1. Basic Definitions and Conventions

When refering to a *graph*, we mean an undirected graph G with vertex set V broken down into interior vertices (V_{int}) and boundary vertices (∂V) where

(1) X is connected, finite, and has no loops
(2) There exists disjoint subsets, ∂V and V_{int}, of V where ∂V is nonempty and $V = \partial V \cup V_{int}$
(3) When identifying vertices, all ∂V labels come before V_{int}.

When drawing such graphs, boundary nodes are represented by a solid black dot and interior nodes by a open dot. Due to the inability to project three dimensional graphs onto paper, we will often draw the same node more than once. If a node is drawn multiple times, it will be denoted by the same number. The following graph in Figure 1 has multiple labelled vertices.

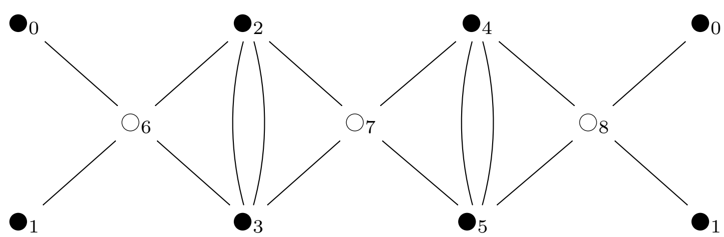


Figure 1. Blah for now

A *conductivity* on a graph G is a function γ which assigns to each edge e a positive real number $\gamma(e)$. Thus, a *resistor network*, $\Gamma(G, \gamma)$, is a graph G with a conductivity function γ [5]. The term *resistor network* is standard for a graph with resistors as edges. The conductance of a resistor is defined as the reciprocal of the resistance.

Suppose $\Gamma = (G, \gamma)$ is a resistor network with n vertices $(v_1, \ldots, v_n)$. Let

$$\gamma_{ij} = \sum_{\substack{\text{all edges } e \\ \text{joining } v_i \text{ to } v_j}} \gamma(e)$$

and $\gamma_{ij} = 0$ if there is no edge between v_i and v_j. If n is the number of nodes in the network (G, γ). then *Kirchhoff Matrix* is defined as

$$K_{ij} = \begin{cases} \gamma_{ij} & i \neq j \\ -\sum_{j \neq i} \gamma_{ij} & i = j \end{cases}$$

We often write the Kirchhoff Matrix in its block form. Let's assume that G has m boundary vertices, then

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

where A is an $m \times m$ matrix and C is an $n - m \times n - m$ matrix. From [5], it is shown that C is an invertible matrix. The definition of the *Kirchhoff matrix* leads to the following characteristics:

(1) The off-diagonal entries are positive or 0 ($\gamma_{ij} \geq 0$ for all $i \neq j$
(2) The row sums equal 0.
(3) K is symmetric.

Because of the symmetry of the Kirchhoff matrix, only the γ_{ij}'s in the upper triangle need to be found in order to define the entire K matrix. Moreove, there is a 1-to-1 correspondence between any matrix that satisfies the above characteristics and a Kirchhoff matrix of a network. (This will especially be useful later on in the paper.)

We now define Λ $(m \times m)$ to be the *response matrix* (G, γ):

$$\Lambda = A - BC^{-1}B^T = K/C.$$

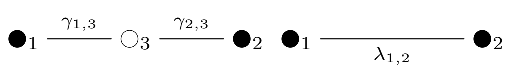
Λ can be interpreted as the Schur complement of C in K. This new matrix, Λ, has the same characteristics as the Kirchhoff matrix:

(1) The off-diagonals are greater than or equal to 0 ($\lambda_{ij} \geq 0$ for $i \neq 0$)
(2) The row sums are 0
(3) Λ is symmetric

Hence, Λ is a Kirchhoff matrix for another resistor network. This is easily shown.

The inverse problem associated with resistor networks is given a response matrix, Λ, and graph, G, find the conductivities (i.e. the entries in the Kirchhoff matrix). This simple definition of the inverse problem leads to the term *recoverability*. The idea is that with any response matrix there exist unique conductivities on G that generated Λ. Prior research shows that given certain characteristics of the graph, there is recoverability. Hence, we can find theoretically find the conductivities with the response matrix. When there are non-unique conductivities corresponding to the response matrix, we say that the graph is *n to 1* (The phrase coined in [2]) where $n > 1$. In this sense with a given Λ there exists n different conductivities that created the response matrix. The simpliest example of such graphs is the "series" connection (View Figure 2). In this case, we can show that Λ is a 2×2 matrix. Then $\lambda_{1,2} = \frac{\gamma_{1,3}\gamma_{2,3}}{\gamma_{1,3}+\gamma_{2,3}}$. Because we can not break $\lambda_{1,2}$ down into two components, $\gamma_{1,3}$ and $\gamma_{2,3}$ can be any value such that this ratio holds. Thus, there is ∞ *to 1* different conductivities that produce the response matrix. The more interesting cases occur when n is a finite number greater than one.

FIGURE 2. The graphs of the Kirchhoff and response matrices for series configurations



2. Motivation

The concept of *n to 1* graphs where n is a finite number has been of great interest. In [3], the 2-to-1 graphs were explored thoroughly (and 2^n) using n-gon in n-gon graphs. They established the existance of at least 2^n-to-1 graphs. It wasn't until the work by [1] and [2], who showed the existance of 3-to-1 graphs, that n-to-1 graphs were possible (n is finite). However, the creation of these graphs and their underlying structure remains relatively limited. In this paper, we explore the construction of n-to-1 graphs (when n does not equal a power of 2) and certain properties that arise.

3. Preliminary Notions

3.1. **Star-K Transformation.** The concept of Star-K Transformation and the enforcement of the quadrilateral rule are essential to constructing n-to-1 graphs. We define a *star* as a graph in which there exists no interior to interior edges and no boundary to boundary edges. An n-star (for $n > 1$) is defined as n boundary vertices connected to a single interior vertex. The K transformation takes the star and creates a complete graph with the interior vertex removed. We will denote K_n as a complete graph with n boundary vertices (This is not to be confused with the Kirchhoff matrix K). Thus the *Star-K Transformation* takes a n-star to a K_n.

The importance of this Transformation lies in understanding what it does to the Kirchhoff matrix. By eliminating interior vertices, we are doing row reductions to reduce B to the 0 matrix. Once all interior vertices have been "row reduced", the upper left-hand matrix is $A - BC^{-1}B^T$, which is defined as the response matrix, Λ.

Knowing this response matrix, we understand the relationship between γ_i and μ_{ij}. It is shown that

$$\frac{\gamma_0\gamma_1}{\sigma} = \mu_{0,1}$$

where $\sigma = \sum_i \gamma_i$. Similarly, $\mu_{1,3} = \frac{\gamma_1\gamma_3}{\sigma}$.

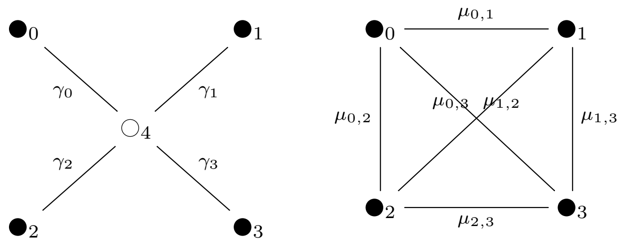


FIGURE 3. 4-star transformed to a K_4

3.2. **Quadrilateral Rule.** The quadrilateral rule derives from the 4-star and K_4 (Figure 3) and the Determinant Connection Theorem found in [5]. The basic idea is that in the 4-star there exists no 2-connection between any 4 boundary vertices. Because of this and the Determinant Connection Theorem, we know that the determinant of any 2×2 off-diagonal sub-matrix (γ_{ii} is not an entry in the sub-matrix) of the Kirchhoff matrix is 0. In the K_4 graph, it is equvialent to $det[2 \times 2] = 0$ or

$$\mu_{0,1}\mu_{2,3} = \mu_{1,3}\mu_{0,2} = \mu_{0,3}\mu_{1,2}$$

In more generality for larger stars, the quadrilateral rule is stated as

$$\mu_{ij}\mu_{kl} = \mu_{ik}\mu_{jl} \quad \text{for all } i \neq j \neq k \neq l. \tag{1}$$

The quadrilateral rule leads to an important theorem developed in [4] regarding computation of the γ from the response matrix.

Theorem 3.1. *A network on a complete graph (K_n) is response-equivalent to a star if and only if its conductivities satisfy Equation 1.*

In particular, [4] discovered a method of recovering the γ's given all sides of the quadrilateral. Define

$$\alpha_i = \sqrt{\frac{\mu_{ij}\mu_{ik}}{\mu_{jk}}}.$$

Then we can formally compute γ_i's by

$$\gamma_i = \alpha_i \sum_j \alpha_j. \tag{2}$$

We must be careful when applying this. **The μ_{ij}'s needed for the calculation do not necessarily come directly from the response matrix.** In some cases, the response matrix has the sum of multiple edges in the quadrilateral. Hence in order to compute the γ_i we must be able to calculate each edge in the quadrilateral separately.

3.3. **Parameterizing the Response Matrix.** This remarkable formula allows us to parameterize the response matrix so that it preserves the quadrilateral rule. A closer examination of Figure 3 reveals how we can choose λ_{ij} and create a valid graph with conductivities. Assume that the μ's for $\mu_{0,1}$ and $\mu_{2,3}$ are fixed. We are now able to select any positive values for $\mu_{0,2}$ and make $\mu_{1,3} = \frac{\mu_{0,1}\mu_{2,3}}{\mu_{0,2}}$ satisfy the quadrilateral rule. Moreover, we can choose a positive value for $\mu_{0,3}$ (independent of $\mu_{0,2}$) and make $\mu_{1,2} = \frac{\mu_{0,1}\mu_{2,3}}{\mu_{0,3}}$. By doing this, we can recover the corresponding γ's by Equation 2. Thus, we can construct a "response matrix" which corresponds to real conductivities on a 4-star.

To avoid confusion with the response matrix and the Kirchhoff matrix, we will defined new terminology.

Definition 3.1. *R-MultiGraph and R Matrix*
The R-MultiGraph is the graph of the star after performing the Star-K Transformation. In Figure 3, the complete graph is the R-MultiGraph. We use the term multi-graph to describe R because as we will see later multiple edges will be allowed in the R-MultiGraph. The R-Matrix is a matrix that stores the values of the μ's or the "conductivities" on the R-MultiGraph. Because multiple edges are allowed in the R-MultiGraph, the

entries in the R-Matrix often will be sets containing the μ's. If a multiple edge occurs in the R-MultiGraph, the R-Matrix separates the multiple edges storing both values as the entry in the matrix. This slightly differs from the response matrix since a multiple edge in the R-MultiGraph results is a sum of the μ's in the response matrix between any two vertices. When there is only a single edge in the R-MultiGraph, the response matrix and the R-Matrix will contain the same values. All off-diagonal entries in the R-Matrix must be positive and the diagonal entries are the same as the response matrix. In addition, the R-Matrix like the response is symmetric.

The R-Matrix is a useful tool because in order to recover the conductivities we must know all the sides on the quadrilateral. In order to better understand this terminology, we will provide some examples in the next section.

3.4. **Connecting Multiple Stars.** Although we can connect different n-stars to each other, we will focus on 4-stars. The following figures illustrate different connections of multiple 4-stars and their corresponding R-MultiGraphs.

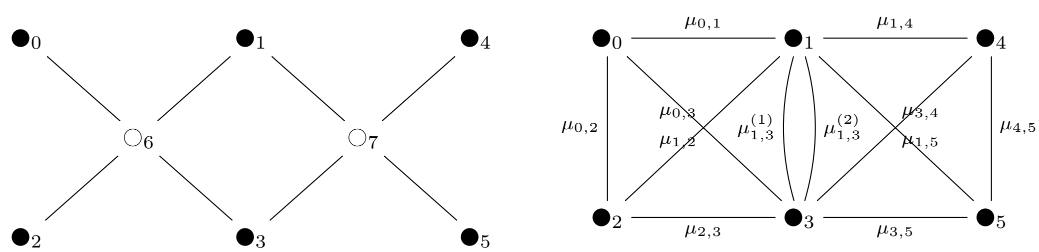


FIGURE 4. Connection of Multiple 4-Stars by Parallel Sides

The response matrix in Figure 4 would be the following:

$$\Lambda = \begin{bmatrix} -\sum \mu_i & \mu_{0,1} & \mu_{0,2} & \mu_{0,3} & 0 & 0 \\ \mu_{0,1} & -\sum \mu_i & \mu_{1,2} & \mu_{1,3}^{(1)} + \mu_{1,3}^{(2)} & \mu_{1,4} & \mu_{1,5} \\ \mu_{0,2} & \mu_{1,2} & -\sum \mu_i & \mu_{2,3} & 0 & 0 \\ \mu_{0,3} & \mu_{1,3}^{(1)} + \mu_{1,3}^{(2)} & \mu_{2,3} & -\sum_i \mu_i & \mu_{3,4} & \mu_{3,5} \\ 0 & \mu_{1,4} & 0 & \mu_{3,4} & -\sum \mu_i & \mu_{4,5} \\ 0 & \mu_{1,5} & 0 & \mu_{3,5} & \mu_{4,5} & -\sum \mu_i \end{bmatrix},$$

whereas the R-Matrix is

$$\begin{bmatrix} -\sum \mu_i & \mu_{0,1} & \mu_{0,2} & \mu_{0,3} & 0 & 0 \\ \mu_{0,1} & -\sum \mu_i & \mu_{1,2} & \{\mu_{1,3}^{(1)}, \mu_{1,3}^{(2)}\} & \mu_{1,4} & \mu_{1,5} \\ \mu_{0,2} & \mu_{1,2} & -\sum \mu_i & \mu_{2,3} & 0 & 0 \\ \mu_{0,3} & \{\mu_{1,3}^{(1)}, \mu_{1,3}^{(2)}\} & \mu_{2,3} & -\sum_i \mu_i & \mu_{3,4} & \mu_{3,5} \\ 0 & \mu_{1,4} & 0 & \mu_{3,4} & -\sum \mu_i & \mu_{4,5} \\ 0 & \mu_{1,5} & 0 & \mu_{3,5} & \mu_{4,5} & -\sum \mu_i \end{bmatrix},$$

Note the distinction between the two. In row 1 column 4, the response matrix has the sum of the μ's whereas the R-Matrix has in entry 1,4 two values of μ. We can see that in the R-MultiGraph we get two edges from vertices 1 and 3, so the R-Matrix and Λ will only differ by this entry.

Figure 3.4 will often be used in creating n-to-1 graphs. Later we will refer to this type of connection of multiple stars as a *switch*. Note that when you see an R-MultiGraph like Figure 3.4 it comes from that resistor network graph and is simply a quadrilateral.

4. Gaining Intuition about R-MultiGraphs

The Triangle-in-Triangle graph has been thoroughly scrutinized by prior REU students [3]. They have determined that most of the time it is a 2-to-1 graph with one exception that corresponds to a root of

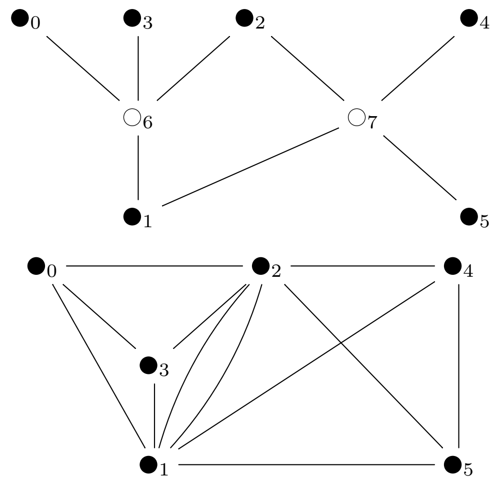


FIGURE 5. Connection of Multiple 4-Stars by Diagonal Side

multiplicity 2 in a quadratic equation. We will use this graph to understand the driving forces behind its 2-to-1 behavior and in turn use our intuition to create n-to-1 graphs.

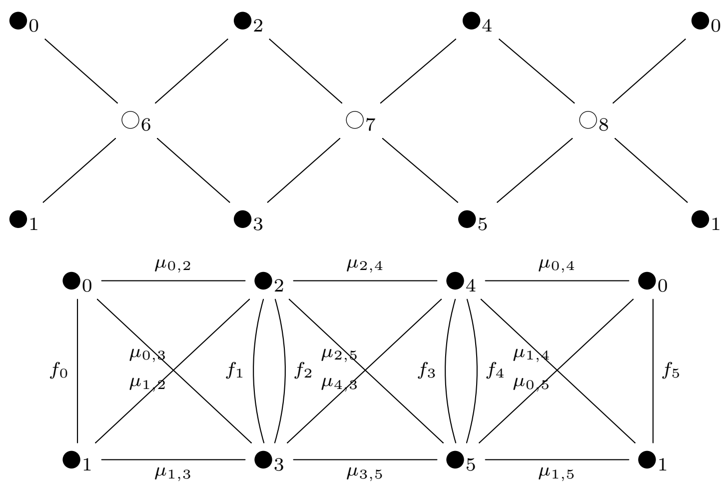


FIGURE 6. Triangle within Triangle Resistor Network Graph and R-MultiGraph

In order to demonstrate Figure 4's 2-to-1-ness, let's assume that we know what the R-Graph looks like, but don't know the values in Λ. We will force Λ to be certain values such that the graph is 2-to-1. However, we do know the edges that are equivalent in Λ and R-Matrix. Next, if we can find that two distinct R-Matrices exist such that quadrilateral rule holds and the Λ relationships hold then we calculate the conductivities by equation 2. With two distinct R-Matrices, some f_j's will vary and hence equation 2 will produce two unique conductivities. Thus for a given response matrix there are two different conductivities that produce the same Λ.

In order to use equation 2 to recover the conductivies, quadrilateral rule must be preserved. Because of this, there are some relationships that must be satisfied between the entries in the response matrix. Note that in Figure 4 the f_i are values in the R-Matrix that correspond to multiple edges. They are the entries that differ from the response matrix and must be found using the quadrilateral rule and other response matrix conditions.

To start assume we know that $f_0 = x$ and write down the quadrilateral and response matrix conditions.

Equation	Sign of the Derivative
$f_0(x) = x$	$+$
$f_1(x) = \frac{\lambda_{0,2}\lambda_{1,3}}{f_0}$	$-$
$f_2(x) = \lambda_{2,3} - f_1$	$+$
$f_3(x) = \frac{\lambda_{2,4}\lambda_{3,5}}{f_2}$	$-$
$f_4(x) = \lambda_{4,5} - f_3$	$+$
$f_5(x) = \frac{\lambda_{0,4}\lambda_{1,5}}{f_4}$	$-$
$\Sigma(x) = f_5 + x = \lambda_{0,1}$	

We must show that there exists two distinct values of x that satisfy these equations. **Moreover because the R-Matrix must have all positive entries, every f_j must be positive for those two values.** We can do this by finding appropriate λ's that ensure the f_j's are positive (e.g. $\lambda_{2,3} > f_1(x)$ for the two roots). The question is *How do we choose appropriate λ's to guarantee that this graph is 2-to-1?*

The system of equations produces specific characteristics:

(1) The f_j's are a continued fractions. ($f_5(x)$ is equal to a linear term over linear term.)
(2) f_5 has a singularity at a point y_0 if and only if $f_4(y_0) = 0$ if and only if $f_3(y_0) = \lambda_{4,5}$.
(3) The sign of the derivative for all f_j is constant and alternates.
(4) Σ can have at most 2 roots.

4.1. **Singularities and Σ.** Let's denote the singularity in f_5 as y_0. Due to the singularity, Σ blows up at y_0. Because the sign of f_5's derivative is negative, then for an interval around y_0 the derivative of Σ is also negative. Moreover, f_5 is a linear term over linear term; thus as x goes to $\pm\infty$, Σ goes to $\pm\infty$ respectively. Therefore, the behavior of Σ can be seen in Figure 7.

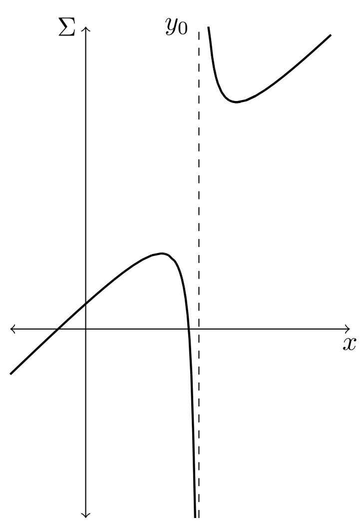


FIGURE 7. Behavior of Σ

If we set Σ to any positive value, we can see it would cross the graph either 0, 1, or 2 times at positive values for x, but some $f_j < 0$. Denote the two roots as x_1 and x_2 where $x_1 < x_2$. Although x_1 and x_2

are positive, we have not shown that the other f_j's are positive. In particular, examine f_4 positivity. The singularity of f_5 is constructed at y_0 if and only if $\lambda_{4,5} = f_3(y_0)$. Replacing this, $f_4(x) = f_3(y_0) - f_3(x)$. Designate $z < y_0$, $f_4(z) = f_3(y_0) - f_3(z)$. The sign of $f_3(x)$ derivative is negative, so $f_3(y_0) - f_3(z) < 0$, which makes $f_4 < 0$ if $z < y_0$. Hence if the roots x_1 and x_2, lie on the left side of the singularity point, f_4 will always be negative and thus not a valid entry in R-Matrix. This forces our x_1 and x_2 to lie on the right side of the singularity to even hope for all the f_j's to be positive Let $z > y_0$ then $f_4 = f_3(y_0) - f_3(z)$ and because $f_3(x)$ is a decreasing function, $f_4 > 0$. *The singularity can be thought of as defining a half-plane, where on one side an f_j will be negative and on the other side positive.* We will designate the side of the singularity on which the value is positive by an arrow (Figure 8). Thus, the roots of a 2-to-1 will lie to the right of a singularity point.

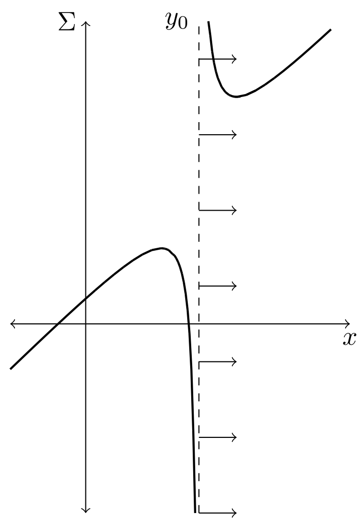


FIGURE 8. Location of Roots Based Only On f_4's Positivity

4.2. **Choosing Appropriate λ's.** There are four steps to producing two distinct R-Matrices:

1. Pick a positive value, y_0, to be the singularity of f_5 and hence Σ
2. Parameterize the λ's to uphold the quadrilateral rule
3. Ensure that all f_j's are positive. In particular, f_2
4. Choose $\lambda_{4,5}$ to create the singularity at y_0 for f_5

In [3], the singularity value, y_0, is always positive. This was based on a determinant criterion which stated that there existed a 2-connection. In fact, we can choose y_0 with the condition that it is positive.

Recall, the parameterization of Λ via the quadrilateral rule. Start by working the way down the f_j. The edge with the value f_1 is in the quadrilateral formed by the vertices (0,1,2,3). Since no values of the quadrilateral are yet determined, we can chose $\lambda_{0,2}$ and $\lambda_{1,3}$ to be any positive values. When we do this, the equation for f_1 forces the quadrilateral rule to hold for $\lambda_{0,2}\lambda_{1,3} = f_0 f_1$. However, there is another quadrilateral rule than needs to be satisfied, but we have not chosen those values so pick a value for $\lambda_{0,3}$ and make it so that $\lambda_{1,2} = \frac{\lambda_{0,2}\lambda_{1,3}}{\lambda_{0,3}}$. Thus the following is true

$$f_0 f_1 = \lambda_{0,2}\lambda_{1,3} = \lambda_{0,3}\lambda_{1,2}$$

and hence all quadrilateral rules are satisfied. This scheme will work for all edges that are perpendicular to each other. For instance, we can do the same parameterization for the quadrilateral whose vertices are (2,3,4,5).

This scheme guarantees that the quadrilateral rule holds. However, we must still show that all the f_j's are positive. We can see that if f_0 is positive then f_1 is positive. Similarly, f_3 will be positive if f_2 is positive and f_5 and Σ will be positive if f_4 is positive. Because we know that the roots x_1 and x_2 are positive then $f_0 > 0$. We only need to ensure that $f_2, f_4 > 0$. First focus on f_2 which will be positive if $\lambda_{2,3}$ is sufficiently large. *But, how large?* To determine the magnitude of $\lambda_{2,3}$, we note that f_2 will be at its smallest when x is at its smallest. The smallest value for x is y_0 for we have already determined that $f_4 > 0$ only if $x > y_0$. Therefore, we set $\lambda_{2,3} = \frac{\lambda_{0,2}\lambda_{1,3}}{y_0} + C_{2,3}$ where $C_{2,3}$ is any value greater than zero. This ensures that f_2 will be positive for any value to the right of y_0.

In order to ensure that f_4 is positive, we will utilize the information we found in Singularities and Σ Section. f_4 will only be positive to the right of y_0. Thus, all we have to do is set the value of the singularity to occur at y_0. We accomplish this by letting $\lambda_{4,5} = f_3(y_0)$. By doing this, $f_4(y_0) = 0$ and hence f_5 and Σ have a singularity at y_0.

Now we have guaranteed that all the f_j's and λ's are positive in the sector to the right of the singularity y_0. Thus because the derivative sign is constant, there exist an $\lambda_{0,1}$ such that there are two distinct values of x and all intermediate f_j's are positive at those two roots.

4.3. **Plugging in Numbers.** Let's attempt to create a 2-to-1 graph by choosing appropriate λ's.

Step 1: Pick a positive value to be the singularity of f_5. Let $y_0 = 1$.

Step 2: Choose values of the λ's in the quadrilateral to uphold the quadrilateral rule. For instance, set $\lambda_{0,2}, \lambda_{1,3}, \lambda_{0,3} = 1$ and hence $\lambda_{1,2} = 1$ by the quadrilateral rule. Now do the same for all other quadrilaterals (i.e. $\lambda_{2,4}, \lambda_{3,5}, \lambda_{2,5} = 1$ and by the rule $\lambda_{3,4} = 1$). When we do this, we get a new set of equations for our f_j's:

Equation	Sign of the Derivative
$f_0(x) = x$	$+$
$f_1(x) = \frac{1}{f_0}$	$-$
$f_2(x) = \lambda_{2,3} - f_1$	$+$
$f_3(x) = \frac{1}{f_2}$	$-$
$f_4(x) = \lambda_{4,5} - f_3$	$+$
$f_5(x) = \frac{1}{f_4}$	$-$
$\Sigma(x) = f_5 + x = \lambda_{0,1}$	

Step 3 Ensure that all the f_j's are positive in some interval. First, let's consider f_2. By the scheme established above, set $\lambda_{2,3} = f_1(y_0) + C_{2,3}$. Since $f_1(1) = 1$, $\lambda_{2,3} = 1 + C_{2,3}$ where $C_{2,3}$ can be any positive value. For simplicity sake, set $C_{2,3} = 1$ so that $\lambda_{2,3} = 2$. Now, we have ensured that all the f_j's up until f_4 will be positive.

Step 4 Set the singularity to be the value y_0. We see that Σ has a singularity at 1 if and only if f_5 has a singularity at 1 if and only if f_4 is zero at y_0 if and only if $\lambda_{4,5} = f_3(y_0)$. Thus, we set $\lambda_{4,5} = f_3(1)$. If we write out the string of continued fractions, we get:

$$\begin{aligned}
f_0(x) &= x \\
f_1(x) &= \frac{1}{x} \\
f_2(x) &= 2 - \frac{1}{x} = \frac{2x-1}{x} \\
f_3(x) &= \frac{x}{2x-1} \\
f_4(x) &= \frac{x-1}{2x-1} \\
f_5(x) &= \frac{2x-1}{x-1} \\
\Sigma(x) &= x + \frac{2x-1}{x-1}
\end{aligned}$$

Plugging in 1 into f_3, $\lambda_{4,5} = 1$. By doing this, we have forced the singularity of f_5 to occur at 1.

Step 5 For fun, let 3 be a root of Σ. In order to do this, plug 3 into Σ and calculate the value of $\lambda_{0,1}$. With $\lambda_{4,5} = 1$, $f_4 = \frac{x-1}{2x-1}$ and $f_5 = \frac{2x-1}{x-1}$. Thus, $f_5(3) = \frac{5}{2}$ and $\Sigma(3) = \frac{11}{2} = \lambda_{0,1}$. Now, we have all the λ's. In summary,

$$\Lambda = \begin{bmatrix}
\frac{-19}{2} & \frac{11}{2} & 1 & 1 & 1 & 1 \\
\frac{11}{2} & \frac{-19}{2} & 1 & 1 & 1 & 1 \\
1 & 1 & -6 & 2 & 1 & 1 \\
1 & 1 & 2 & -6 & 1 & 1 \\
1 & 1 & 1 & 1 & -5 & 1 \\
1 & 1 & 1 & 1 & 1 & -5
\end{bmatrix}$$

Let's now solve Σ in terms of x, knowing that one of the roots of the quadratic is 3:

$$\begin{aligned}
\Sigma = x + \frac{2x-1}{x-1} &= \frac{11}{2} \\
x^2 - \frac{9}{2}x + \frac{9}{2} &= 0
\end{aligned}$$

Using the quadratic formula, $x = 3$ and $x = \frac{3}{2}$. Leaving this to the reader, you can go back and verify that all the f_j's are positive at these values (Remember we have guaranteed that the f_js are all positive and uphold the quadrilateral rule for any $x > 1$). With the same Λ, we get two different R-Matrices.

For $x = \frac{3}{2}$,

$$R-\text{Matrix} = \begin{bmatrix}
\frac{-19}{2} & \{\frac{3}{2}, 4\} & 1 & 1 & 1 & 1 \\
\{\frac{3}{2}, 4\} & \frac{-19}{2} & 1 & 1 & 1 & 1 \\
1 & 1 & -6 & \{\frac{2}{3}, \frac{4}{3}\} & 1 & 1 \\
1 & 1 & \{\frac{2}{3}, \frac{4}{3}\} & -6 & 1 & 1 \\
1 & 1 & 1 & 1 & -5 & \{\frac{3}{4}, \frac{1}{4}\} \\
1 & 1 & 1 & 1 & \{\frac{3}{4}, \frac{1}{4}\} & -5
\end{bmatrix}$$

For $x = 3$,

$$R-\text{Matrix} = \begin{bmatrix}
\frac{-19}{2} & \{3, \frac{5}{2}\} & 1 & 1 & 1 & 1 \\
\{3, \frac{5}{2}\} & \frac{-19}{2} & 1 & 1 & 1 & 1 \\
1 & 1 & -6 & \{\frac{1}{3}, \frac{5}{3}\} & 1 & 1 \\
1 & 1 & \{\frac{1}{3}, \frac{5}{3}\} & -6 & 1 & 1 \\
1 & 1 & 1 & 1 & -5 & \{\frac{3}{5}, \frac{2}{5}\} \\
1 & 1 & 1 & 1 & \{\frac{3}{5}, \frac{2}{5}\} & -5
\end{bmatrix}$$

Left to the reader, you can verify that all the quadrilateral rules hold. Now, we have produced two distinct R-Matrices deriving from the same Λ and hence will two different sets of conductivities.

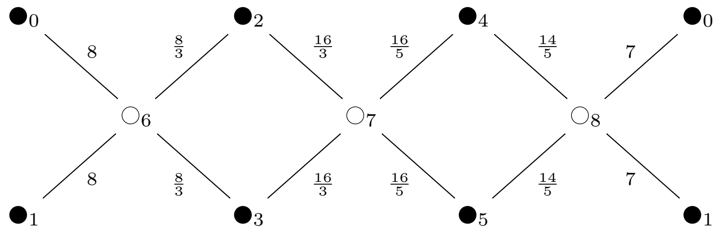


FIGURE 9. Conductivities corresponding to $x = 3$

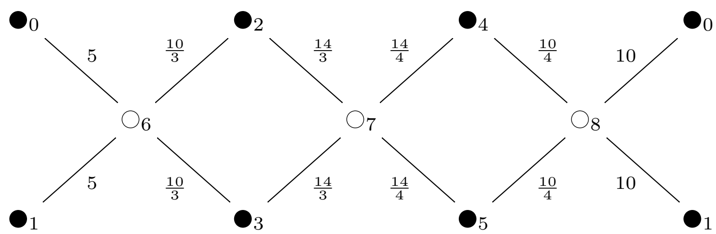


FIGURE 10. Conductivities corresponding to $x = \frac{3}{2}$

In summary, we have seen that

(1) Because of the quadrilateral rule, the f_j's have constant sign derivatives
(2) There is a multitude of different choices for the λ's as long as we preserve quadrilateral rule and response conditions.
(3) The singularities in Σ act like half-planes: on one side one of the f_j's is positive and on the other it is negative.
(4) We can choose the value of the singularity, y_0, as long as it is positive

Goal: Knowing the above, can we construct backwards a R-MultiGraph and Λ so that it produces n distinct conductivities?

5. The Process

When using the quadrilateral rule and response conditions, there are three different forms the f_j's can take. Assume f_{j-1} is on the left most side of the quadrilateral. If f_j is adjacent or along the diagonal edge with all the other edges known, then $f_j = Cf_{j-1}$ where C is a constant formed by the other edges in the quadrilateral. If f_j is a parallel edge, then $f_j = \frac{C}{f_{j-1}}$. The third type occurs when multiple edges are formed in R-MultiGraph and thus $f_j = C - f_{j-1}$. In summary, the three different ways for f_j are

(1) $f_j(x) = Cf_{j-1}(x)$
(2) $f_j(x) = \frac{C}{f_{j-1}(x)}$
(3) $f_j(x) = C - f_{j-1}(x)$

Because of the computation from multiple quadrilaterals strung together, the $f_j(x)$ will alternate between (1) or (2) then (3) back to (1) or (2) and always terminating at a (1) or (2). Assume that $f_0 = x$ then at each stage $f_j(x)$ will be a linear term over linear term.

Proposition 5.1. *Each $f_j(x)$ has a constant sign derivative*

Proof. Because the $f_j(x)$ are linear term over linear term,

$$f_j(x) = \frac{ax + b}{cx + d},$$

where $a, b, c, d \in \mathbb{R}$. Then

$$f_j'(x) = \frac{ad - bc}{(cx + d)^2},$$

and hence since $(cx + d)^2 \geq 0$, the sign of $f_j'(x)$ is determined by the constants a, d, b, c, which don't depend on x. Thus, $f_j'(x)$ is constant sign. □

5.1. **Constructing the Graph of $\Sigma(x)$.** The central idea behind constructing n-to-1 graphs is to generate "arms" that correspond to n edges between the same pair of vertices. For instance, the 2-to-1 case had a pair of vertices $\{0, 1\}$ that were repeated twice on the R-MultiGraph. For a 3-to-1 graph and keeping notation, the pair $\{0, 1\}$ would be repeated 3 times. If starting with an edge between $\{0, 1\}$, then there would be two different computation paths that end with an edge between the vertices $\{0, 1\}$ to generate the 3-to-1 graph. For an n-to-1 graph, there would be $n-1$ different computation paths that end with an edge between vertices $\{0, 1\}$. Let $g_j(x)$ indicate the last linear term over linear term in the computational path which ends at $\{0, 1\}$. Moreover, order the $g_j(x)$ such that $g_0(x)$ has the smallest singularity (y_0) in $\Sigma(x)$ and $g_{n-1}(x)$ has the largest singularity (y_{n-2}) in Σ Hence,

$$\Sigma(x) = x + \underbrace{g_0(x) + g_1(x) + g_2(x) + \cdots + g_{n-2}(x)}_{n-1} \tag{3}$$

where $g_j(x)$ are the continued fraction that end with the edge between vertices $\{0, 1\}$ (e.g. In the 2-to-1, $g_0(x) = f_5(x)$ and all the others would be not exist). All prior linear term over linear term function are designated by $f_j(x)$

Because every $g_j(x)$ in Σ is a linear term over linear term results in equation 3 having $n - 1$ singularities for an n-to-1 graph. When equation 3 is set equal to a constant value, Σ has at most n *roots.* This result follows by clearing the denominator of $\Sigma(x)$ which generates a monic polynomial of degree n.

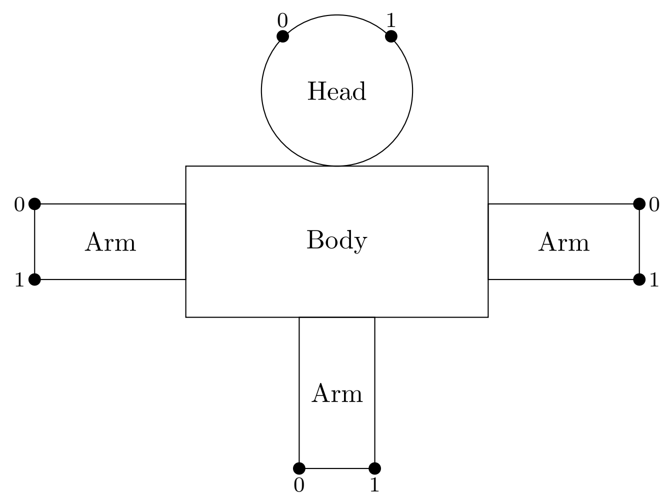


FIGURE 11. Diagram of Constructing 3-to-1 Graph

Definition 5.1 (Sector)**.** *A sector is a section of the graph Σ that lies between 2 singularities with no other singularities in between or to the left or right of all singularities. In essence, sections of the graph of Σ where it is a continuous function.*

In equation 3, there are $n - 1$ $g_j(x)$ equations. Because each $g_j(x)$ is a linear term over linear term, there are $n - 1$ singularities (different), resulting in n sectors of the graph of $\Sigma(x)$.

Lemma 5.1. *All singularities in the $f_k(x)$ and $g_l(x)$ are created from another $f_j(x)$'s root where $(j < k)$ and $f_j(x)$ is a term in the computational path of $f_k(x)$ or $g_l(x)$.*

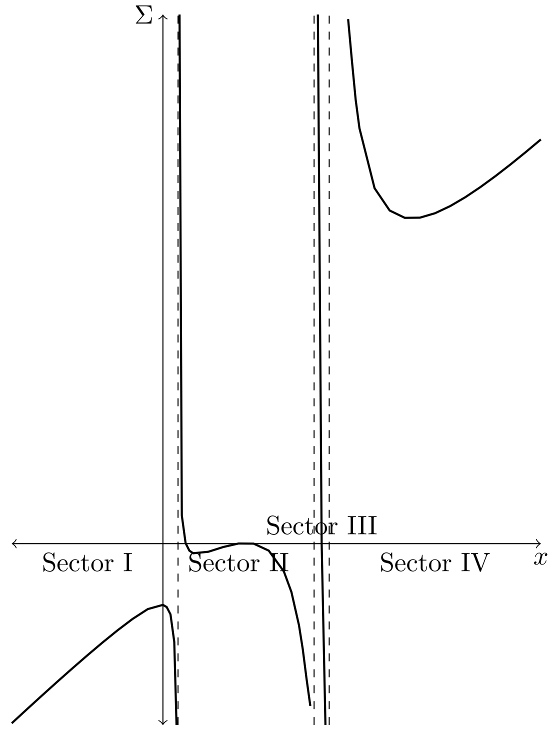


FIGURE 12. Example of Sectors

Proof. Let $f_j(x)$ denote a prior term used in the creation of $f_k(x)$. If $f_k(x)$ has a singularity, then either $f_k(x)$ looks like $\frac{C}{f_j}$ where C is a constant or there exists an $f_{j+1}(x)$ that looks like the above with $j+1<k$ and $f_{j+1}(x)$ is in $f_k(x)$'s computational path. The singularity of $f_k(x)$ is created when $f_j(x)=0$. Hence, the singularity of $f_k(x)$ is generated from the root of a prior term in its computational path. □

Lemma 5.2 (Key)**.** *Every singularity for $f_k(x)$ and $g_l(x)$ defines a half space where on one side a $f_j(x)$ ($j<k$) is negative and on the other side positive.*

Proof. From Lemma 5.1, every singularity comes from the root of $f_j(x)$, which is in the form $C - f_{j-1}(x)$. The quadrilateral rule dictates that $f_j(x)'s$ alternate between $Cf_{j-1}(x)$ or $\frac{C}{f_{j-1}(x)}$ and $C - f_{j-1}(x)$. Hence,

$$f_{j-1}(x) = C_{j-1}f_{j-2}(x) \quad \text{or} \quad \frac{C_{j-1}}{f_{j-2}(x)}$$
$$f_j(x) = C_j - f_{j-1}(x)$$

If the singularity comes from the root of $f_j(x)$ then $C_j = f_{j-1}(y)$ where y is the singularity. Replacing this yields,

$$f_j(x) = f_{j-1}(y) - f_{j-1}(x).$$

Since $f_{j-1}(x)$ has constant sign derivative,

$$f_j(x) > 0 \begin{cases} \text{if } f_{j-1}(x) \text{ is a decreasing function and } x > y \\ \text{if } f_{j-1}(x) \text{ is an increasing function and } x < y \end{cases}$$

Hence, $f_j(x)$ will only be greater than 0 for either $x>y$ or $x<y$, which will define the half-space. □

The singularities determine half spaces where on one side the $f_j(x)>0$ and on the other $f_j(x)<0$. Pointing *arrows* along the singularity y indicate the half plane in which $f_j(x)>0$.

Lemma 5.3. *Positivity of all $f_j(x)$'s occurs in at most one sector of the graph of Σ.*

Proof. Applying Lemma 5.2 to one $g_k(x)$, we know that at least one $f_j(x) > 0$ for either $x < y_k$ or $x > y_k$ where y_k is the singularity of $g_k(x)$. Repeat the process for another k. When Lemma 5.2 is applied to all k, we get a series of inequalities that if non-empty define a region in which some of the $f_j(x)$ are positive. Since the inequalities depend on the singularities of $g_k(x)$, the region of positivity for the $f_j(x)$'s will occur within a sector of the graph. If the inequalities formed an empty interval, then at least one $f_j(x)$ will be negative for any root of Σ. □

This is not to say that by ensuring all the roots of Σ lie within one sector guarantees n-to-1, but it is necessary condition.

If we want a graph of conductivities to be n-to-1 with n different computational paths to $\{0, 1\}$, then all n roots must lie within one sector of the graph of Σ. If a root lies outside of a feasible sector, it violates the positivity of at least one $f_j(x)$ even if the root is positive.

Proposition 5.2. *Positivity of all $f_j(x)$'s occurs only in sectors that define intervals with at least one strictly positive x value.*

Proof. Assume we have a strictly negative sector and all $f_j(x)$ are positive. For all x in that sector, $x < 0$. However, $f_0(x) = x$, which can not be greater than 0 for any x in this sector. Thus, we have reached a contradiction. □

This statement leads to the idea that there must be at least one singularity of Σ that defines the boundary of the feasible sector, which is greater than 0. Moreover, the roots of Σ must be positive. For the rest of this paper, we will chose the singularities of Σ to all be greater than 0 to ensure that the real roots of Σ that lie in the feasible sector will always be positive.

As noted before, $\Sigma = x + g_0(x) + g_1(x) + \cdots + g_{n-2}(x)$ has exactly n roots. Hence when $\Sigma = C$, where C is a constant, then the horizontal line at C can cross at most n times of the graph of Σ. Combined with the dominating sign of the derivative (which are constant) near the singularities of each $g_k(x)$, limits the construction of the graphs of Σ.

5.2. **Drawing the R-MultiGraph.** If we know the graph of Σ, we can draw a R-MultiGraph that has the features of the Σ graph around the singularity points of each g_k. The features we will utilize are the sign of the derivative and the *arrow* direction or the values of x for which the $g_g(x) > 0$. Note that there are only four possibilities for the direction of the arrow and the sign of the derivative. The sign of the derivative, ($+$ for positive and $-$ for negative) is located at the top next to the singularity point. Figure 5.2 shows the different possibilities.

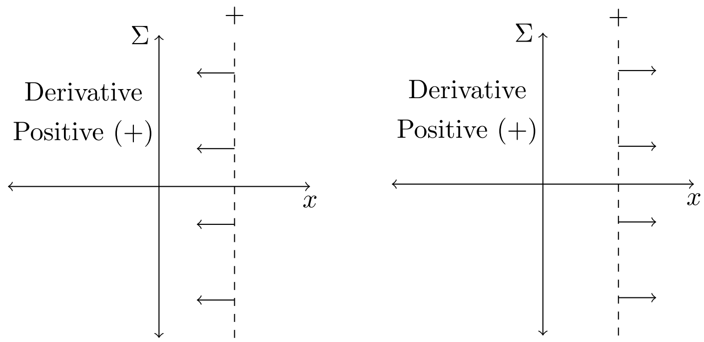
To construct R-MultiGraphs that generate graphs of Σ that look like Figure 5.2, we discovered some R-MultiGraphs that follow the Σ conventions.

Definition 5.2 (Arm)**.** *An arm is a R-MultiGraph constructed from connecting two 4-stars that contain the vertices of $\{0, 1\}$ in the right most quadrilateral. There are 4 distinct types corresponding to the 4 different possibilities in Figure 5.2. Each of these arms produces the correct derivative sign and half-plane defining features. The arms will be the building blocks for constructing the R-MultiGraphs. For an n-to-1 graph, there are $n - 1$ arms. The last occurance of vertices $\{0, 1\}$ is in the R-MultiGraph is designated as the head.*

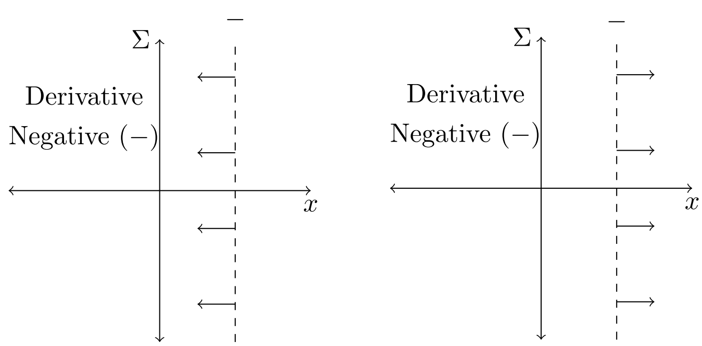
We indicate the different arms by Type (I or II) and derivative sign ($+$ or $-$) of g_k. In addition, it is necessary to understand the feature that connects all the arms together. We designate the signs of the derivatives of each edge in the R-MultiGraph with a $(+)$ for positive and a $(-)$ for negative. The most significant derivative sign is the left most, which will be utilized later. Also note the location of 0 and 1. The edge between the vertices $\{0, 1\}$ is the g_k and hence the sign of its derivative is a component of the Σ function.

In Figure 14, the left most derivative sign of the 4 types alternates between positive and negative. Denote the quadrilateral left most edge as f_i. We need a way to string these arms together such that the sign of f_i is correct. We can then hook up the arms to this graph so that they produce the correct derivative sign for f_1^k. In [1], the tool used to generate the correct derivative sign for f_i was called a *inversion*, but we refer to it as a *switch*.

Definition 5.3 (Switch)**.** *A switch is a R-MultiGraph that enters with a positive (negative) derivative and leaves with a negative (positive) derivative. Thus, switching the sign of the derivative.*



(a) Derivative Positive



(b) Derivative Negative

FIGURE 13. The Four Possible Half Planes and Derivative Combinations for a Singularity in the graph of Σ

Figure 15 is a quadrilateral in disguise. The sign of the derivatives is the result of the quadrilateral rule applied to non-parallel edges to the entry edge. In Figure 3 assume that $\mu_{0,2}, \mu_{2,3}$ are known and $\mu_{0,1} = f_{0,1}(x)$ with a positive derivative, then

$$f_{1,3}(x) = \mu_{1,3} = \frac{\mu_{2,3} f_0(x)}{\mu_{0,2}}.$$

Because $\mu_{2,3}, \mu_{0,2}$ are positive constants, the derivative of $f_{1,3}(x)$'s derivative is the same as $f_{0,1}(x)$. Applying the quadrilateral rule like this to the diagonal edges produces the same positivity in derivatives. If we have multiple stars connected together, we will always go through two vertices that have a multiple edge. To proceed through the multiple edge as shown in Figure 4

Note that we can match up the f_i's in Figure 14 with the f_i in Figure 15 and chose the appropriate entering sign of the derivative.

Let's assume the entry edge of the *switch* is always positive. To produce f_i's that have positive derivative, we can string together switches. Now if we enter the first switch with a positive derivative, we will exit with a negative derivative, which starts the next switch with a negative derivative so it ends with a positive derivative. Thus, we can attach the f_i of Type I (+) and Type II (+) at the end of the second switch. If we continue this, we conclude that

(1) Every odd number of switches, exits with a negative derivative
(2) Every even number of switches, exits with a positive derivative

A switch always has two exit sides. When strung together, only one of the exit edges is used and the other exit edge can be utilized to attach various arm types so that the derivative of $\{0, 1\}$ edge is whatsoever desired. We call multiple switches strung together a *Switch Yard*. The switch yard forms the body of the

R-MultiGraph. The number of switches necessary depends entirely on the types of arms needed to produce the Σ graph.

Figure 11 is almost complete with the exception of the *head*.

Definition 5.4 (Head)**.** *A single quadrilateral graph that contains an edge between the vertices* $\{0, 1\}$ *and connects directly to the entering edge of the switch.*

The tools (a head, a switch, and the arms) are the three gadgets that construct n-to-1 graphs. By using these tools, we guarantee certain conditions that are necessary for the existance of n-to-1 graphs. However, we haven't concluded that there are appropriate λ choices that ensure n-to-1 behavior.

5.3. **Process for Choosing the λ's to Create n-to-1.**

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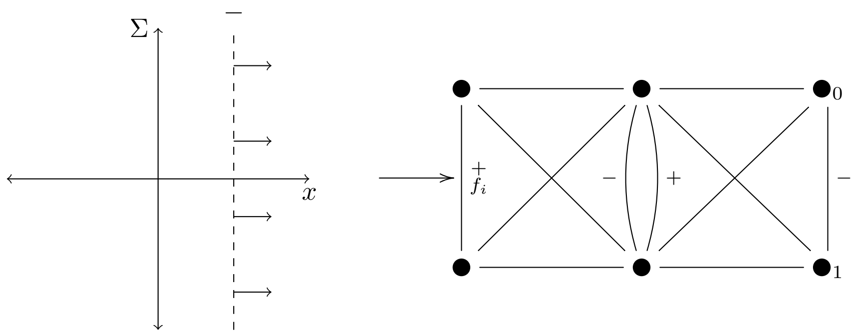
[3] Jennifer French and Shen Pan, 2^n *to 1 Graphs*, University of Washington Math REU (2004).

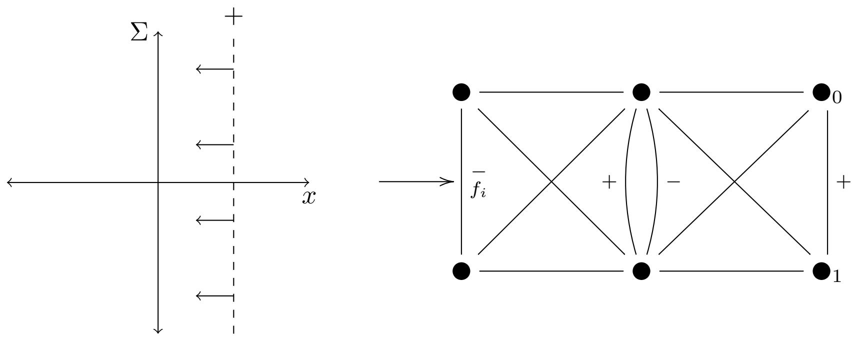
[4] Jeffrey Russell, *Star and K Solve the Inverse Problem*, University of Washington Math REU (2003).

[5] Edward Curtis and James Morrow, *Inverse Problems for Electrical Networks*, World Scientific, Singapore, 2000.

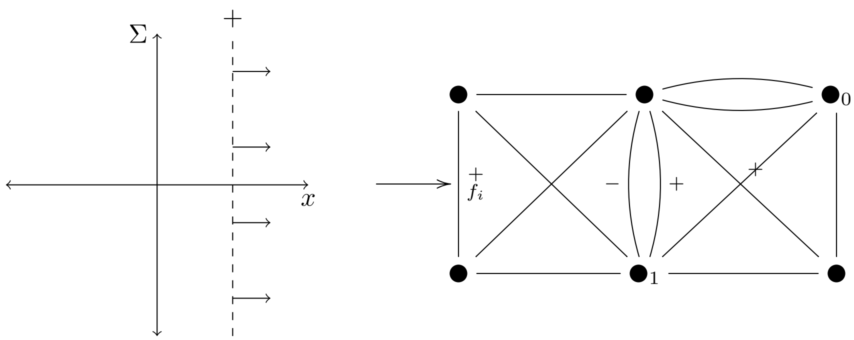
Figure 14. Arm Types



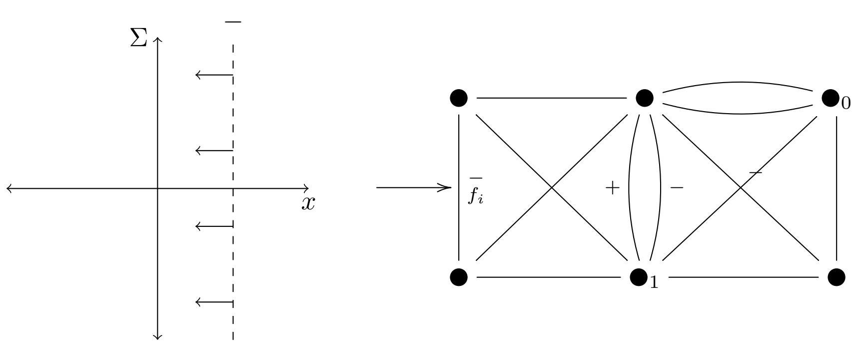
(a) Type I and −



(b) Type I and +



(c) Type II and +



(d) Type II and −

FIGURE 15. Drawing of a Switch

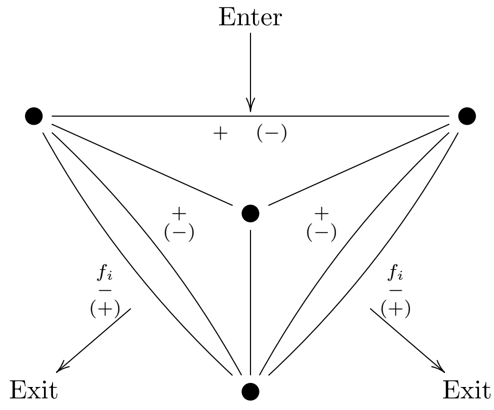


FIGURE 16. An Example of a Switch Yard

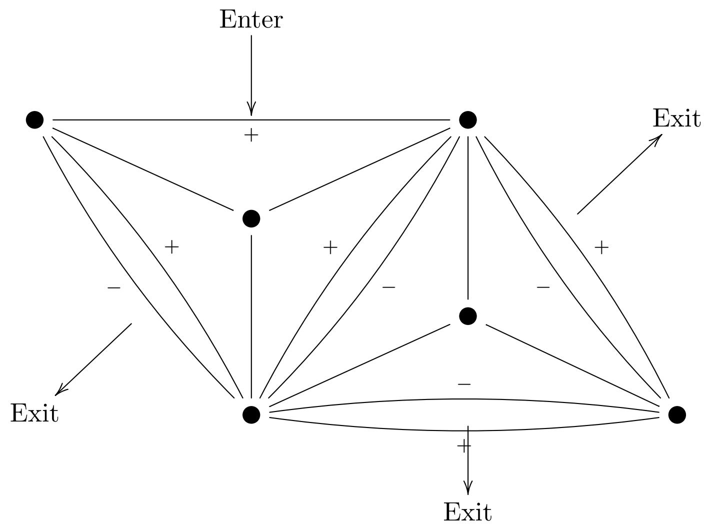
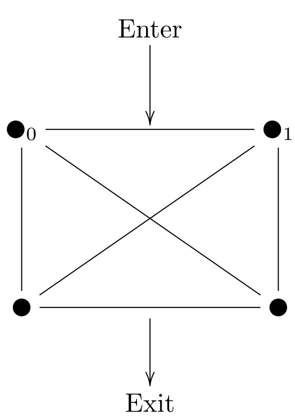
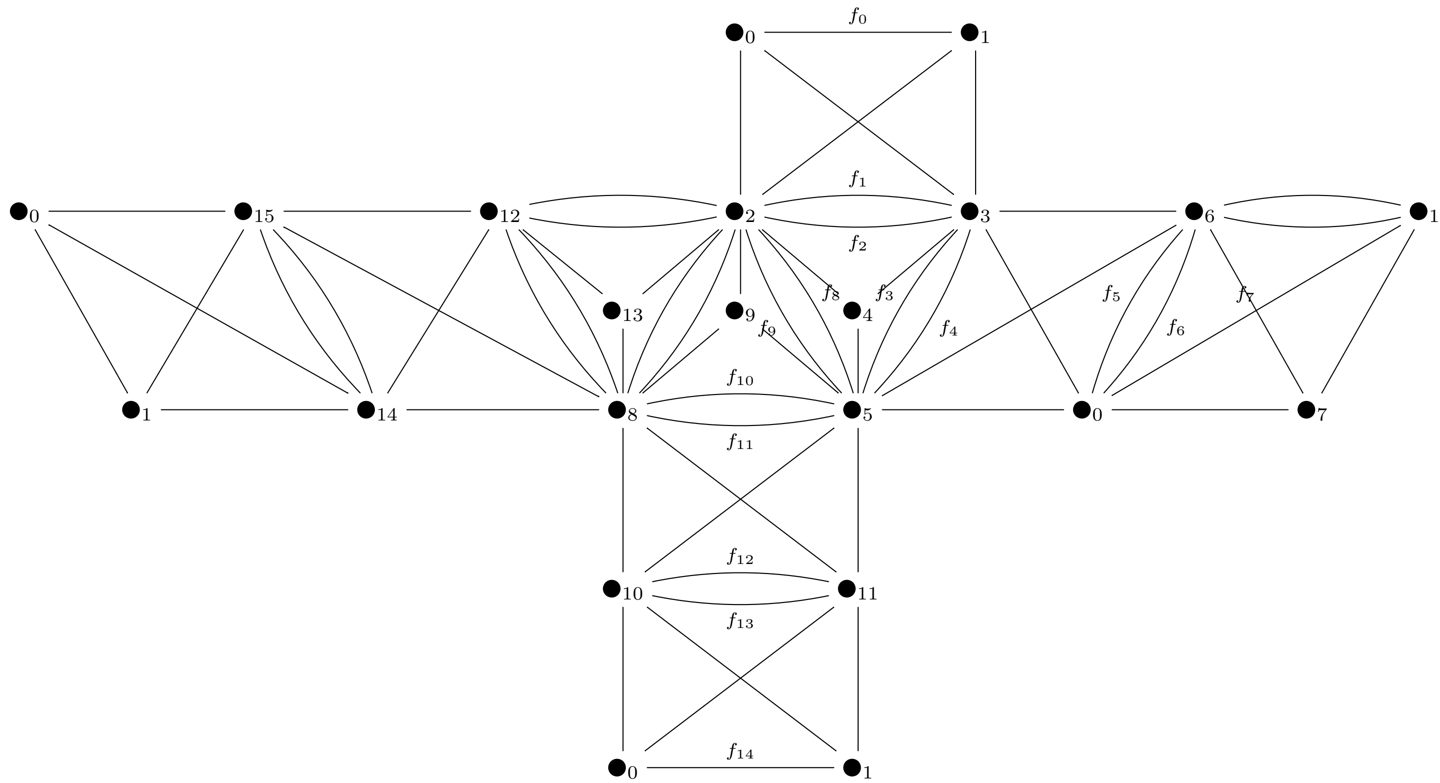


FIGURE 17. Head





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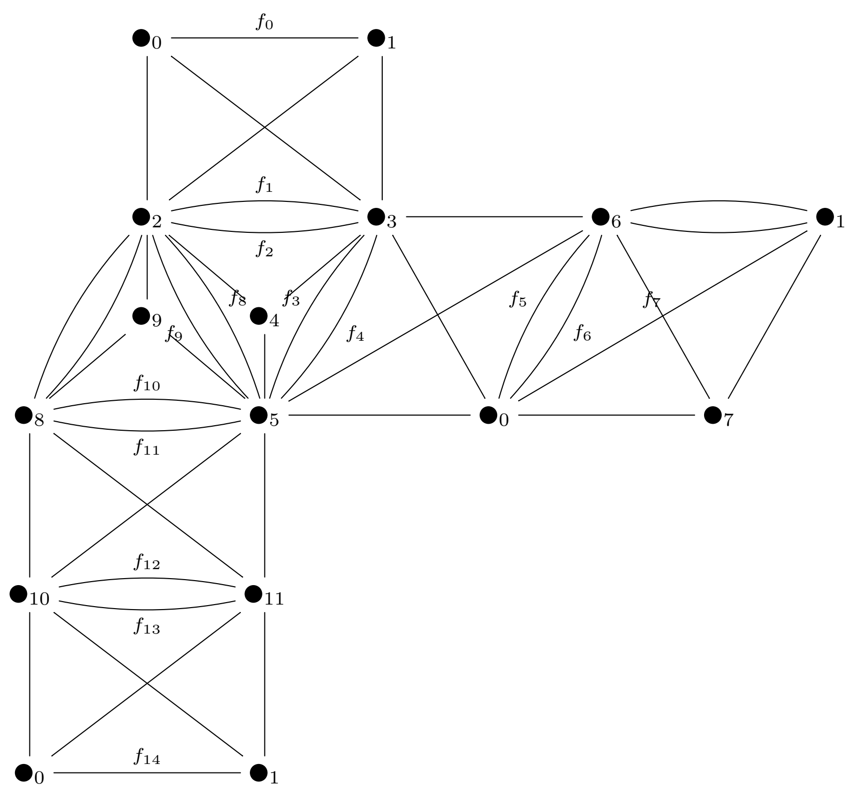

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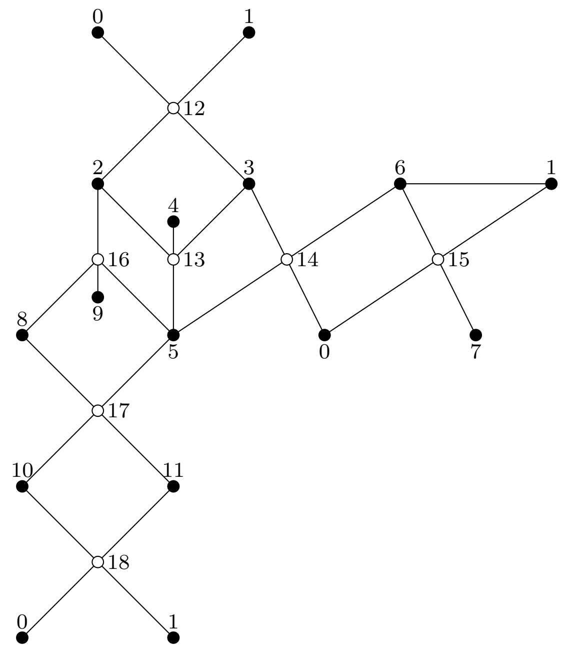
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