A FRAMEWORK FOR THE ADDITION OF KNOT-TYPE COMBINATORIAL GAMES

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ABSTRACT. We consider two-player nonloopy games in which both players have the same option at each point of the game, and the parity of the length of the game is determined, but the winner is not determined by the last move. We develop the additive theory of such games, where the sum of two games is a biased operation favoring one of the two players, in case the outcomes of the two games differ. This arrangement occurs naturally in the context of knot games. We show that modulo indistinguishability, there is a finite commutative monoid of games, with 37 elements.

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1. INTRODUCTION

In the paper *A Midsummer Knot's Dream* [3], Henrich et alintroduce a game *To Knot or Not to Knot* played on a knot pseudodiagram, in which two players, King Lear and Ursula, alternatively resolve precrossings. The game ends when all precrossings have been resolved, with **K**ing Lear winning if the resultant knot is knotted, and **U**rsula winning if the knot is the unknot.

Given two pseudodiagrams, we can play *To Knot or Not to Knot* on the connected sum. Equivalently, we could place the two games next to each other and have each player on her or his turn choose one game to play in, with Ursula winning only if she wins in both of the shadows.

This sort of combined game, where two games are played in parallel, is very similar to the concept of a sum of games used in Combinatorial Game Theory, as developed in Conway's *On Numbers and Games* [2] and Guy et al.'s *Winning*

Ways [1]. The critical difference is that CGT focuses on games where the winner is determined by who moves last. Therefore, there is no need to specify the winner of the game. In games like *To Knot or Not to Knot*, on the other hand, the last player is not necessarily the winner – in fact the player to move last can be predicted from the outset, because the game has fixed length. Instead, the winner in each endgame must be specified by the game itself, and the sum of two games becomes an asymmetric affair, favoring King Lear.

Knowing the outcome of two games under perfect play does not necessarily determine the outcome of the sum of two games. Consequently, one of the main focuses of CGT is to assign a *value* of some sort to each game, such that the outcome of a game is determined by the value, and the value of a sum of games is the sum of the values. While some games can be studied using numerical values, other games take their values in other algebraic structures.

This paper presents a similar approach for the addition of knot-type games, defined in the next section. In a way, this is similar to the alternative ways of combining games, such as conjunctive and selective sums, studied in Chapter 14 of *On Numbers and Games,* and Volume 2 of *Winning Ways.* It is also similar to the theory of Misere games – in some situations, such as Misere Kayles, positions take values in a finite monoid, as happens here (this was shown by Sibert and Conway).

2. Knot-type Games

We consider non-loopy games between two players – Knotter and Unknotter – which end after a finite number of moves. We impose two important caveats which are satisfied by all positions that occur in *To Knot or Not to Knot.*

- The length of the game must be determined in advance. Actually, we just require that the parity of the length be known in advance. This occurs in *To Knot or Not to Knot* because the number of moves remaining in any position is simply the number of precrossings left in the diagram.
- At each position, both players have the same options. This makes knot-type games similar to the "impartial" games of CGT, except for the fact that the endgame specifies which of two players wins. One might call such games "semipartial."

Formally, then, we use the following recursive definitions:

Definition 2.1. *An* odd knot-type game *G is a non-empty set of even knot-type games, the* options *of G.*

Definition 2.2. *An* even knot-type game *G is either one of the symbols u or k, or a non-empty set of odd knot-type games, the* options *of G.*

Definition 2.3. *A* knot-type game *G is an even or odd knot type game.*

Definition 2.4. *An* endgame *is one of the games k or u.*

Henceforth, "game" will refer to knot-type games.

The outcome of a game is defined recursively:

Definition 2.5. *(half-outcomes)*

- *A game is a* win for K playing second (K2) *if it is equal to k, or if all options are wins for K playing first.*

- *A game is a* win for K playing first (K1) *if it is equal to k, or if some option is a win for K playing second.*
- *A game is a* win for U playing second (U2) *if it is equal to u, or if all options are wins for U playing first.*
- *A game is a* win for U playing first (U1) *if it is equal to u, or if some option is a win for U playing second.*

Definition 2.6. *The outcome of a game* outcome(G) *is*

- *K if $G \in K1 \cap K2$, that is, G is always a win for the Knotter*
- *1 if $G \in K1 \cap U1$, that is, G is always a win for the first player*
- *U if $G \in U1 \cap U2$, that is, G is always a win for the Unknotter*
- *2 if $G \in K2 \cap U2$, that is, G is always a win for the second player.*

Because games are defined recursively, we can use induction to prove theorems about them. For example:

Theorem 2.7. *A game G is in K2 iff it is not in U1; it is in U2 iff it is not in K1.*

Proof. Proof by induction on G. For the base cases, if $G = k$, then G is in K2 but not U1 and K1 but not U2. If $G = u$, then G is U1 but not K2 and U2 but not K1. Otherwise, suppose the hypothesis is true for all options of G. Then (G is in K2) $\iff$ (all options of G are in K1) $\iff$ (no option of G is in U2) $\iff$ (G is not in U1). Similarly, (G is in U2) $\iff$ (all options G are in U1) $\iff$ (no option of G is in K2) $\iff$ (G is not in K1). □

It is clear from the previous theorem that each game has exactly one outcome from the four possibilities.

We use G' to represent an option of G, in a way similar to that used in *ONAG* and *Winning Ways*. For example:

Definition 2.8. *Let G and H be games. The sum $G+H$ is the game $\{G'+H, G+H'\}$, unless G and H are both endgames, in which case $G+H$ is u iff both G and H are u, and k otherwise.*

The notation $\{G' + H, G + H'\}$ is shorthand for

$$\{G' + H \ : \ G' \in G\} \cup \{G + H' \ : \ H' \in H\}.$$

Theorem 2.9. *Addition of games is commutative: $G + H = H + G$.*

Proof. This is clear in the case that G and G are both endgames, and otherwise

$$G + H = \{G' + H, G + H'\} = \{H + G', H' + G\} = H + G,$$

where the middle step follows by induction. □

Theorem 2.10. *If G is a game, then $G + u = G$*

Proof. This is clear if G is an endgame, and otherwise,

$$G + u = \{G' + u, G + u'\} = \{G' + u\} = \{G'\} = G,$$

by induction and the fact that u has no options. □

Theorem 2.11. *Addition of games is associative: $(F + G) + H = F + (G + H)$*

Proof. If any of F, G, H are the endgame u, then this follows from the previous two theorems. Otherwise, if F and G are both the endgame k, then

$$(F+G)+H = \{k+H'\} = \{(k+k)+H'\} =$$
$$\{k+(k+H')\} = \{k+(k+H)'\} = k+(k+H) = F+(G+H),$$

using induction to equate $(k+k)+H'$ with $k+(k+H')$. A similar proof works if G and H are both k. Otherwise, neither $F+G$ nor $G+H$ is an endgame, so we have

$$(F+G)+H = \{(F+G)'+H, (F+G)+H'\} = \{(F'+G)+H, (F+G')+H, (F+G)+H'\}$$
$$= \{F'+(G+H), F+(G'+H), F+(G+H')\} = \{F'+(G+H), F+(G+H)'\} = F+(G+H),$$

using induction as usual. □

To summarize, games form a commutative monoid, with u as the identity. We also have the following:

Theorem 2.12. *The sum of two even games or two odd games is even. The sum of an even and odd game is odd.*

Proof. Left as an exercise to the reader. □

3. Equivalence and the Monoid of Values

The key definition we are going to make is the following:

Definition 3.1. *Two games G and H are* equivalent *(denoted $G \cong H$) iff G and H have the same parity, and for every game X,*

$$\text{outcome}(G+X) = \text{outcome}(H+X).$$

In other words, two games are equivalent whenever they are interchangeable in a sum, without affecting the outcome of the sum. We also require as a caveat that the games have the same parity. This simplifies the statements of some theorems, and in §10 we will see that this makes almost no difference.

Theorem 3.2. *Equivalence is an equivalence relationship.*

Proof. Assign to each game G the function f_G from Games to Outcomes by $f_G(X) = \text{outcome}(G+X)$. Then clearly, $G \cong H$ if and only if $f_G = f_H$ and G and H have the same parity. □

Definition 3.3. *The set of* values *V is the set of games modulo equivalence. The value of a game $|G|$ is its image in V.*

Theorem 3.4. *The set V inherits the structure of a commutative monoid. Also, there is a map g from values to outcomes, and the map from games to outcomes factors through g, so that* $\text{outcome}(G) = g(|G|)$.

Proof. Suppose that G and H are equivalent. Then $\text{outcome}(G) = \text{outcome}(G+u) = \text{outcome}(H+u) = \text{outcome}(H)$. So the outcome of a game is a function of its value.

Now suppose that $G' \cong G$ and $H' \cong H$. Then for any K,

$$\text{outcome}(G+H+K) = \text{outcome}(G'+H+K) =$$
$$\text{outcome}(H+G'+K) = \text{outcome}(H'+G'+K),$$

so $G+H \cong H'+G' = G'+H'$. Therefore, equivalence is a congruence relationship, and the quotient space V inherits the structure of a commutative monoid. □

4. Star

Let $*$ denote the odd game $\{u\}$. This game lasts one move, and the Unknotter always wins. The sum $* + * = \{*' + *\} = \{u + *\} = \{*\}$.

Lemma 4.1. *If X is K2, then $X + * + *$ is K2. If X is K1, then $X + * + *$ is K1.*

Proof. Proof by induction on X. If $X = k$, then $k + * + * = \{\{k\}\}$ which is K1 and K2. The endgame u is neither K1 nor K2.

Otherwise, suppose X is K2. Then every option X' is K1. Meanwhile every option of $X + * + *$ is either of the form $X' + * + *$ or $X + *$. The former is K1 by induction, and the latter is K1 because it has $X \in$ K2 as an option.

Likewise, suppose X is K1. Then some option X' is K2, so by induction, $X' + * + *$ is K2, and this is an option of $X + * + *$, which is therefore K1. □

Lemma 4.2. *If X is U2, then $X + * + *$ is U2. If X is U1, then $X + * + *$ is U1.*

Proof. Analogous to the previous lemma. □

Theorem 4.3. $* + * \cong 0$

Proof. This follows from the previous two lemmas. □

Roughly speaking, the game $* + *$ is equivalent to 0 because any player who has a strategy in a game G can use the same strategy in $G + * + *$, and reply to moves in $*$ by replying in $*$. Once G ends, the player may be forced to move in one of the stars, but by this point the outcome of the game has already been determined.

Because $* + * \cong u$, the identity element of the monoid of values, it follows that $*$ acts as an involution on V. In fact, $*$ establishes a one-to-one correspondence between even and odd elements of V. V can be expressed as a direct product

$$V \cong V_0 \times \mathbb{Z}_2$$

where V_0 denotes the even elements of V. So we could focus on merely V_0 to determine the structure of V.

Definition 4.4. *Let G be a game or value. The* odd projection of G, $\mathrm{o}(G)$, *is G if G is odd, and $G + *$ if G is even. The* even projection of G, $\mathrm{e}(G)$, *is G if G is even, and $G + *$ if G is odd.*

5. Gradings

In this section we begin the technical work of dividing games into classes and seeing how they interact. Section 9 summarizes the results of the next four sections.

Definition 5.1. *A* grading *is a partition of the set of games into three sets G_0, G_1, and G_2, such that*

- *$u \in G_0$*
- *$k \in G_2$*
- *If $X \in G_0$, then all options of X are in $G_0 \cup G_1$*
- *If $X \in G_2$, then all options of X are in $G_1 \cup G_2$*
- *A game X is in G_1 iff some option of X is in G_0 and some option is in G_2.*

Note that the class a game falls into is determined recursively by its options, unless all its options are in G_1.

Gradings have the following general properties:

Lemma 5.2. *If $X \in G_2$ and Y is any game, then $X + Y \in G_2$*

Proof. Any option of $X+Y$ is of the form $X'+Y$ or $X+Y'$. The latter is in G_2 by induction. Since $X \in G_2$, $X' \in G_1 \cup G_2$, so $X' + Y \in G_1 \cup G_2$ by the next lemma. Therefore, all options of $X + Y$ are in $G_1 \cup G_2$, so $X + Y \notin G_1$. It remains to show that $X + Y \in G_2$ or some option of $X + Y$ is in G_2, because the latter implies that $X + Y \notin G_0$.

If $Y = u$, then $X + Y = X + u = X \in G_2$, and we are done. If $Y = k$, then either X is an endgame, in which case $X + Y = k \in G_2$, or X has some option X'. Then $X' + k \in G_2$ by induction, so some option of $X + Y$ is in G_2. If Y is not an endgame, then there is some option Y', and then $X + Y'$ is in G_2 by induction. Therefore, some option of $X + Y$ is in G_2. □

Lemma 5.3. *If $X \in G_1 \cup G_2$ and Y is any game, then $X + Y \in G_1 \cup G_2$.*

Proof. If $X \in G_2$, this follows from the previous lemma. So suppose $X \in G_1$. Then some option $X' \in G_2$, and so $X + Y$ has an option $X' + Y$ which is in G_2 by the previous lemma. Therefore, $X + Y \notin G_0$. □

Lemma 5.4. *If $X, Y \in G_1 \cup G_2$ then $X + Y \in G_2$.*

Proof. If both X and Y are endgames, then $X = Y = k = X+Y \in G_2$. Otherwise, any option of $X+Y$ is of the form $X+Y'$ or $X'+Y$. Since X and Y are in $G_1 \cup G_2$, so are all options of $X + Y$, by the previous lemma. Therefore, $X + Y \notin G_1$. We already knew by the previous lemma that $X + Y \notin G_0$. Thus $X + Y \in G_2$. □

We summarize the results so far in the following theorem:

Theorem 5.5. *If G is a grading and X and Y are games, with $X \in G_i$ and $Y \in G_j$, then $X + Y \in G_k$, where*

$$k \geq \min(2, i + j)$$

Theorem 5.6. *If G is a grading, the classes G_0, G_1, and G_2 are invariant under addition by $*$.*

Proof. We already know that G_2 and $G_1 \cup G_2$ are closed under addition by $*$. But addition by $*$ is actually symmetric between the two players, and so is the definition of a grading. So by symmetry, G_0 and $G_0 \cup G_1$ are closed under addition by $*$. It follows that each of G_0, G_1, and G_2 is closed under addition by $*$.

Alternatively, suppose X is some game. If $X \in G_0$, then all options of $X + *$ of the form $X' + *$ are in $G_0 \cup G_1$ by induction, and the remaining option $X + u$ is in G_0 by assumption, so $X + *$ cannot be in G_1 or G_2. The case when $X \in G_2$ is handled similarly. Finally, if $X \in G_1$, then some $X' \in G_0$, so the option $X' + *$ is in G_0 by induction. Similarly, some other option of X is in G_2, so the corresponding option of $X + *$ is in G_2. Then $X + *$ has options in both G_0 and G_2, and is in G_1. □

6. The gradings X, Y, and Z

We recursively partition games into classes as follows:
The endgame u is in X_0, Y_0, and Z_0. The endgame k is in X_2, Y_2, and Z_2.
For an even non-endgame A,

- $A \in X_0$ iff no option of A is in X_2

- $A \in X_1$ iff some option of A is in X_0 and some is in X_2
- $A \in X_2$ iff some option of A is in X_2 but none is in X_0

For an odd game B,

- $B \in X_0$ iff some option of B is in X_0 but none is in X_2
- $B \in X_1$ iff some option of B is in X_0 and some is in X_2
- $B \in X_2$ iff no option of B is in X_0

For an even non-endgame A,

- $A \in Y_0$ iff some option of A is in Y_0 but none is in Y_2
- $A \in Y_1$ iff some option of A is in Y_0 and some is in Y_2
- $A \in Y_2$ iff no option of A is in Y_0

For an odd game B,

- $B \in Y_0$ iff no option of B is in Y_2
- $B \in Y_1$ iff some option of B is in Y_0 and some is in Y_2
- $B \in Y_2$ iff some option of B is in Y_2 but none is in Y_0

For an even non-endgame A,

- $A \in Z_0$ iff no option of A is in Z_2 but some is in Y_0
- $A \in Z_1$ iff some option of A is in Z_0 and some is in Z_2
- $A \in Z_2$ iff A is not in Z_0 or Z_1

For an odd game B,

- $B \in Z_0$ iff some option of B is in Z_0 but none is in Z_2
- $B \in Z_1$ iff some option of B is in Z_0 and some is in Z_2
- $B \in Z_2$ iff no option of B is in Z_0

Theorem 6.1. *X, Y, and Z are gradings.*

Proof. It is easily checked from the definitions that each of X, Y, and Z is a partition of the class of games, and all further requirements are easily checked, except for the requirement on Z_2. We need to show that if $C \in Z_2$, then all options of C are in $Z_1 \cup Z_2$. This is clear if C is odd, so suppose C is even. Then if some $C' \in Z_0$, the next lemma implies $C' \in Y_0$. Then if some other option of C is in Z_2, $C \in Z_1$, and otherwise $C \in Z_0$. So no $C' \in Z_0$. $\square$

Lemma 6.2. *If C is an even game in Z_0, then $C \in Y_0$*

Proof. If $C = u$, then $C \in Y_0$ and we are done. Otherwise, all options of C are in $Z_0 \cup Z_1$, so by the following lemma no option of C is in Y_2. Also, $C \in Z_0$ implies that some option of C is in Y_0. So some option of C in Y_0 but none is in Y_2. Therefore $C \in Y_0$. $\square$

Lemma 6.3. *If C is an odd game in $Z_0 \cup Z_1$, then $C \in Y_0 \cup Y_1$.*

Proof. By the definition of Z_0 and Z_1, some option of C is in Z_0. By the previous lemma, this option is in Y_0. Therefore, $C \notin Y_2$. $\square$

Using the gradings X, Y, and Z, we define three functions on games:

$$X(A) = i \iff A \in X_i$$
$$Y(A) = i \iff A \in Y_i$$
$$Z(A) = i \iff A \in Z_i$$

Theorem 6.4. *For any game A, $Y(A) \leq Z(A)$*

Proof. By Theorem 5.6, $Y(A) = Y(A + *)$ and $Z(A) = Z(A + *)$. So $Y(A) = Y(\mathrm{o}(A)) = Y(\mathrm{e}(A))$ and $Z(A) = Z(\mathrm{o}(A)) = Z(\mathrm{e}(A))$. We consider three cases:

- If $Z(A) = 2$, there is nothing to show.
- If $Z(A) = 1$, then $Z(\mathrm{o}(A)) = 1$, so $\mathrm{o}(A)$ is an odd game in $Z_1 \subset Z_0 \cup Z_1$. Then $\mathrm{o}(A)$ is also in $Y_0 \cup Y_1$ by Lemma 6.3. So $Y(A) = Y(\mathrm{o}(A)) \leq 1 = Z(A)$.
- If $Z(A) = 0$, then $Z(\mathrm{e}(A)) = 0$, so $\mathrm{e}(A)$ is an even game in Z_0. Then by Lemma 6.2, $\mathrm{e}(A) \in Y_0$, so $Y(A) = Y(\mathrm{e}(A)) = 0 \leq 0 = Z(A)$.

□

Analogously for X, we have

Lemma 6.5. *Let C be an even game in Z_0. Then $C \in X_0$.*

Proof. If $C = u$ then $C \in X_0$. Otherwise, all options of C are in $Z_0 \cup Z_1$, so by the following lemma, all options are in $X_0 \cup X_1$, implying that $C \in X_0$. □

Lemma 6.6. *Let C be an odd game in $Z_0 \cup Z_1$. Then $C \in X_0 \cup X_1$.*

Proof. Since C is in $Z_0 \cup Z_1$, some $C' \in Z_0$. Then the previous lemma implies $C' \in X_0$, so $C \in X_0 \cup X_1$. □

Theorem 6.7. *For any game A, $X(A) \leq Z(A)$*

Proof. Analogous to the proof that $Y(A) \leq Z(A)$. □

The importance of X and Y lies in the following:

Theorem 6.8. *An even game A is in X_0 iff it is in U2, and an odd game B is in X_2 iff it is in K2.*

Proof. It is clear from the definitions that

- An even game A is in X_0 iff it is u or all of its options are in $X_0 \cup X_1$.
- An odd game B is in $X_0 \cup X_1$ iff one of its options is in X_0.

Meanwhile,

- An even game A is in U2 iff it is u or all of its options are in U1
- An odd game B is in U1 iff one of its options is in U2

So clearly these define the same sets. □

Theorem 6.9. *An even game A is in Y_2 iff it is in K2, and an odd game B is in Y_0 iff it is in U2.*

Proof. Analogous to the previous theorem. □

Thus, we could have alternatively defined X and Y by

- $A \in X_0$ iff $\mathrm{e}(A) \in U2$ and $\mathrm{o}(A) \in U1$
- $A \in X_1$ iff $\mathrm{e}(A) \in K1$ and $\mathrm{o}(A) \in U1$
- $A \in X_2$ iff $\mathrm{e}(A) \in K1$ and $\mathrm{o}(A) \in K2$
- $A \in Y_0$ iff $\mathrm{e}(A) \in U1$ and $\mathrm{o}(A) \in U2$
- $A \in Y_1$ iff $\mathrm{e}(A) \in U1$ and $\mathrm{o}(A) \in K1$
- $A \in Y_2$ iff $\mathrm{e}(A) \in K2$ and $\mathrm{o}(A) \in K1$

7. Properties of X, Y, and Z

From Theorems 5.5 and 5.6, we know that

$$X(A+B) \geq X(A) \oplus X(B) \tag{1}$$
$$Y(A+B) \geq Y(A) \oplus Y(B)$$
$$Z(A+B) \geq Z(A) \oplus Z(B)$$
$$X(A+*) = X(A)$$
$$Y(A+*) = Y(A)$$
$$Z(A+*) = Z(A)$$

where $i \oplus j = \min(2, i+j)$. Also, from Theorems 6.4 and 6.7 we know that

$$X(A) \leq Z(A) \geq Y(A)$$

In this section, we develop rules that nearly determine the values of X, Y, and Z for sums of games.

Lemma 7.1. *If A and B are even games in X_0, then $A+B \in X_0$.*

Proof. If $A+B$ is an endgame, then so are A and B, so both equal u, and $A+B = u \in X_0$. Otherwise, we only need to show that all options of $A+B$ are in $X_0 \cup X_1$. Any option might as well be of the form $A'+B$. Since $A \in X_0$, $A' \in X_0 \cup X_1$, so the following lemma implies that $A'+B \in X_0 \cup X_1$. □

Lemma 7.2. *If A is an even game in X_0, and B is an odd game in $X_0 \cup X_1$, then $A+B \in X_0 \cup X_1$.*

Proof. Since B is in $X_0 \cup X_1$, some option B' is in X_0. Then $A+B'$ is in X_0 by the previous lemma. So some option of $A+B$ is in X_0, and $A+B \in X_0 \cup X_1$. □

These imply the following theorem:

Theorem 7.3. *If A and B are arbitrary games, then*

$$X(A+B) = X(A) \oplus X(B)$$

Proof. Using Equation (1) we only need to show that $X(A+B) \leq X(A) \oplus X(B)$. This is trivial unless $X(A) \oplus X(B) < 2$. This only happens if both $X(A)$ and $X(B)$ are less than 2, and at least one is 0. So there are essentially two cases:

- If $X(A) = 0$ and $X(B) = 1$, then $X(\mathrm{e}(A)) = 0$ and $X(\mathrm{o}(B)) = 1$, so $\mathrm{e}(A) \in X_0$ and $\mathrm{o}(B) \in X_0 \cup X_1$. By the previous lemma, $\mathrm{o}(A+B) \cong \mathrm{e}(A) + \mathrm{o}(B) \in X_0 \cup X_1$. So $X(A+B) = X(\mathrm{o}(A+B)) \leq 1 = X(A) \oplus X(B)$. Here we use the fact that $\mathrm{o}(A+B)$ and $\mathrm{e}(A)+\mathrm{o}(B)$ differ at most by the addition of some stars.
- If $X(A) = 0$ and $X(B) = 0$, then $X(\mathrm{e}(A)) = X(\mathrm{e}(B)) = 0$, so $\mathrm{e}(A), \mathrm{e}(B) \in X_0$. Then by a previous lemma, $\mathrm{e}(A+B) \cong \mathrm{e}(A) + \mathrm{e}(B) \in X_0$. Thus $X(A+B) = X(\mathrm{e}(A+B)) = 0 \leq X(A) \oplus X(B)$, using the fact that $\mathrm{e}(A+B)$ differs from $\mathrm{e}(A)$ and $\mathrm{e}(B)$ by the addition of some stars.

□

Unfortunately, we cannot prove the same result for Y. However, there are some relations between Y and Z:

Lemma 7.4. *If A is an odd game in Y_0 and B is an even game in Z_0, then $A+B$ is in Y_0.*

Proof. Since $A+B$ is odd and not an endgame, we only need to show that no option is in Y_2. This follows from the following two lemmas, and the facts that all options of A are in $Y_0 \cup Y_1$, and all options of B are in $Z_0 \cup Z_1$. □

Lemma 7.5. *If A is an odd game in Y_0 and B is an odd game in $Z_0 \cup Z_1$, then $A+B$ is in $Y_0 \cup Y_1$.*

Proof. Since $A+B$ is even, we only need to show that some option of $A+B$ is in Y_0. This follows from the fact that some $B' \in Z_0$, and so $A+B' \in Y_0$ by the previous lemma. □

Lemma 7.6. *If A is an even game in $Y_0 \cup Y_1$ and B is an even game in Z_0, then $A+B$ is in $Y_0 \cup Y_1$.*

Proof. If A is an endgame, then $A = u$, and we need $A+B = B$ to be in $Y_0 \cup Y_1$. This follows by Lemma 6.2. Otherwise, A is not an endgame, so some option A' is in Y_0. Then $A'+B \in Y_0$ by Lemma 7.4. □

These imply the following:

Theorem 7.7. *If A and B are arbitrary games, then*

$$Y(A+B) \leq Y(A) \oplus Z(B)$$

Proof. We break the situation into cases:

- If $Y(A) \oplus Z(B) = 0$, then $Y(A) = Y(\mathrm{o}(A)) = 0$ and $Z(B) = Z(\mathrm{e}(B)) = 0$. So $\mathrm{o}(A) \in Y_0$ and $\mathrm{e}(B) \in Z_0$. Then by Lemma 7.4, $\mathrm{o}(A) + \mathrm{e}(B) \in Y_0$. Meanwhile, $A+B$ differs from $\mathrm{o}(A) + \mathrm{e}(B)$ solely by the addition of some stars, so $Y(A+B) = Y(\mathrm{o}(A) + \mathrm{e}(B)) = 0 \leq Y(A) \oplus Z(B)$
- If $Y(A) = 1$ and $Z(B) = 0$, then $\mathrm{e}(A) \in Y_0 \cup Y_1$ and $\mathrm{e}(B) \in Z_0$, so by Lemma 7.6 $\mathrm{e}(A) + \mathrm{e}(B) \in Y_0 \cup Y_1$. Thus
$$Y(A+B) = Y(\mathrm{e}(A) + \mathrm{e}(B)) \leq 1 = Y(A) \oplus Z(B)$$
- If $Y(A) = 0$ and $Z(B) = 1$, then $\mathrm{o}(A) \in Y_0$ and $\mathrm{o}(B) \in Z_0 \cup Z_1$, so by Lemma 7.5, $\mathrm{o}(A) + \mathrm{o}(B) \in Y_0 \cup Y_1$. Then
$$Y(A+B) = Y(\mathrm{o}(A) + \mathrm{o}(B)) \leq 1 = Y(A) \oplus Z(B)$$
- In all other cases, $Y(A) + Z(B) \geq 2$, so $Y(A+B) \leq 2 = Y(A) \oplus Z(B)$.

□

It turns out that the same addition law works for X as for Z.

Lemma 7.8. *If A and B are even games in Z_0, then $A+B$ is in Z_0*

Proof. If A an endgame, then it is u, so $A+B = B \in Z_0$, and we are done. Otherwise, we need to show that no option of $A+B$ is in Z_2, and some is in Y_0. Any option is, without loss of generality, of the form $A'+B$. But A' is not in Z_2, so $A'+B$ is neither, by the following lemma. So no option of $A+B$ is in Z_2. Also, A is an even non-endgame in Z_0, so some option A' is in Y_0. Then A' is an odd game in Y_0 and B is an even game in Z_0, so by Lemma 7.4, $A'+B \in Y_0$. So some option of $A+B$ is in Y_0, and we conclude that $A+B \in Z_0$. □

Lemma 7.9. *If A is an even game in Z_0 and B is an odd game in $Z_0 \cup Z_1$, then $A+B$ is in $Z_0 \cup Z_1$.*

Proof. By definition of Z, B has an option $B' \in Z_0$. Then $A + B' \in Z_0$ by the preceding lemma. So $A + B \in Z_0 \cup Z_1$, since $A + B$ is odd. □

This implies the following:

Theorem 7.10. *For arbitrary games A and B,*

$$Z(A + B) = Z(A) \oplus Z(B)$$

Proof. Analogous to Theorem 7.3. □

There is one final result about sums, Y, and Z.

Lemma 7.11. *Let A and B be odd games in Z_2. Then $A + B$ is in Y_2.*

Proof. Since $A+B$ is not an endgame, we only need to show that any option of $A+B$ is not in Y_0. Consider an arbitrary option, $A' + B$ without loss of generality. Then $A' \in Z_1 \cup Z_2$ by definition of Z. So by the following lemma, $A' + B \in Y_1 \cup Y_2$. □

Lemma 7.12. *Let A be an even game in $Z_1 \cup Z_2$ and B be an odd game in Z_2. Then $A + B \in Y_1 \cup Y_2$.*

Proof. If A is an endgame, then $A = k$, so $Y(A+B) \geq Y(A) = 2$ and so $A+B \in Y_2$. Otherwise, we either have

- Some $A' \in Z_2$, in which case $A' + B \in Y_2$ by the previous lemma. Then since $A + B$ is an odd game with an option in Y_2, it is in $Y_1 \cup Y_2$ itself.
- No $A' \in Y_0$, in which case $A \in Y_2$, so $A + B \in Y_2 \subseteq Y_1 \cup Y_2$ by Lemma 5.2.

□

These imply the following:

Theorem 7.13. *If A and B are arbitrary games, then*

$$Y(A + B) \geq Z(A) \odot Z(B),$$

where $i \odot j = max(0, i + j - 2)$.

Proof. As usual, we break into cases:

- If $Z(A) = Z(B) = 2$, then $\mathrm{o}(A)$ and $\mathrm{o}(B) \in Z_2$, so $\mathrm{o}(A) + \mathrm{o}(B) \in Y_2$, and so
$$Y(A + B) = Y(\mathrm{o}(A) + \mathrm{o}(B)) = 2 = Z(A) \odot Z(B).$$
- If $Z(A) = 1$ and $Z(B) = 2$, then $\mathrm{e}(A) \in Z_1 \cup Z_2$ and $\mathrm{o}(B) \in Z_2$, so by the previous lemma $\mathrm{e}(A) + \mathrm{o}(B) \in Y_1 \cup Y_2$, and so
$$Y(A + B) = Y(\mathrm{e}(A) + \mathrm{o}(B)) \geq 1 = Z(A) \odot Z(B).$$
- The case where $Z(A) = 2$ and $Z(B) = 1$ is similar.
- In all other cases, $Z(A) \odot Z(B) = 0$, so there is nothing to show.

□

8. The leftover cases

We have shown the following facts about X, Y, and Z:

$$X(A) \leq Z(A)$$
$$Y(A) \leq Z(A)$$
$$X(A+B) = X(A) \oplus X(B)$$
$$Z(A+B) = Z(A) \oplus Z(B)$$
$$Y(A+B) \geq Y(A) \oplus Y(B)$$
$$Y(A+B) \leq Y(A) \oplus Z(B)$$
$$Y(A+B) \leq Z(A) \oplus Y(B)$$
$$Y(A+B) \geq Z(A) \odot Z(B)$$

These almost determine the rules for combining X, Y, and Z. From now on we disregard X because it is mostly unrelated to Y and Z, and we already know how it behaves with respect to sums. We denote the class of games A which have $Y(A) = i$ and $Z(A) = j$ by ij. For example, we have 00, 01, 02, 11, 12, and 22. Then

- $ii + jk$ is $j'k'$, where $j' = i \oplus j$ and $k' = i \oplus k$, since $Z(ii + jk) = Z(ii) + Z(jk) = i \oplus k$, and
$$i \oplus j = Y(ii) \oplus Y(jk) \leq Y(ii + jk) \leq Z(ii) \oplus Y(jk) = i \oplus j$$
- $02 + jk$ is $k2$, since $Z(02 + jk) = Z(02) \oplus Z(jk) = 2 \oplus k = 2$, and
$$k = 2 \odot k = Z(02) \odot Z(jk) \leq Y(02 + jk) \leq Y(02) \oplus Z(jk) = 0 \oplus k = k$$
- $i2 + j2$ is 22, since
$$Y(i2 + j2) \geq Z(i2) \odot Z(j2) = 2 \odot 2 = 2$$

These determine all sums except for the remaining two:

- $01 + 01$ can be either 02 or 12.
- $01 + 12$ can be either 12 or 22.

Unfortunately, the behavior of these two sums depends on a finer subdivision of the classes 01 and 12.

Definition 8.1. *A game A in* 01 *is in the set* 01^- *if no option is in* 12^+*, and in* 01^+ *otherwise.*

Definition 8.2. *A game A in* 12 *is in the set* 12^+ *if no option is in* 01^-*, and in* 12^- *otherwise.*

Lemma 8.3. *If A and B are two games in* 01^-*, then $A + B$ is in* 02.

Proof. We already know that $A + B$ is either in 02 or 12. Thus we only need to show that $A + B \notin Y_1$. Suppose it was. Then some option $A' + B$ is in Y_2. Now $A \in Y_0$, so any option $A' \in Y_0 \cup Y_1$. If $A' \in Y_0$, then

$$Y(A' + B) \leq Y(A') \oplus Z(B) = 0 \oplus 1 = 1,$$

a contradiction. Otherwise, $A' \in Y_1$, so A' is in either 11 or 12^-. In the first case, $A' + B$ is in $11 + 01$ which is in 12, so $A' + B \notin Y_2$. In the latter case, $A' + B$ is in 12 by the following lemma, so $A' + B \notin Y_2$. $\square$

Lemma 8.4. *If A is in* 01^- *and B is in* 12^-*, then $A + B$ is in* 12.

Proof. We already know that $A+B$ is in either 12 or 22. So we only need to show that $A+B \notin Y_2$. Now since $B \in 12^-$, some option B' is in 01^-. Then $A+B'$ is in $01^- + 01^-$, so by the previous lemma $A+B' \in 02 \subseteq Y_0$. Then an option of $A+B$ is in Y_0, so $A+B \notin Y_2$. □

Lemma 8.5. *If $A \in 01^+$ and $B \in 01$, then $A+B \in 12$.*

Proof. We already know that $A+B$ is in either 02 or 12, so we only need to show that $A+B \notin Y_0$. Since $A \in 01^+$, it has an option $A' \in 12^+$. Then by the following lemma, $A'+B \in 22 \subseteq Y_2$, so $A+B$ cannot be in Y_0. □

Lemma 8.6. *If $A \in 12^+$ and $B \in 01$, then $A+B \in 22$*

Proof. We already know that $A+B$ is either in 12 or 22, so we only need to show that $A+B \notin Y_1$. Suppose that it was. Then some option of $A+B$ is in Y_0. However, an option of the form $A+B'$ cannot be in Y_0, because $Y(A+B') \geq Y(A) > 0$. So suppose $A'+B$ is in Y_0. Now since $A \in Z_2$, $A' \in Z_1 \cup Z_2$. If $A' \in Z_2$, then

$$Y(A'+B) \geq Z(A') \odot Z(B) = 2 \odot 1 = 1,$$

so $A'+B \notin Y_0$. Otherwise, $A' \in Z_1$. Also, $Y(A') \leq Y(A'+B) = 0$, so $A' \in Y_0$. Thus A' is in 01, and in 01^+ in particular, by definition of 12^+. Then by the preceding lemma, $A'+B$ is in 12, and therefore not in Y_0. □

Lemma 8.7. *If $A \in 12$ and $B \in 01^+$, then $A+B \in 22$.*

Proof. We already know that $A+B$ is either in 12 or 22, so we only need to show that $A+B \notin Y_1$. Suppose that it was. Then some option of $A+B$ is in Y_0. However, an option of the form $A+B'$ cannot be in Y_0, because $Y(A+B') \geq Y(A) > 0$. So suppose $A'+B$ is in Y_0. Now since $A \in Z_2$, $A' \in Z_1 \cup Z_2$. If $A' \in Z_2$, then

$$Y(A'+B) \geq Z(A') \odot Z(B) = 2 \odot 1 = 1,$$

so $A'+B \notin Y_0$. Otherwise, $A' \in Z_1$. Also, $Y(A') \leq Y(A'+B) = 0$, so $A' \in Y_0$. Thus A' is in 01. However, $A'+B$ is in $01 + 01^+$, which by Lemma 8.5 is in 12, and therefore not in Y_0. □

In summary, we have

- $01^- + 01^-$ is 02
- $01^- + 01^+$ is 12
- $01^+ + 01^+$ is 12
- $01^- + 12^-$ is 12
- $01^- + 12^+$ is 22
- $01^+ + 12^-$ is 22
- $01^+ + 12^+$ is 22

Unfortunately, we do not yet have a complete method for adding values, since we don't know, for example, whether $01^- + 01^+$ is in 12^- or 12^+. However, it turns out that all questions are answered by the use of associativity.

Lemma 8.8. *Some game W is in 01^-.*

Proof. Define the following games

- $J = \{*, k + *\}$, which is an even game in 11 (because it is in G_1 for any grading G).

- $K = \{J\}$, an odd game in 02 (because no option is in Y_2 or Z_0 – see the definitions)
- $W = \{u, K\}$, an even game in 01 (because it has options in both Z_0 and Z_2, but no option is in Y_2 and some options are in Y_0).

Since no option of W is in 12, certainly no option is in 12^+, and $W \in 01^-$. □

The game W can now be used as a test to tell whether something is in 12^- or 12^+.

Lemma 8.9. *If $A \in 02$ and $B \in 01$, then $A + B \in 12^+$*

Proof. We already know that $A+B \in 12$, so we only need to show that for $W \in 01^-$, $(A + B) + W$ is 22, not 12. But $B + W$ is in $01 + 01 \subseteq 02$, and so $A + (B + W) \in 02 + 02 \subseteq 22$ □

Lemma 8.10. *If $A \in 11$ and $B \in 01^-$, then $A + B \in 12^-$*

Proof. We already know that $A+B \in 12$, so we only need to show that for $W \in 01^-$, $A + B + W$ is 12, not 22. But $B + W$ is in 02, and $02 + 11 \subseteq 12$. □

Lemma 8.11. *If $A \in 11$ and $B \in 01^+$, then $A + B \in 12^+$*

Proof. We already know that $A+B \in 12$, so we only need to show that for $W \in 01^-$, $A + B + W$ is 22, not 12. But $B + W$ is in 12, and $12 + 11 \subseteq 22$. □

Lemma 8.12. *If $A \in 11$ and $B \in 02$, then $A + B \in 12^+$*

Proof. Certainly $A + B \in 12$, so we only need $A + B + W \in 22$. Now $B + W \in 02 + 01 \subseteq 12$, so $A + (B + W) \in 11 + 12 \subseteq 22$. □

Lemma 8.13. *If $A \in 01^+$ and $B \in 01^-$, then $A + B \in 12^-$*

Proof. We need $A+B+W \in 12$. But $B+W \in 01^- + 01^- \subseteq 02$, and $02+01 \subseteq 12$. □

Lemma 8.14. *If $A \in 01^+$ and $B \in 01^+$, then $A + B \in 12^+$.*

Proof. With W as usual, $A + B + W \in 01^+ + (01^+ + 01^-) \subseteq 01^+ + 12^- \subseteq 22$, so $A + B$ must be in 12^+. □

9. The Monoid of Values

In summary, we have

Theorem 9.1. *There is an eight-element commutative monoid M defined on the set*

$$\{00, 01^-, 01^+, 11, 02, 12^-, 12^+, 22\},$$

and a function f from knot-type games to M such that $f(A + B) = f(A) + f(B)$ for all games A and B. A game with $f(A) \in ij^\pm$ or ij has $Y(A) = i$, $Z(A) = j$. The monoid M has the following addition table:

+	00	01^-	01^+	11	02	12^-	12^+	22
00	00	01^-	01^+	11	02	12^-	12^+	22
01^-	01^-	02	12^-	12^-	12+	12^+	22	22
01^+	01^+	12^-	12^+	12^+	12^+	22	22	22
11	11	12^-	12^+	22	12^+	22	22	22
02	02	12^+	12^+	12^+	22	22	22	22
12^-	12^-	12^+	22	22	22	22	22	22
12^+	12^+	22	22	22	22	22	22	22
22	22	22	22	22	22	22	22	22

Theorem 9.2. *There are at most 38 equivalence classes of games. Every game is determined up to equivalence by its X-value, its parity, and its representative in the monoid M.*

Proof. We must have $X \leq Z$. So there are nineteen possible combinations, ignoring parity: one for 00, two for each of 01^-, 01^+, and 11, and three for each of 02, 12^-, 12^+, and 22. If we define the value of a game A to be $(parity(A), X(A), f(A))$, then the value of $A+B$ is completely determined by the value of A and the value of B. Also, the outcome of a game A is determined by its value as follows: If A is even, then $A \in K1$ iff $X(A) > 0$, and $A \in U1$ iff $Y(A) < 2$. If A is odd, then $A \in K1$ iff $Y(A) > 0$, and $A \in U1$ iff $X(A) < 2$.

Thus, if A and A' have the same value, then for any game B, the value of $A+B$ is the same as the value of $A'+B$, and so the outcomes are also equal. Furthermore, A and A' have the same parity. Therefore, they are equivalent. ☐

In fact, it is tedious, but straightforward to verify that all nineteen possibilities occur. It then follows:

Theorem 9.3. *There are exactly 38 classes of games up to equivalence.*

Proof. We just need to show that if two even games have different values of X, Y, or Z, or different representatives in M, then they are not equivalent. Now if A is an even game, then $X(A)$ and $Y(A)$ is determined by the outcomes of A and $A+*$, which depends only on the value of A. So $X(\cdot)$ and $Y(\cdot)$ are well defined on values. Now, let E be some game in 02. Then

$$Z(A) = Z(A) \odot Z(E) \leq Y(A+E) \leq Z(A) \oplus Y(E) = Z(A),$$

so $Z(A)$ is determined by the value of $A+E$. Therefore, it is determined by the value of A. So $Z(\cdot)$ is well defined on values. Finally, we can tell whether a game is in 01^- or 01^+, or whether it is in 12^- or 12^+, by adding it to a game $W \in 01^-$, and seeing what the Y-value of the resulting game is. ☐

10. Weak Equivalence

The usual definition of “equivalence” or “indistinguishability” is simply that G and H are equivalent if outcome$(G+X)$ = outcome$(H+X)$ for all games. We modified this slightly, to require that G and H have the same parity. In this section, we show that without this extra stipulation, there are 37 equivalence classes of games. In fact, the only equivalence classes which merge are the even and odd games in $X_2 \cap Y_2 \cap Z_2$.

Definition 10.1. *Two games G and H are* weakly equivalent *(or* indistinguishable*) if for every game X,* outcome$(G+X)$ = outcome$(H+X)$.

As before, this is an equivalence relationship, and the quotient space has the structure of a commutative monoid.

Theorem 10.2. *There are 37 equivalence classes modulo indistinguishability. These correspond to the 38 equivalence classes of $\cong$, except that the even and odd classes of games G satisfying $X(G) = Y(G) = Z(G) = 2$ are merged.*

Proof. It is clear that if G and H are both in $X_2 \cap Y_2 \cap Z_2$, then so is any game of the form $G + X$ or $H + X$, using Lemma 5.2. Also, every game in $X_2 \cap Y_2$ is a win for the knotter. So certainly outcome$(G + X) \cong$ outcome$(H + X)$.

Conversely, we need to show that if G and H are indistinguishable but not equivalent, then G and H are both in $X_2 \cap Y_2 \cap Z_2$, which equals $X_2 \cap Y_2$ by Theorem 6.7 or Theorem 6.4.

If G and H have the same parity, then they are equivalent, by definition of equivalence. So they do not have the same parity. If we add G to a game in the class 022, we will get a game in the class 022, 122, or 222, according to the value of $X(G)$. By the comments at the end of §6, these classes can be alternatively described as follows:

- 022 is all games G for which outcome(e(G)) $= 2$ and outcome(o(G)) $= 1$
- 122 is all games G for which outcome(e(G)) $= k$ and outcome(o(G)) $= 1$
- 222 is all games G for which outcome(e(G)) $= k$ and outcome(o(G)) $= k$

So we can tell which of these a game G is in by looking at the outcomes of G and $G + *$. Thus, if G and H are indistinguishable, and R is an arbitrary game in the class 022, then by examining the outcomes of $G + R$ and $G + R + *$, which must equal the outcomes of $H + R$ and $H + R + *$, we conclude that $X(G) = X(H)$.

Again using the comments at the end of §6, it turns out that the outcome of G and $G + *$ tells us the *unordered* pair $\{X(G), Y(G)\}$. But since we also know $X(G)$ just by knowing G's value modulo indistinguishability, we can also determine $Y(G)$. In other words, if G and H are indistinguishable, they must belong to the same $X_i \cap Y_j$.

If $G \in X_i \cap Y_j$, the comments at the end of §6 completely determine the outcome of o(G) and e(G) in terms of i and j. Therefore, we can tell whether $G =$ o(G) or $G =$ e(G), unless o(G) and e(G) have the same outcome. This turns out to occur when $i = j$, i.e., when $X(G) = Y(G)$.

So, only knowing G up to indistinguishability allows us to determine the parity of G, unless $G \in X_0 \cap Y_0$ or $G \in X_1 \cap Y_1$ or $G \in X_2 \cap Y_2$. In the first two cases, however, we can add G to an arbitrary game $R \in 022$, and get a game in $X_i \cap Y_2$ for $i = X(G) < 2$. Then the parity of $G+R$ can be determined, and we know the parity of R, so the parity of G can be determined. So if G and H are indistinguishable, and either is not in $X_2 \cap Y_2$, then they must be equivalent. □

11. Future Work

The current exposition of the theory in this paper seems very long and tedious. One wonders if there is a better way to prove the main results. There is certainly a way to automate the bulk of the proofs, as follows:

- Write down the candidate monoid of values V
- For each $x \in V$, write down the *forbidden set* $F_x \subseteq V$ of values that the options of x may not take, and the collection $\mathcal{R}_x \subseteq \mathcal{P}(V)$ of sets of values that the options of x must hit. For example, if G is an even game, it belongs

to the class 101^+ iff (a) none of its options belong to the classes 022, 122, or 222, (b) one of its options belongs to one of the classes 012^+, 112^+, 212^+, (c) one of its options belongs to 000, (d) one of its options belongs to either 202 or 212. So we would have

$$F_{101^+} = \{022, 122, 222\}$$

$$\mathcal{R}_{101^+} = \{\{012^+, 112^+, 212^+\}, \{000\}, \{202, 212\}\},$$

roughly.

- Show that for each $x, y \in V$, $F_x + y \subseteq F_{x+y}$ and $x + F_y \subseteq F_{x+y}$ (here, $+$ denotes a Minkowski sum).
- Show that for each $R \in \mathcal{R}_{x+y}$, there is either an $R_1 \in \mathcal{R}_x$ such that $R_1 + y \subseteq R$ or an $R_2 \in \mathcal{R}_y$ such that $x + R_2 \subseteq R$.
- Show that each game belongs to exactly one class, if we classify games recursively using the F_x and $\mathcal{R}_x$. (This step seems harder to automate).
- Conclude by a massive inductive argument that V is a quotient space on games.
- Show that the classes corresponding to each outcome have the correct games.

There are also some technicalities involving sums of the form $G + u$ and $G + k$. For example, we need u to be assigned to the identity element of V, and k to be assigned to an essentially nilpotent element.

However, this approach is no more enlightening. There might also be symbolic ways of describing all the lemmas in this paper, which could significantly reduce the lengths of the proofs. The chief difficulty seems to be in formalizing the sort of induction that is used throughout.

As far as actually applying this theory, the only evident application is to the case of knot games. The easiest knot games to analyze are ones corresponding to the shadows or pseudodiagrams of rational knots, since in this case we have a method for telling whether a knot is the unknot. This line of inquiry is carried out in another paper by the same author. Oddly enough, some experimental evidence suggests that the only values which occur for knot games are 000, 011, 022, 111, 122, and 222. In fact, these may be the only values which occur for a wide class of games, suggesting that the rest of the theory developed here is somewhat frivolous.

There are several ways to generalize knot-type games. One of the most obvious is to remove the bizarre parity requirement. Unfortunately, this breaks many of the mirroring strategies and arguments used in this paper. Two more fruitful possibilities may be

- Allowing the two players to have separate moves. This is not as much of a leap as the leap from impartial to partizan games in Combinatorial Game Theory, since we already have an asymmetry between the two players, and we can already compare games in a certain sense. In particular, in many cases, a game has two options, but one is definitely better for the unknotter and the other is definitely better for the knotter. This happens with the game $\{u, k\}$ for example. In this case, we could just as well assume that the unknotter will only use the move that is better for her, and the knotter will only use his preferred move. Consequently, the place where complications might occur in this extended theory is the case where each player's option is better for her opponent. For example, if we construct a game in which the Unknotter can move to k and the Knotter can move to u, this game will not be equivalent to one of the ones considered so far.

- Another possibility is to add another operation into the mix: a sum of games where the unknotter only needs to win one of the two games. This is the dual operation to the one considered in this paper. Allowing both operations brings symmetry back into the picture, but may complicate things. For example, the definition of equivalence needs to be something like this: two games G and H are equivalent iff, for every sequence $\{X_1, X_2, \ldots X_n\}$,
$$\text{outcome}(G +_k X_1 +_u X_2 +_k \ldots X_n) =$$
$$\text{outcome}(H +_k X_1 +_u X_2 +_k \ldots X_n),$$
where $+_k$ and $+_u$ are the two sums available.

The symmetry introduced by the second of these possibilities might shed some light on what a "grading" is. It is interesting to note that in the dual theory, Y plays the role of X, and X plays the role of Y. Consequently, we will have conditions like $Y(G +_u H) = Y(G) \odot Y(H)$, and $X(G +_u H) \leq W(G) \oplus W(H)$, where W is the dual grading to Z. The fact that we have already seen the dual of $\oplus$, namely $\odot$, also seems peculiar.

12. References

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