electrical netwok from the $\mathbb{Z}$-sequence rather than the graph. This can be easily generalized to arbitrary medial graphs.

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1 Background

We use this section to give some initial definitions. Throughout this paper, let $G = (E, V)$ be a circular planar electrical network, by which we mean that we can decompose V into sets ∂V and $intV$, such that the vertices ∂V can be embedded on $\partial\mathbb{D}$ such that G remains a planar graph.

We will define the medial graph as follows, taking the definition from [1].

Definition Let G be a connected circular planar graph. with n boundary nodes $v_1, \cdots v_n$ in counterclockwise order around $\partial\mathbb{D}$,and the rest of G staying in $int\mathbb{D}$. For each edge e of G, let m_e be its midpoint. Now place $2n$ points $t_1, \cdots t_{2n}$ around $\partial\mathbb{D}$ such that

$$t_1 < v_1 < t_2 < t_3 < v_2 \cdots < t_{2n-1} < v_n < t_{2n} < v_1$$

within the embedding on $\partial\mathbb{D}$,

(1) The vertices of the medial graph M consist of the points m_e for all edges e.

(2) If e and f are edges in G with a common vertex, incident to the same face in G, the line $m_e m_f$ is an edge in M. For each t_j on the boundary circle, t_{2i} is joined to m_e where e is the edge of the form $e = v_i w$ which comes first after the line $v_i t_{2i}$ in counterclockwise order around v_i; the point t_{2i-1} ic joined by an edge to m_f where f is the edge of form $v_i w$ which comes first after the line $v_i t_{2i-1}$ in clockwise order around v_i.

We begin by citing some theorems of [1] which link networks and medial graphs.

Theorem 1.1 (Theorem 8.3) *A circular planar graph G is critical if and only if its medial graph M_G is lensless.*

Recall that we call a circular planar graph G critical if the deletion of any edge in E breaks a connection in G.

Theorem 1.2 (Lemma 9.3) *Suppose that a medial graph M has one or more lenses. Then by a finite sequence of uncrossing arcs, M can be reduced to be lensless.*

Combined, this means that any medial graph M is equivalent to a medial graph M_G of a critical circular planar graph G. Throughout, we will only consider such medial graphs M corresponding to circular planar graphs. Therefore, they are lensless.

Another theorem will establish a unifying characteristic of such graphs.

Theorem 1.3 (Lemma 8.6) *Suppose $M \subset \mathbb{D}$ is lensless, and that each geodesic in M intersects at least one other chord in M. Then there are at least three empty boundary triangles with disjoint interiors.*

Three empty boundary triangles with disjoint interiors correspond to any combination of three or more boundary edges or spikes. Therefore, the resulting graph from M is critical, as necessary.

Lastly we establish the usage of the $\mathbb{Z}$-sequence as an invariant.

Theorem 1.4 (Lemma 8.7) *Suppose that $\mathcal{A} = \{\alpha_i\}$ and $\mathcal{B} = \{\beta_i\}$ are two lensless families of n geodesics in $\mathbb{D}$ and for each i the endpoints of α_i are the endpoints of β_i. Then $\mathcal{A}$ and $\mathcal{B}$ are equivalent by switches of chords.*

Remark We will always use the designation that the $\mathbb{Z}$-sequence tabulates the geodesic endpoints in a counterclockwise manner.

The above theorem implies that the $\mathbb{Z}$-sequence produces a class of electrically equivalent medial graphs, i.e. a network and its $Y - Delta$ equivalents. We wish to examine graphs with no $Y - Delta$ equivalents.

Y-Delta Transformation A Y is a 3-star, and a Delta is K_3, i.e. a triangle. A Y-Delta transformation takes a 3-star, deletes the root vertex, and turns the other three vertices into the complete graph on three vertices. The inverse is similarly defined.

That is to say,

and

We can also write the conductivities of the edges of the Y as rational functions of the conductivities of the *Delta*, and vice versa. This establishes the idea that the two graphs are electrically equivalent, as they also give the same response matrix.

In terms of the medial graph, we have the following two local images.

and after the motion

We say that the two medial graphs have undergone an <u>motion</u>, since we have shifted a single geodesic by changing the relative positions of the intersections.

Using the same coloring, this corresponds to a Y-Delta transformation. The careful reader will note that the Y is the dual of the Delta, but duality is not an issue which concerns us at this point. We will also note that we have colored the same upper-left cell in the medial graph in both cases, so what we did does not correspond to taking the dual of the graph.

If no motions exist for M_G, then the original graph G has no Y-Delta equivalents.

This gives rise to the following claim.

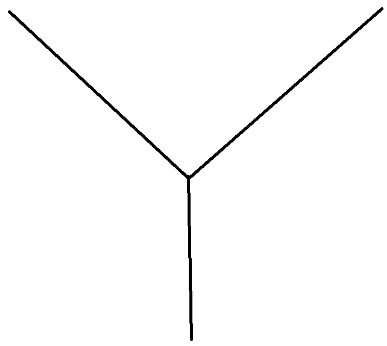


Figure 1: Example Y

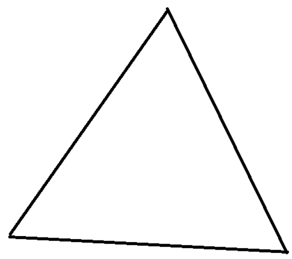


Figure 2: Example Delta

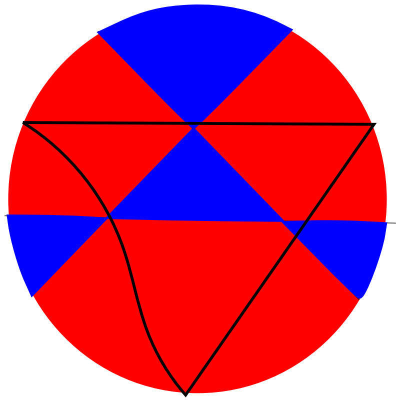
Claim A medial graph has no Y-Delta equivalents if and only if there exists no crossing of three geodesics as above.

Proof If a crossing of three geodesics as above exists, this corresponds to a Y-Delta transformation as demonstrated.

Conversely, if no such crossing exists, then there is no Y-Delta equivalent.

In particular, if the graph has a Y or a Δ, then the medial graph has an empty triangle. In particular, the medial graph has a triangle, and if the triangle isn't empty, there is a geodesic passing through it, that intersects two edges of the triangle, creating a smaller triangle. Therefore, there will be an empty triangle in the medial graph.

We will seek to give a count of medial graphs with n geodesics with no such crossings of three geodesics.



Graph 1.pdf

Figure 3: Initial medial graph

2 Definitions

Throughout, let M be a medial graph that represents a critical circular planar graph. In particular, it is lenseless.

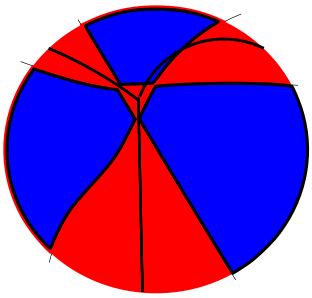
Definition Let α and β be geodesics in M. Say that α and β are parallel if they do not intersect, and incident if they intersect. On this basis, we can also define the incidence graph as follows.

If M has n geodesics, number them 1 to n in some fashion. Draw an edge between i and j if geodesics i and j are incident. The incidence graph is connected as long as the medial graph is connected.

Any two geodesics are either incident or parallel. We can also define these in terms of the $\mathbb{Z}$-sequence. Suppose our medial graph has n geodesics, and the $\mathbb{Z}$-sequence takes some form of $1 \cdots_1 1 \cdots_2$, where $\cdots_1$, $\cdots_2$ are sequences of integers, possibly empty.

A geodesic k taking both vertices in one of the $\cdots_1$ or $\cdots_2$, assuming lenseless-ness in the medial graph, will not intersect geodesic 1. Conversely, a geodesic k taking one vertex in $\cdots_1$ and one in $\cdots_2$ will intersect geodesic 1.

Given our assumption of a lenseless medial graph, we now define the parallel and incidence graphs.



Graph 1'.pdf

Figure 4: Post-Motion medial graph

Remark Parallel-ness is not an equivalence relation, as if geodesics γ and β are parallel to geodesic α, they need not necessarily be parallel, by the following example.

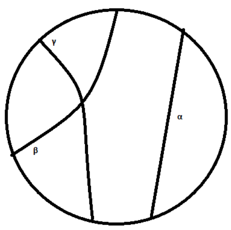


Figure 5

Despite parallel-ness not being an equivalence relation, it is clear that three geodesics intersecting is equivalent to there being a triangle in the incidence graph. Therefore, a medial graph with no three geodesics mutually intersecting corresponds to a triangle-free incidence graph.

Of course, at this point we have no idea whether the map between these medial graph and triangle-free incidence graphs is injective, surjective, or both. We will seek to prove that the map is both injective and surjective, thereby giving us a count for Y-Delta free critical circular planar graphs with n vertices.

3 The Main Theorem

Claim There is a correspondence between Y-Delta free critical circular planar graphs with n vertices and connected triangle free graphs with n vertices modulo certain knot-theoretic invariants.

Proof We first state a lemma establishing the equivalence between medial graphs and incidence graphs in our triangle-free case.

Lemma 3.1 *There exists a one-to-one, surjective correspondence between Y-Delta free critical circular planar graphs and lenseless medial graphs with no three geodesics mutually crossing.*

This result is clear.

Remark There exist connected triangle-free graphs that are not intersection graphs for any medial graph.

We will now proceed to work with these lenseless medial graphs and their $\mathbb{Z}$-sequences.

Suppose we have a given incidence graph I. Place some order on the vertices of I from 1 to n. We will prove that it arises from a unique medial graph M by examining the $\mathbb{Z}$-sequence that arises.

Consider I. Construct its incidence matrix (a_{ij}) as follows. It is an n by n matrix, and for $i \neq j$,$a_{ij} = -1$ if there is an edge between i and j, and $a_{ij} = 0$ otherwise. For a_{ii}, simply make the row and column sums to be 0. Then the matrix is Kirchhoff, and each diagonal entry is strictly positive since no vertex is isolated. The incidence matrix corresponds in a one-to-one and surjective manner with the incidence graph I, hence we construct a bijective correspondence between valid $\mathbb{Z}$-sequences for medial graphs of critical circular planar graphs.

Specifically, we wish to construct a correspondence between connected graphs and valid medial graphs. We do so by constructing a correspondence between disconnected graphs and invalid medial graphs by usage of the extremal principle.

Let I be such a disconnected graph. Suppose i and j are two vertices in distinct connected components, and let C_i and C_j denote those components respectively. Suppose that C_i contains vertices $\{i_1, i_2, \cdots i_k\}$ and C_j contains vertices $\{j_1, j_2, \cdots j_l\}$. In the parallel graph, therefore, we obtain that each geodesic i_α is parallel to each geodesic j_β. Then the geodesics $\{i_1, i_2, \cdots i_k\}$ and $\{j_1, j_2, \cdots j_l\}$ are parallel in the sense that any i_α is parallel to any j_β. Thus there exists a cut of $\mathbb{D}$, where M is embedded, such that $\{i_1, \cdots i_k\}$ lies on one side and $\{j_1, \cdots j_l\}$ lies on the other side. Then the resulting graph is not connected, contradiction.

4 Applicable Results in the Theory of Chord Diagrams

Remark From now on, we will use the terminology of chord diagrams (chord diagrams and chords) and the terminology of electrical networks (medial graphs and geodesics) interchangeably. They are equivalent, and we will use whichever designation fitting the discussion on hand.

A chord diagram consists of a circle, and 2n points on the circle, paired with a curve inside the disk connecting them. In this regard, they are simply medial graphs if we perturb them slightly to avoid intersections of valence greater than 2. Suppose in addition that no three curves intersect. Then this construction is equivalent to a medial graph.

The following Theorem, found as Theorem 4.28 of [2], is instrumental in establishing the relationship between intersection graphs to chord diagrams/medial graphs.

Theorem 4.1 (Chmutov-Duzhin-Mostovoy) *Two chord diagrams have the same intersection graph if and only if they are related by a sequence of mutations.*

To make sense of this assertion, we define the notion of a share.

Definition A *share* is a part of a chord diagram consisting of two arcs of the outer circle with the following property: each chord one of whose ends belongs to these arcs has both endpoints on these arcs.

Definition A *mutation of a chord diagram* is another chord diagram obtained by the flip of a share.

Note that the complement of a share is also a share. The fact that a share consists of two arcs should give some sort of restriction. Also, note that if there is a triangular cell in the medial graph, flipping the share also corresponds locally to a Y-Delta transformation.

5 Applying the Theory of Chord Diagrams to Electrical Networks with no Y-Delta equivalences

It is clear that any medial graph corresponds to a chord diagram. In fact, they correspond to chord diagrams with no so-called *short chords*, that is to say, chords which are not obligated to intersect any other chord because the medial graph of a recoverable graph is lenseless.

Now we move on to investigate shares in medial graphs with no Y-Delta equivalents. Let S be a share, with components S_1 and S_2, thereby defining two components of their complement, R_1 and R_2. Then each geodesic with one endpoint in S_1 and one endpoint in S_2 will intersect each geodesic with one component in R_1 and one component in R_2. From this we also see that the complement of a share is a share.

This implies that if S is a share, then the set of geodesics that go from R_1 to R_2 do not intersect at all, hence they are mutually parallel. But then similarly, the set of geodesics going from S_1 to S_2 must also be mutually parallel, giving us a restriction on the number of shares.

Note that it is pointless to think about a share containing a single geodesic, since flipping it accomplishes nothing. Therefore, the only shares we need to consider are those with at least two geodesics. In fact, both shares need to have at least two geodesics, and the graph at least four geodesics.

It is clear that the number of equivalent medial graphs corresponding to a given triangle-free intersection graph is equal to 2 raised to the power of viable share transforms.

Furthermore, let S be a share with arcs S_1 and S_2, then its complement R is a share with arcs R_1 and R_2, and suppose there exists a geodesic in the medial graph with one endpoint in S_1 and one endpoint in S_2. Then it intersects each geodesic in the share R with one endpoint in R_1 and one endpoint in R_2. Using the characterization of the medial graph that no three geodesics intersect, it is abundantly clear that the geodesics in R that have one vertex each in R_1 and R_2 are all mutually parallel. Similarly, this holds for those going from S_1 to S_2. Hence the geodesics in a share consists a set of mutually parallel geodesics plus those which stay in a single arc of the share. We call geodesics who have one vertex on each arc of the share a "crossing geodesic."

For example, the below

Chord Diagram.png

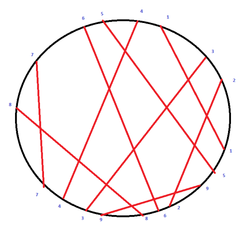


Figure 6

is certainly triangle-free by inspection, with $\mathbb{Z}$-sequence $1, 2, 3, 1, 4, 5, 6, 7, 8, 7, 4, 3, 9, 8, 6, 2, 9, 5$. Does it have nontrivial shares? Yes, for example we may take the portion of the $\mathbb{Z}$-sequence given by 787 and the single 8 to get a share. However, by the condition that the crossing geodesics of any share are all mutually parallel, taking more than a single crossing geodesic often becomes very difficult.

Remark The number of shares is bounded above by the number of sets of mutually parallel geodesics.

By the connectedness of S_1 and S_2, it is also relevant to note that the set of crossing geodesics automatically defines a "minimal" share by taking portions of the $\mathbb{Z}$-sequence that bound the relevant geodesics.

First, viewing the medial graph M as a chord diagram, the medial graph produces a connected graph if and only if each one of the geodesics intersects at least one other geodesics. If this does not happen, the chord diagram is termed degenerate. In his 2000 paper *On the Number of Chord Diagrams*, Alexander Stoimenow computes the number degenerate chord diagrams with respect to the total number of chord diagrams.

Remark The total number of chord diagrams with n chords is $\prod_{k=1}^{n}(2k-1) = (2n-1)!!$. This is clear, as we are obligated to partition a set of $2n$ vertices into n sets of 2 vertices in some order, where all vertices are indistinguishable. The result is precisely $\frac{(2n)!}{n!2^n}$, or $(2n-1)!!$.

and we can similarly compute the number of degenerate chord diagrams.

Definition A linearized chord diagram is formed by taking a chord diagram and making a cut along a diameter of the circle to "linearize" it.

Definition The length of a chord in a linearized chord diagram is equal to the number of endpoints it encloses, plus 1. Say that a chord is minimal if it encloses no other chord, and maximal if no other chord encloses it.

Claim The number of degenerate linearized chord diagrams is given by the following formula in [5]. If the chord diagram has n chords, then the number of degenerate linearized chord diagrams φ_n is

$$\varphi_n = \sum_{i=1}^{n}(-1)^{i-1} \sum_{j_1,\cdots j_i,k\geq 0,\sum j_l+k=n-i} \lambda_{k,i+1} \prod_{l=1}^{i}(\lambda_{j_l} - \varphi_{j_l})$$

where i is the number of minimal isolated choards, j_1 up to j_i the degrees of the linear chord diagrams enclosed by those i chords, and k the degree of the remaining diagram.

For chord diagrams the number of degenerate chord diagrams is $\omega_n = \frac{1}{2n}\sum_{d\cdot c=2n}\phi(c)\tilde{\gamma_{d,c}}$, where the $\tilde{\gamma_{d,c}}$ are given by a recursive form. Generally speaking the number $\frac{\omega_n}{(2n-1)!!}$ converges. This gives us a bound for the number of connected chord diagrams, i.e. valid medial graphs for electrical networks.

In Stoimenow's paper, the ratio of connected (linearized) chord diagrams to (linearized) chord diagrams is claimed to be $\frac{1}{e}$, albeit in a very slow fashion.

We have the following bound on the $\gamma_{d,c}$.

Theorem 5.1

$$\gamma_{d,c} \leq (1+\sqrt{c})^d (d-1)!!$$

This gives us a very useful bound on $\gamma_{d,c}$, which we will now use to do some computations relating to shares. We have that the bound on the number of shares is related to $\gamma_{d,c}$ as follows.

Let M_G be a medial graph, or, equivalently, a chord diagram. Then we see immediately by our prior observation that if $S = (S_1, S_2)$ forms a share, so does the complement $\partial\mathbb{D} - S = (R_1, R_2)$, and that the geodesics in M_G are partitioned in the following manner.

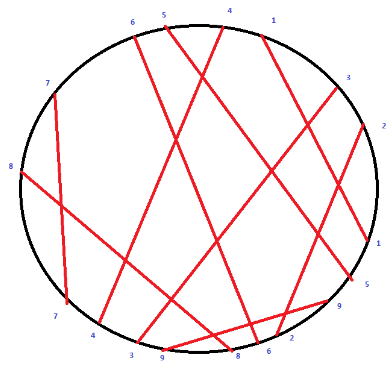
$$M_G = R_{R_1} \cup R_{R_2} \cup R_{S_1} \cup R_{S_2} \cup R_{1,2} \cup S_{1,2}$$

What does the above partition mean? R_K, for K an arc, denotes the re-entrant geodesics of K, while $R_{1,2}$ and $S_{1,2}$ denote the geodesics with one endpoint in S_1 and one endpoint in S_2. Note that the above holds for any chord diagram. What is unique about this situation for triangle-free chord diagrams is that the geodesics in $R_{1,2}$ and in $S_{1,2}$ form mutually parallel sets, and that no geodesic in $R_{1,2}$ is parallel to any geodesic in $S_{1,2}$.

Therefore, this gives a good restriction on possible shares in a chord diagram.

We now work through an example.

Chord Diagram.png



6 Creating and Identifying Shares within a Chord Diagram

Identifying a share will first require us to identify two sets of mutually parallel geodesics, that is to say, a bipartite subgraph of the intersection graph.

Firstly, taking two intersecting geodesics satisfies the conditions. To create a share, all that is left is to take some other possible geodesics only intersect one of the two intersecting ones. But this is not very exciting and the construction is fairly trivial.

For example, here we might take geodesics $\{3, 4\}$ and $\{5, 6\}$ as a basis for our share. But then one portion of the share S, WLOG S_1, has to contain the endpoint 1 on the upper half of the circle, and so it has to contain the other endpoint of 1, but then between them is an endpoint labelled 2, and so on and so forth, and we see that the shares "spread" until they encompass the entirety of the circle, which is not what we want. Generally, this gives us an idea of the obstacles we encounted by constructing shares.

So to create a share, our idea is as follows

- Find a bipartite graph in the intersection graph. Denote the components by P and Q
- Consider all vertices connected to vertices in P, they ought to be in the share too
- Hope that they are not connected to any vertices in Q
- Repeat for Q to obtain a share (hopefully not the entire set of vertices is considered in the intersection graph)

The components P and Q identify the crossing geodesics of the share. The triangle-free condition ensure that geodesics corresponding to P are all mutually parallel.

Note that fairly trivial shares are always possible, taking a single edge (and the two corresponding vertices) is the most trivial example, and flipping them does nothing. Generally, since the complement of a share is also a share, although we still need to consider flips independently.

Since we are working with flips of shares, trivial shares involving only a single geodesic are irrelevant for creating a different medial graph. Shares that consist solely of crossing geodesics are also irrelevant.

Now let us suppose that we have a bipartite subgraph of the intersection graph, $B_{m,n}$ for $m, n > 1$.

Definition There exist two sets of vertices B_m, B_n which partition the vertices of $B_{m,n}$ for which there exists an edge from each vertex of B_m to each vertex of B_n. Call B_m, B_n the vertex sets of $B_{m,n}$.

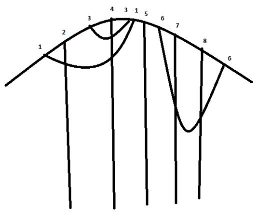
Definition Define the graph $I - B_{m,n}$ as the graph obtained by deleting all vertices in the vertex sets of $B_{m,n}$ as well as all edges which take a vertex in the vertex set as one of its endpoints.

Definition Say a vertex $v \in V(I - B_{m,n})$ is connected to B_m through $I - B_{m,n}$ if there exists a path $v \to i \in B_m$ that whose vertices are all in $I - B_{m,n}$.

Remark Since our intersection graph I is connected, it follows that each $v \in V(I - B_{m,n})$ is connected to at least one of B_m, B_n. For our given $B_{m,n}$, if no $v \in V$ is connected to both B_m and B_n, then it follows that $B_{m,n}$ generates a share. Conversely, if some $v \in V$ is connected to both B_m and B_n, then $B_{m,n}$ cannot generate a share.

Since our medial graph admits some $\mathbb{Z}$-sequence representation, we may take a representation of the $\mathbb{Z}$-sequence such that in the components of our bipartite graph $B_{m,n}$, B_m and B_n do not include both geodesic n and geodesic 1, for ease of notation. Denote the vertices of B_m as $i_1 < i_2 \cdots < i_m$ and of B_n as $j_1 < j_2 \cdots < j_n$.

For now, consider the share generated by B_m. The vertices of B_m generate something like



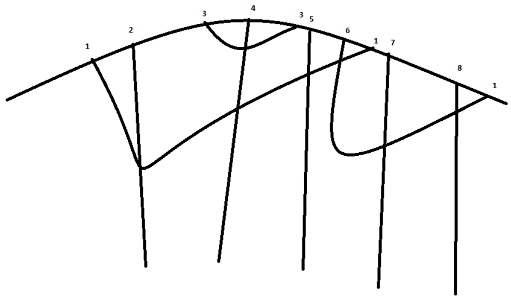
Chord Diagram.png

Then the B_m portion consists of geodesics marked $2, 4, 5, 7, 8$. It is immediately clear that geodesics numbered 1, 3, and 6 are obligated to be in the share and to remain in the same portion of the geodesic. This is what we mean when we talk about B_m "generating" a share. Of course, not every geodesic intersecting those vertices in B_m have to be in the same portion of the share. So 1, 3, 6 can be "permuted" in that manner to either the "upper" half of the circle as envisioned here or to the"lower" half. Then each geodesic can be manuvered to each half of the share independently, giving

However, note that these geodesics have a "parallel" condition in the sense that none of the geodesics 1, 3, and 6 intersect. Doing the same thing with n shares, since they do not intersect while all in the same arc of the share, putting them on one or the other arc of the share does not influence the part of the intersection graph corresponding to the drawn portion of the medial graph, implying that there are 2^n possible shares that have the same local intersection graph (flips of shares are also considered here). Suppose now that 1 and 6 intersect, in such a manner as below

Then 1 and 6 must lie in the same arc of that share, since they intersect and are both re-entrant. Therefore, there exists a unique flip in the creation of a share between maximal classes of "intersecting" geodesics.

Example 1.png



Keep in mind that the $\mathbb{Z}$-sequence of a chord diagram or medial graph is numbered in order, and that there is a correspondence with the vertices of the intersection graph, in particular B_m.

From a bipartite graph $B_{m,n}$, consider the resulting graph obtained by deleting all vertices of $B_{m,n}$ and all edges emanating from those vertices, but remember the vertices that were connected to $B_{m,n}$. Since the intersection graph is connected, from each vertex in $I(M_G)$ there exists a path from that vertex to both B_m and B_n. In particular, there either exists a path not passing through B_m to B_n or a path not passing through B_n to B_m, by terminating the path either when it first hits B_n or B_m. We may also assume that said path contains no cycles, as if it does we can just delete the cycle and obtain a similar path.

Definition A share component is a set of vertices in $I - B_{m,n}$ that are all connected through $I - B_{m,n}$.

Given $B_{m,n}$, say that vertices i and j not in $B_{m,n}$ lie in the same share component if there exists a path connecting vertices i and j not passing through B_m or B_n.

The maximality concept for re-entrant geodesics is somewhat more complicated than what might appear on the surface. If two geodesics intersect the same crossing geodesic, then they do not intersect by the triangle-free nature of the graph. Suppose i is a re-entrant geodesic that intersects the set of crossing geodesics A_i, and similarly for j. Then if $A_i \subset A_j$ or $A_j \subset A_i$, the two are parallel and can lie in the same or different arcs of the share. If not, then they must lie in different arcs and hence are "dependent" in a sense.

However, in both cases the re-entrant geodesics do not intersect, but are somehow "connected" through $I - B_{m,n}$. This leads to the first preliminary characterization:

- Let i_1, i_2 be vertices of the intersection graph I which are only connected to $B_m \subset B_{m,n}$, and suppose that $B_{m,n}$ generates a share S. Let A_1, A_2 be the sets of vertices that i_1, i_2 have an edge to, respectively. If $A_1 \subset A_2$ or $A_2 \subset A_1$, then

i_1, i_2 are parallel. Hence they can occur in (be re-entrant in) either the same or different arcs of the share.

- If $A_1 \not\subset A_2$ and $A_2 \not\subset A_1$, then the two geodesics must be re-entrant in different arcs of the share.

In the former case, the flips of i_1, i_2 can occur independently, whereas in the latter case flipping i_1 results in a flip of i_2 also, so in some sense the flips are not "independent."

If we have maximal share components, we can still take the same characterization, since if two geodesics intersect, then their flips are dependent in order to preserve that intersection point.

- If geodesics i_1, i_2 intersect, then their flips are dependent.
- If geodesics i_1, i_2 intersect the same crossing geodesic, they are parallel to each other. They are re-entrant in the same arc if and only if $A_1 \subset A_2$ or $A_2 \subset A_1$ per the definitions above.

We take the following method. Let C be the set of all vertices that have an edge to some vertex of B_m and let V be the set of all vertices connected to B_m. Write out the vertices as $\{v_1, \cdots v_n\}$. Then if v_i, v_j are connected through $I - B_{m,n}$, then v_i, v_j must lie in the same maximal share component.

For each v_i, we may define C_i as the set of vertices that are connected to v_i. For each $v \in V$, we may define D_v as the set of vertices of C connected to v.

If $C_i \cap C_j \neq \emptyset$, then v_i, v_j are in the same maximal share component. Two vertices $v_1, v_2 \in V$ are in the same connected component if and only if D_{v_1} and D_{v_2} do not contain each other and are not disjoint.

Claim Given a share generated by a bipartite graph $B_{m,n}$ for some intersection graph I, the number of lenseless medial graphs/chord diagrams corresponding to said intersection graph I is 2^k, where k is the number of sets of disjoint maximal share components.

Definition Let X and Y be two maximal share components. Call them disjoint if there do not exist a path from both X and Y to the same vertex in $B_{m,n}$ through $I - B_{m,n}$. Otherwise, call them intersecting. Therefore, we can partition the maximal share components of $B_{m,n}$ into classes of intersecting maximal share components. Each set is maximal, hence disjoint.

Proof Each maximal share component induces a reflection as described above. Since the share components are maximal, they do not intersect each other, and hence reflections of those geodesics are independent, rendering 2^k equivalent medial graphs.

As a corollary, if $B_{m,n}$ does not generate a share, then it is irrelevant to our computation. If we can tabulate the number of bipartite graphs for each triangle-free intersection graph along with the corresponding number of maximal share components, then we will have calculated the number of medial graphs corresponding to that chord diagram.

The formula for the number of triangle-free medial graphs corresponding to a given intersection graph is

$$\sum_{B_{m,n}} 2^{k_{B_{m,n}}}$$

by summing over all maximal disjoint bipartite subgraphs of the intersection graph, and k is the number of sets of disjoint intersecting maximal share components associated to $B_{m,n}$.

Reflection of a single maximal share component can be thought of as reflecting a "sub-share" of the larger share, corresponding to those vertices that have a path to the maximal share component. This is the reason we take maximal bipartite subgraphs, as each bipartite subgraph is contained in a maximal bipartite subgraph and hence we will count exactly the number of possible flips.

7 Differing Notions of parallel geodesics

Example Examine the following chord diagram

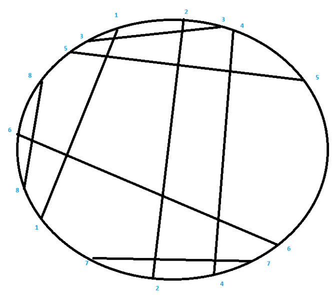


Diagram Example 1.png

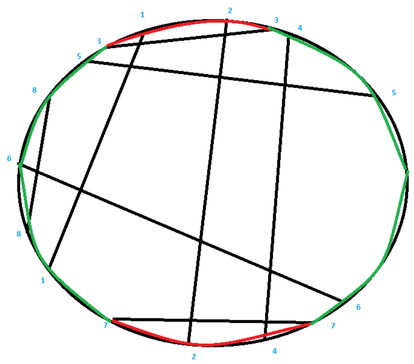
Figure 7

There are two notions of "parallel" geodesic one might draw from here. For example, geodesics $3, 5, 7$ are all mutually parallel, but so are geodesics $3, 8, 7$. But we might note that they are parallel in different ways.

Both sets contain geodesics 3 and 7. We can use the endpoints of those two geodesics to define four arcs on the boundary of the circle as below.

Since 5 and 8 are both parallel both of their endpoints lie in the green area of the boundary. But the vertices for 5 lie in different areas, while the vertices for 8 lie in the same area.

Diagram Example 2.png



In this case we say that 8 is “trivially parallel” with 3 and 7, that is to say, the vertices of 8 lie in the same green area, and 5 is“strictly parallel” with 3 and 7, that is to say, the vertices of 5 lie in different green areas. If any other geodesic parallel to 3 and 7 has both vertices lying in red areas, we do not care.

Remark We clearly need two parallel geodesics to define these criteria of strictly and trivially parallel.

Now, in this case, the bipartite graph $B_{m,n}$ with $B_m = \{3, 7\}$ does not generate a share, because the only possible outcome is $B_n = \{2\}$, and then the maximal share component is $\{1, 4, 5, 6, 8\}$ in the intersection graph, which has paths to both B_m and B_n.

Recall that our construction of a share is given by taking two sets $\mathcal{A}$ and $\mathcal{B}$ of mutually parallel chords, with each chord in $\mathcal{A}$ intersecting each chord in $\mathcal{B}$, thereby defining disjoint arcs A_1, A_2 and B_1, B_2 with all other geodesics re-entrant in one of those arcs.

Hence if some geodesic not in $B_{m,n}$ is strictly parallel to two geodesics in either B_m or B_n, it cannot be re-entrant and hence B_m cannot generate a share. If, however, there were trivially parallel geodesics, we would be unaffected. Generally, given a bipartite graph $B_{m,n}$ we have to verify that there are no geodesics that are strictly parallel with two geodesics in B_m or B_n. We can easily identify those geodesics: a geodesic i strictly parallel to geodesics j_1, j_2 in B_m must intersect each crossing geodesic in B_n. If this is indeed the case, this would mean that $B_{m,n}$ is contained in a bigger bipartite graph where the components are $B_m \cup \{i\}$ and B_n.

Geodesic i is situated in between j_1, j_2 if and only if $j_1 < i < j_2$ where $<$ is given by the circular ordering on the graph that places j_1 before j_2 in one component of the share. This condition is also satisfied on the other component of the share, because all crossing geodesics in $B_m \cup \{i\}$ are mutually parallel.

Nevertheless, as long as we take maximal classes of parallel geodesics we will not be bothered by the issue of some strictly parallel geodesics.

Using the above chord diagram/medial graph, we construct a share.

Example Take the bipartite graph $B_{m,n}$ with $B_m = \{1, 2, 4\}$ and $B_n = \{5, 6\}$. There are three maximal share components, $\{3\}$, $\{7\}$, and $\{8\}$.

Each maximal share component admits a flip, but if two maximal share components are both connected to the same vertex, then the flip is not independent. Rather, if one is flipped, so is the other. We immediately see that $\{3\}$ and $\{7\}$ form a maximal set of intersecting maximal share components, since they both intersect $\{2\}$ and no other share component does.

Then there are two maximal sets of intersecting maximal share components, giving $2^2 = 4$ possible chord diagrams with the same intersection graph using that share as a basis, which we can physically verify by drawing them out.

Therefore, given a possible generating disjoint bipartite subgraph $B_{m,n}$ for a share, we can identify whether it actually generates a share by (1) examining the maximal share components and (2) examining the possibility of strictly parallel geodesics.

Remark It is probably easier to look at the number of chord diagrams with the same intersection graph by taking some chord diagram that has that intersection graph, identifying the maximal share components, and then computing it.

Under share reflection, the notions of strict and trivial parallelism are preserved, as can also be evidenced by the intersection graph. Notions of strict and trivial parallelism are also reflected in the construction of a share, as each maximal share component can only intersect a single B_m or B_n.

Example It is necessary to discuss maximal bipartite graphs $B_{m,n}$ used to generate shares. These maximal bipartite graphs are important because if they generate a share, we don't need to worry about other possible crossing geodesics interfering with our ability to create a share.

Since any bipartite subgraph is contained in a maximal bipartite subgraph, if given some $B_{m,n}$ not maximal there exists a maximal bipartite subgraph containing it, which might automatically trigger some strictly parallel geodesics. If $B_{m,n}$ only induces trivially parallel geodesics, then it could still generate a share. Hence for each maximal $B_{m,n}$, there are only m^2n^2 possible choices of subgraphs to examine, rather than the 2^{m+n} total subgraphs of the maximal graph. As we can tell, for large m, n this is an extremely significant reduction in the number of graphs to consider.

To recap, given a chord diagram and its associated intersection graph, we can construct and compute the chords as follows

- For the intersection graph, identify all maximal bipartite subgraphs of the intersection graph.

- Examine the maximal share components of each bipartite subgraph, and partition them into disjoint maximal components.
- The number of share flips for the bipartite subgraphs is 2^k where k is the number of disjoint maximal components.
- The total number of chord diagrams with that intersection graph is given by summing over all these 2^k.

However, we have no idea whether a given graph is the intersection graph of a chord diagram.

8 Identifying and Characterizing valid Intersection Graphs

Let I be a connected graph with n vertices, numbered 1 to n. Suppose in addition that I is triangle-free, so that should it correspond to an intersection graph of a medial graph M, the critical network associated to M will have no $Y - Delta$ equivalencies. There exist certain graphs which are not intersection graphs of any chord diagram, such as

Intersection Graph.png

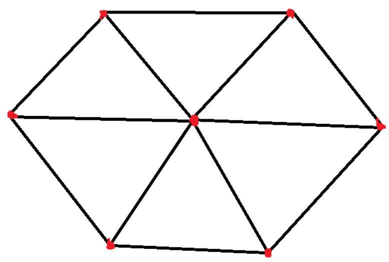


Figure 8

For example, we can easily identify that this graph cannot correspond to a chord diagram. The middle vertex (corresponding to some geodesic) has an edge connecting it to each other vertex so draw it as $[-1, 1]$ in the plane.Hence that geodesic intersects each other geodesic, yet the intersection conditions will require the leftmost geodesic to intersect the rightmost geodesic (by examining their endpoints in the upper half-plane), a contradiction.

In [7], it is shown that there is an $O(n^2)$ algorithm to identify whether any given graph is the intersection graph of a chord diagram (the intersection graphs are known as "circle graphs"), where n is the number of vertices of the intersection graph. The algorithm relies on placing the vertices one by one in some order, until either a chord diagram is created or a contradiction in terms of intersections is obtained.

However, it is still unknown as to which graphs are actual circle graphs or do not represent the intersection graph of a chord diagram.

9 Conclusions and Remarks

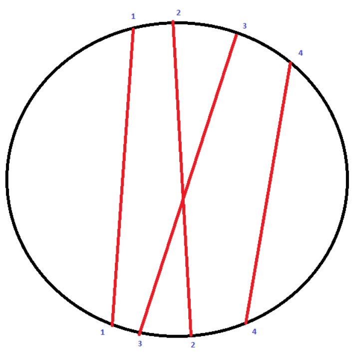
We have given a characterization of how to construct shares, as well as computing share reflections, which will enable us to identify the number of chord diagrams corresponding to some given triangle-free graph as its intersection graph. We have also cited an algorithm (of order $O(n^2)$) which enables us to identify chord diagrams.

It remains to give a characterization of triangle-free graphs that are intersection graphs, and the answer to this question has consistently eluded us. The algorithm described allows us to look at a cycle in the intersection graph and attempt to re-create the graph from there.

9.1 General Intersection Graphs

Most of the tools outlined here can be used to discuss general intersection graphs. Share reflection as a tool can be used for general intersection graphs. Shares are still constructed by bipartite subgraphs of the intersection graph, and the idea of maximal share components stays the same. Unfortunately, the notion of strict and parallel geodesics will often go away.

Examine the following set of crossing geodesics.



2.png

Figure 9

Then 2 and 3 are both strictly parallel to 1 and 4. This is fine. But 2, while parallel to 1, is not parallel to 3. We will say that in this case, geodesics 1, 3 do not generate a share, since if they were crossing geodesics it is obvious that 2 will also be a crossing geodesic. We do not encounter this problem for triangle-free intersection graphs.

Therefore, we have an additional condition on the bipartite subgraphs used to create shares. Le t$B_{m,n}$ be such a bipartite subgraph. It is contained in some maximal bipartite subgraph $B_{m',n'}$, where possibly there are edges between vertices of $B_{m'}$ or $B_{n'}$, with $B_m \subset B_{m'}$ and $B_n \subset B_{n'}$.

As we see in the above example, if there are no edges between $B_{m'} - B_m$ and B_m, then B_m can generate a share, and similarly for B_n.

10 Further Questions

The results outlined in this paper pertain only to triangle-free medial graphs. In particular, the idea of taking a bipartite subgraph of the intersection graph can only be limited to triangle-free medial graphs. This should generalize easily to all medial graphs, as none of our tools relied on the triangle-free nature of either the medial graph or the intersection graph.

We have given a formula to count the number of medial graphs corresponding to a given intersection graph, as well as a condition for which graphs can possibly be intersection graphs for a medial graph. However, it is not easy to identify offhand whether a given graph is actually an intersection graph or not, if we have to examine all possible bipartite subgraphs of the intersection graph. We would like a systematic way to tell whether a graph is an intersection graph or not.

Additionally, it is evident that generally speaking, the presence of a greater number of intersections in the intersection graph will bring with it a greater number of shares for the chord diagram, as more bipartite subgraphs of the intersection graph exist, and more connections through $I - B_{m,n}$ also exist. In [7], it is shown that there is an $O(n^2)$ algorithm to determine whether a given graph is the intersection graph of a chord diagram, so again we have, theoretically, a way to determine the possible intersection graphs, but we cannot construct the chord in the way that they do.

However, we can still construct crossing geodesics by constructing a bipartite subgraph $B_{m,n}$ as we did for the triangle-free case. The difference is that the vertices in B_m and B_n need not all be totally disconnected, though this does not change the construction because of the deletion of the triangle-free condition.

Either way, the construction should take the same route despite the removal of the triangle-free condition. The triangle-free condition manifests itself in the sense that if two vertices both have an edge to another one (like in a bipartite graph), then there isn't an edge between the two vertices.

It still remains to give a count for the number of connected triangle-free graphs that correspond to intersection graphs, or a characterization beyond the $O(n^2)$ algorithm in [7].

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