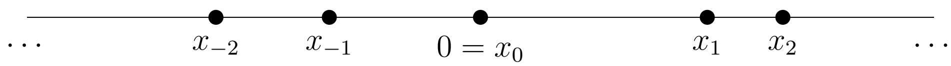
A Random Walk in a Dynamic Environment

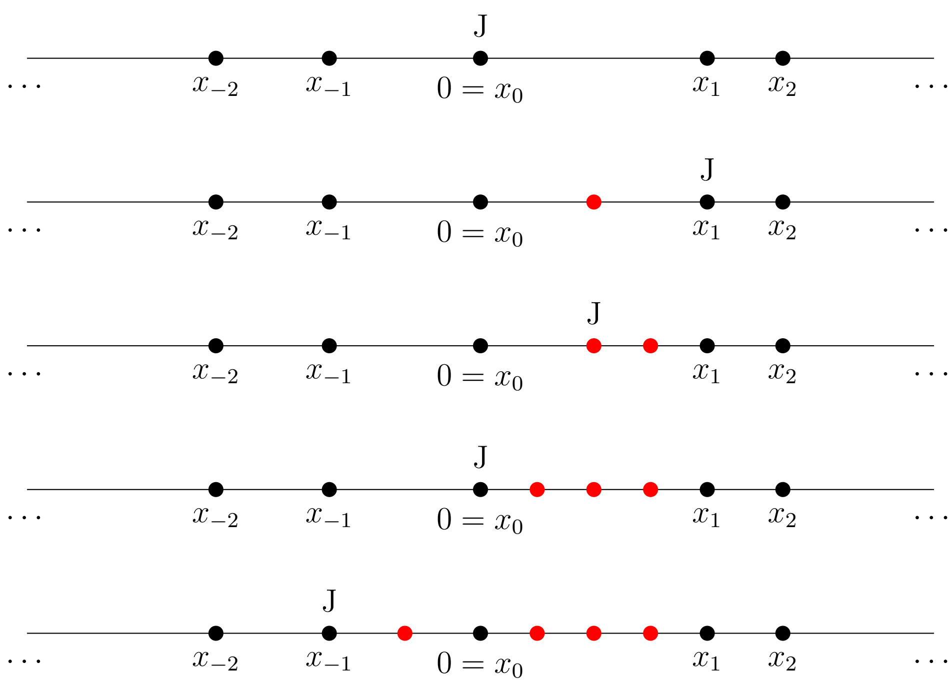
Infinitely many points ("trees") are chosen "randomly" on $\mathbb{R}$, including one at 0. By randomly we mean a normal distribution: the probability that any given tree lies between a and b is $\frac{1}{\sqrt{\pi}}\int_a^b e^{-x^2}dx$. Call these trees $x_0, x_{-1}, x_1, \cdots$ in the following manner:



Our "Johnny Appleseed" begins at $t = 0$ at $0 = x_0$. He performs a walk in a dynamic environment in the following manner:

- He walks towards the furthest of his two neighboring trees. If both neighbors are an equal distance away, he chooses randomly. (In the future, we will ignore this case, since the probability that any three randomly chosen numbers are equally distanced is 0).
- Halfway to that tree, he plants a new tree.
- Repeat.

Note that while the placement of the initial trees is random, Johnny's walk is pre-determined. For example, his first few steps on the above distribution would look like:



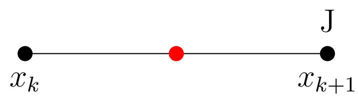
and so on. The red dots represent new trees that Johnny planted. Henceforth, we will refer to inital trees as *black trees* and trees planted by Johnny as *red trees*.

We'll now introduce some notation:
Let s_t represent the sequence of steps Johnny takes.

Let I_k represent the interval $[x_k, x_{k+1}]$ between trees, with length d_k.

This problem seems hopelessly complex, since the trees that Johnny travels between become increasingly numerous as Johnny plants more and more trees behind him. However, we can view the problem from a different standpoint that makes it much easier to handle.

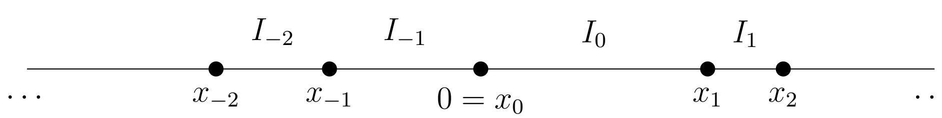
Note that the first time Johnny travels through any interval, he divides it in half with a red tree:



When Johnny ends up at x_k and moves right, or at x_{k+1} and moves left, we say Johnny is *traversing* I_k. Now, if Johnny moves left and traverses I_k, he will evenly divide I_k into four equal parts with red trees. It's not hard to show that Johnny must travel exactly from x_{k+1} to x_k or vice versa. In other words, it is never possible that Johnny "turns around" in-between two black trees.

Let a_k represent the number of times Johnny has completely passed through the interval I_k. Clearly the number of red trees in I_k is exactly $2^{a_k} - 1 = R_k$. Call the *density* of red trees in any I_k, $d_k/R_k = \rho_k$. Be wary of the abuse of notation: a_k, R_k, and therefore ρ_k are constantly changing, everything else is fixed.
So now, we see that the problem can be viewed as Johnny simply traveling from black tree to black tree, while increasing the density of the red trees in-between them. We'll also prove something now that will be useful later:
Lemma: Johnny cannot oscillate back and forth forever in some I_k.
WLOG, let the closest neighbor of I_k be x_{k+2}, with $d_{k+1} = \epsilon$. Then every time Johnny starts at x_{k+1} and traverses I_k twice to return to his starting position, he reduces ρ_k by a factor of 4. So, there exists some N such that, after N pairs traversals of I_k, $\rho_k/4^N < \epsilon$, and Johnny leaves I_k to go to x_{k+2}.

We'll now restate the problem. We have some intervals I_k on the real line, separated by endpoints x_k.

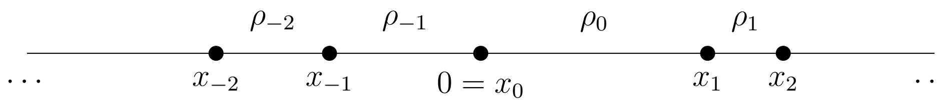


We can associate with each interval its density: initially, they are all zero. Now, Johnny moves from x_k to x_{k+1} if and only if $\rho_k > \rho_{k-1}$. With each move he halves the density of the corresponding interval. So now the problem is this: we have a doubly infinite array of densities: $\mathcal{R} = \{\cdots, \rho_{-2}, \rho_{-1}, \rho_0, \rho_1, \rho_2, \cdots\}$. We also have a position, Johnny's position, initially between ρ_{-1} and ρ_0 at $x_0 = 0$. Initially, for every element in $\mathcal{R}$, $\rho_k = d_k$. Our random walk is:

- Starting at some x_k, move to x_{k-1} or x_{k+1} if ρ_{k-1} or ρ_{k+1} is larger than the other, respectively.

- Halve the respective density.
- Repeat.

We prove the following lemma that nicely orders $\mathcal{R}$:
Lemma: WLOG, if Johnny is at some $x_k > 0$ for the first time, then $\rho_0 < \rho_1 < \cdots < \rho_k$. We can assume strict inequality since we assume no d_k are the same.
Furthermore, once the ρ's are ordered up to k, they can never be unordered. That is, they might sometimes be unordered for "a few" steps, and immediately ordered afterwards.



Well how did Johnny get to x_k? Since he started from 0, and this is his first time at x_k, he must have gotten there from x_{k-1}. So $\rho_{k-1} < \rho_k$. Well how did he get to x_{k-1}? He couldn't have walked backwards from x_k, since we assume it's his first time there. So he must have gotten there from x_{k-2}, so $\rho_{k-2} < \rho_{k-1} < \rho_k$. So far we know Johnny's walking sequence must be at least $\cdots x_{k-2}x_{k-1}x_k$. Well, here Johnny might have oscillated between x_{k-2} and x_{k-1} for an arbitrarily long amount of steps, depending on d_{k-1}. But since Johnny could not have oscillated in I_{k-2} forever, as shown above, he must have, at one point moved from x_{k-3} to x_{k-2}, meaning $\rho_{k-3} < \rho_{k-2} < \rho_{k-1} < \rho_k$. Finally, we can repeat the above argument as many times as it takes to get back to $x_0 = 0$, and we get that $\rho_0 < \rho_1 < \cdots < \rho_k$.
The second part of the lemma will be proved later when we talk about relatively ordered arrays.

An extremely similar argument shows that $\rho_0 < \rho_{-1} < \cdots < \rho_{-k}$. We now state our first big theorem:
Theorem: Johnny goes to $\pm\infty$. More formally, for any k, Johnny eventually reaches x_k.
To begin the proof, we'll need to rewrite $\mathcal{R}$. First, look at $\rho_0 \cdots \rho_k$ and take the maximum, call it ρ_a. Then take $\rho_{-1}, \rho_{-2}, \cdots$ until we've found some $\rho_{-b} > \rho_a$. Now we'll take the array $\{\rho_{-b}, \rho_{-b+1}, \cdots, \rho_{a-1}, \rho_a\}$ and transform into a relatively ordered array. This is best explained by example:

$$\{13, 17, -1, 10, 5, 0, -7, -25, 3, -12\} \implies \{9, 10, 4, 8, 7, 5, 3, 1, 6, 2\}$$

We just replace the kth smallest element by k. We'll call this new relatively ordered array of densities $\mathcal{M} = \{\mu_{-k}, \cdots, \mu_k\}$. Now after a certain operation, if one ρ becomes less than another, we just swap the (necessarily) consecutive μ's in $\mathcal{M}$.
Now we can easily prove the lemma that wasn't proven above: Once the ρ's, or equivalently μ's, from 0 to k are ordered, they can only be unordered for at most k steps, after which they will be immediately ordered again. We prove by some examples that are immediately generalizable to any case. Take $\{\rho_1, \rho_2, \rho_3\} \implies \{\mu_1, \mu_2, \mu_3\} = \{1, 2, 3\}$ with Johnny at x_3. We have that $\rho_2 > \rho_3/2$, so that 2 and 3 will switch on Johnny's next move. Our two cases are $\rho_1 > \rho_2/2$ and $\rho_1 \not> \rho_2/2$. Our examples will be $\{4, 6, 8\}$ and $\{1, 6, 8\}$. The steps immediately follow:

$$\{4, 6, 8\} \to \{4, 6, 4\} \to \{4, 3, 4\} \to \{2, 3, 4\}$$

$$\{1,6,8\} \to \{1,6,4\} \to \{1,3,4\}$$

Finally, we can prove a lemma that directly implies our theorem:
Lemma: If Johnny reaches x_k, he will reach x_{k+1}. Similarly for x_{-k}.
If Johnny is at x_k, we know that $\rho_0 < \rho_1 < \cdots < \rho_{k-1} \implies \mu_0 < \mu_1 < \cdots < \mu_{k-1}$. We also know one more thing: since WLOG Johnny moved right on the first step, $\mu_0 > \mu_{-1}$, and will not have changed, by the lemma above. So now, if $\mu_k > \mu_{k-1}$, then Johnny will move to the right, and we are done. Otherwise $\mu_k < \mu_{k-1}$, and we move to the left.

Other problems: This variety of random walk is classified as a maximum-half random walk. That is, Johnny looks to moves towards neighbor with the *maximum* distance from him, and places a tree *half* way through his walk. Other walks to consider are minimum-half and maximum-uniform. In the latter case, Johnny places a tree a point uniformly randomly over the distance he travels.
It turns out that the minimum-half problem is easy to completely solve. Formally, we say Johnny gets *stuck* if his sequence s_t converges. In the minimum-half problem, Johnny will always get stuck.

Assume he won't. WLOG, on his first move away from 0, he moves right, to the number $1 = x_1$. If $d_1 > 1/2$, then Johnny moves back to the red tree at $1/2$, and will "quickly" become stuck at the point

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{2}{3}$$

In fact, it's easy to see that if Johnny now moves left at any point in his walk, he will become stuck. Therefore, he must always move right during his walk. However, since $d_1 \leq 1/2$, $s_2 \leq 1 + 1/2 = 3/2$. Similarly,

$$d_2 \leq \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} \implies s_3 \leq 1 + \frac{1}{2} + \frac{1}{4}$$

$$d_n \leq \frac{1}{2^n} \implies s_{n+1} \leq 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$$

and so Johnny's walk is monotone and bounded, therefore it converges.
Further questions: