Annular Networks: Elimination of Type 1 Geodesics

David Jekel

July 15, 2013

1 Introduction

Definitions are explained in "Annular Networks: Preliminaries," but some of the most nonstandard definitions are repeated here.

In this paper, I consider annular networks without simply connected lenses or lenses involving type 1 geodesics. I prove that geodesic elimination can be used to remove all type 1 geodesics from the network. The problem of recoverability is thus reduced to the case of radial networks (networks with only type 2 geodesics).

The proof has three parts. First, we show how to construct an empty boundary triangle at which one of the geodesics is type 1. Second, we show how to recover the conductivity on the spike or boundary-to-boundary edge corresponding to the triangle. Finally, we argue by induction that all type 1 geodesics can be removed from the graph.

2 Constructing an Empty Boundary Triangle

Definition 2.1. Suppose g is a type 1 geodesic in M, which does not form a one-pole lens. Then g divides the annulus into two components. Let $S(g)$ be the component which does not include the hole. Let $\widehat{g}$ denote the arc of the outer circle which lies along $S(g)$. A geodesic h is said to lie inside g if $h \neq g$ and $S(h) \subset S(g)$. These terms are not defined for any other geodesics.

Definition 2.2. $g : h_1, h_2$ means that the geodesic g intersects geodesic h_1 and immediately afterward intersects the geodesic h_2. In other words, "h_1, h_2" appears in the crossing sequence of g. If no orientation is specified on g, $g : h_1 h_2$ is equivalent to $g : h_2, h_1$.

Lemma 2.3. *Suppose Γ is an annular planar network with at least one type 1 geodesic; suppose Γ has no type 0 geodesics and no simply connected lenses,* *and that no type 1 geodesic intersects any other geodesic more than once. Then by motions of the medial graph we can transform Γ into a network with an empty boundary triangle, such that at least one of the two geodesics which form the triangle is type 1.*

Proof. Assume without loss of generality that there is an outer-to-outer geodesic and we wish to work on the outer boundary. The argument for the inner boundary is exactly the same.

There exists an outer-to-outer geodesic g_0 with no outer-to-outer geodesics inside it. This is obvious and can easily be proved by contradiction. Suppose there is no such g_0. Then we can find an infinite sequence of geodesics $\{g_n\}$ such that $S(g_{n+1}) \subsetneq S(g_n)$. Clearly, $g_i \neq g_j$ for any $i \neq j$, so this contradicts the assumption that there are only finitely many geodesics.

Given this g_0, let x be the left (further clockwise) endpoint of $\widehat{g_0}$. Let y be the first geodesic endpoint reached by traveling clockwise from x along $\widehat{g_0}$. If y is another endpoint of g_0, then g_0 does not intersect any other geodesics, which is impossible. Thus, y is the endpoint of some other geodesic g_1.

We know that the other endpoint of g_1 does not lie on $\widehat{g_0}$. If it did, then g_1 would either cross g_0 twice (which contradicts our assumptions about lenses) or else it would be an inner-to-inner geodesic inside of g_0 (which contradicts our choice of g_0). By the Jordan curve theorem and our assumptions about lenses, g_1 must cross g_0 exactly once.

I claim that we can make g_0 and g_1 form an empty boundary triangle using Y-Δ transformations.

Let z be point where g_0 and g_1 intersect. Let T be the region inside $\widehat{xy} \cup \widehat{xz}[g_0] \cup \widehat{yz}[g_1]$. Then every geodesic h which intersects T must enter at a vertex on g_0 and exit at a vertex on g_1. It cannot exit along $\widehat{xy}$ because by construction there are no endpoints of geodesics there. Also, it cannot enter along one of the g's and exit along the same one because then it would form a simply connected lens.

Let $h_1, \ldots, h_n$ be the geodesics intersecting T, oriented so that the positive direction of h_j moves from the endpoint lying in $\widehat{xy}$ to g_1, then to g_0, and eventually to the other endpoint. Consider the crossings of h_i and h_j which lie inside T. I claim there exist some h_i and h_j such that $h_i : h_j, g_0$ and $h_j : h_i, g_0$.

Orient g_1 with the starting point at y. Let h_1 be the last geodesic before g_0 in the crossing sequence of g_1 that intersects any other geodesics within T. Let p_1 be the point where h_1 intersects g_0. Let h_2 be the last the last geodesic before g_0 in the crossing sequence of h_1, and let p_2 be the point where h_2 intersects g_0. We know that $p_2 \in \widehat{p_1 z}[g_0]$; this is because h_1 was

Figure 1: Construction of g_0 and g_1.

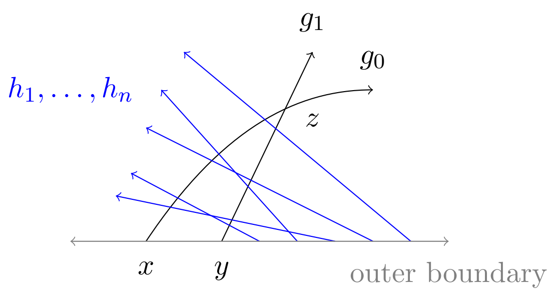
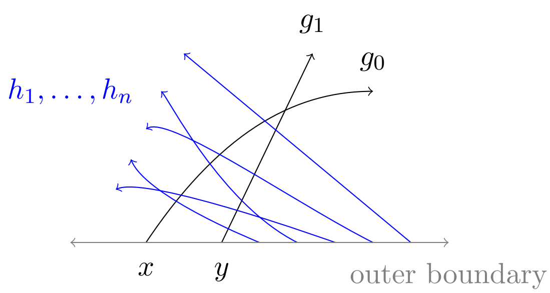


Figure 2: Moving the crossings of $h_1, \dots, h_n$ out of T.



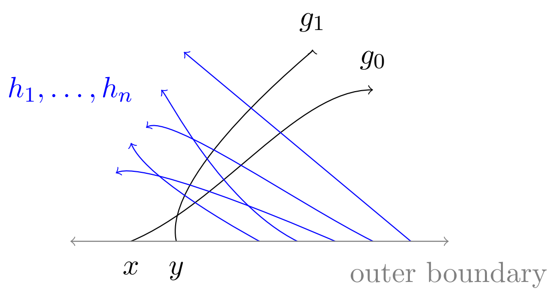
the last geodesic along g_1 before g_0 and because h_2 cannot intersect h_1 more than once. Next, let h_3 be the last geodesic along h_2 before g_0. By a similar argument, we know $p_3 \in \widehat{p_1 p_2}$.

Suppose there are no h_i and h_j such that $h_i : h_j, g_0$ and $h_j : h_i, g_0$. Then continuing inductively, we can create an infinite sequence of geodesics such that $\widehat{p_{2n-1}p_{2n}}[g_0] \subsetneq \widehat{p_{2n+1}p_{2n+2}}[g_0]$. This is a contradiction because there are only finitely many geodesics.

Therefore there is such an h_i and h_j, and we can legally move the crossing of h_i and h_j across g_0 and out of T. Then apply the above argument to the remaining h_n's and move the next crossing out of T. Since there are only finitely many crossings inside T, we will eventually move them all out of T.

At this point g_0 comes immediately after g_1 in the crossing sequence of each h_k. Therefore, we move the crossing of g_0 and g_1 across each h_k toward the outer boundary circle. Once we do this, g_0 and g_1 form an empty boundary triangle.

Figure 3: Moving the crossing of g_0 and g_1 toward the boundary.



$\square$

3 Recovering the Conductivity

Definition 3.1. Let $\mathcal{G}_0$ be the family of type 0 geodesics, $\mathcal{G}_1$ the family of type 1 geodesics, $\mathcal{G}_2$ the family of type 2 geodesics. Let $\mathcal{G}_o$ be the family of outer-to-outer geodesics and $\mathcal{G}_i$ be the family of inner-to-inner geodesics.

Lemma 3.2. *Let Γ and g_0 be as in the previous lemmas. Let Y be the set of medial cells along the inner boundary. Then $\overline{Y} = M \setminus \bigcup_{g \in \mathcal{G}_o} S(g)$.*

Proof. Consider the network as embedded on the universal cover. Let Y be the set of all cells on the whole upper boundary of the strip. Y is connected, so by Will Johnson's lemma, $\overline{Y}$ is the minimal intersection of half-planes containing Y. All inner-to-inner and all type 2 geodesics are in $\mathcal{I}(\overline{Y})$, so $\mathcal{B}(\overline{Y}) \subset \mathcal{G}_o$ because $\overline{Y}$ is the intersection of half-planes. $\overline{Y}$ is the minimal intersection of half-planes containing Y, and the half-planes $S(g)$ for $g \in \mathcal{G}_o$ do not intersect Y, so $\overline{Y} = M \setminus \bigcup_{g \in \mathcal{G}_o} S(g)$. $\square$

Claim. *Let X be the set of medial cells along the outer boundary that do not lie in $S(g_0)$. Then $\overline{X} \cup \overline{Y}$ is connected.*

The proof of this claim is complex enough that it will be broken down into several lemmas. We consider the set W of cells lying along $S(g_0)$ which are not in $S(g_0)$, and we prove that $\overline{X} \cup \overline{Y}$ is connected in the case when $W \subset \overline{X}$ and the case where $W \not\subset \overline{X}$. The first case is easier:

Lemma 3.3. *If $W \subset \overline{X}$, then $\overline{X} \cup \overline{Y}$ is connected.*

Figure 4: $\overline{Y} = M \setminus \bigcup_{g \in \mathcal{G}_o} S(g)$. Not all geodesics are shown.

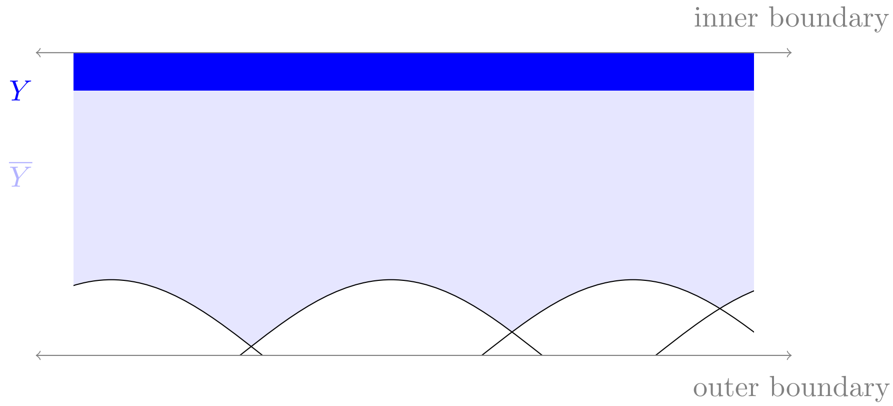
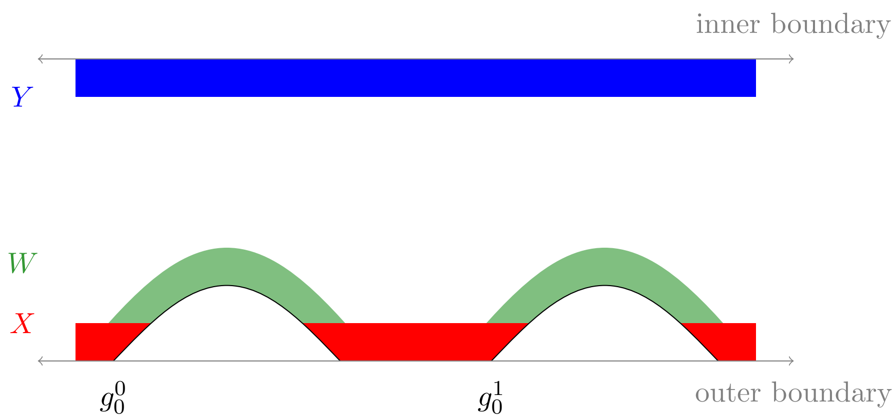


Figure 5: Y, X, and W.



Proof. Observe $\overline{W \cup X} = \overline{X}$. Also, $W \cup X$ is a connected set, even when we consider Γ as embedded on the universal cover and consider W and X over the whole strip.

Suppose g is a type 2 geodesic. If g does not intersect g_0 its vertex on the outer boundary must lie on the complementary arc to $\widehat{g_0}$, so there are cells in X on both sides of g; if g intersects g_0, then there are cells in W on both sides of g. Thus, every type 2 geodesic g is in $\mathcal{I}(\overline{X})$. Since $\overline{X}$ is the intersection of half-planes, $\mathcal{B}(\overline{X})$ must only include g_0 and inner-to-inner geodesics. If $\overline{X} = M \setminus S(g_0)$, we are done; otherwise, $\overline{X}$ includes at least one cell along some inner-to-inner geodesic h. But we know that $S(h) \subset \overline{Y}$, so there are cells $c_1 \in \overline{X}$ and $c_2 \in \overline{Y}$ such that c_1 and c_2 are adjacent. Thus, $\overline{X} \cup \overline{Y}$ is connected. □

We now have to prove that $\overline{X} \cup \overline{Y}$ is connected in the case where $W \not\subset \overline{X}$. We will suppose $\overline{X} \cup \overline{Y}$ is not connected and show that there must be lens involving type 1 geodesics. But before we can produce the lens, we need several constructions.

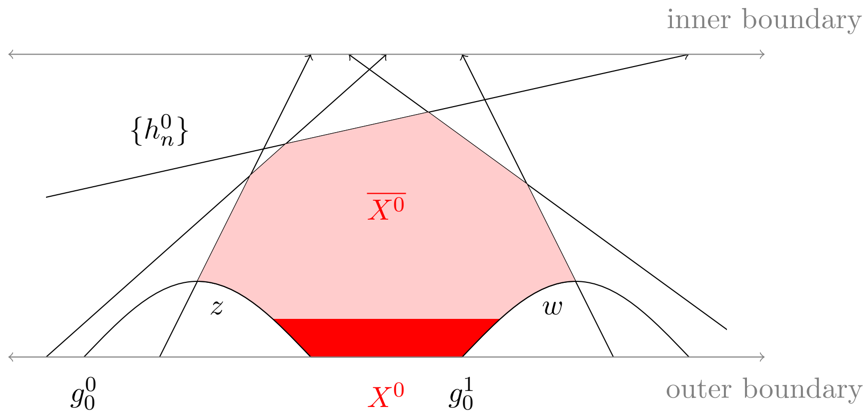
Let x be a point on the lower boundary of the strip such that $e^{ix} \in \widehat{g_0}$. Let X^0 be the set of cells in X in the fundamental domain with real part between x and $x + 2\pi$. X^0 is connected, so we know $\overline{X^0}$ is the intersection of half-planes. X^0 includes adjacent cells on both sides of every outer-to-outer geodesic except g_0. Therefore, $\mathcal{B}(\overline{X})$ contains only type 2 geodesics, inner-to-inner geodesics, and g_0. If $\mathcal{B}(\overline{X_0})$ contains any inner-to-inner geodesics, then we are done because $\overline{X_0}$ has cells adjacent to cells in $\overline{Y}$. So suppose $\mathcal{B}(\overline{X_0})$ contains only type 2 geodesics and g_0.

We assumed there are cells in W that are not in $\overline{X}$. Let g_0^0 be the copy of g_0 immediately to the left of X^0. Let y be the right endpoint of g_0^0. If C is an oriented curve starting at y and moving along g_0^0, it will eventually be alongside a cell in $W \setminus \overline{X}$. Let c be the first cell in $W \setminus \overline{X}$. Then c is located at a corner z of $\overline{X^0}$ because the cells on the other side of g_0^0 are not in X. At the corner, g_0^0 meets another geodesic h_1^0 along $\partial\overline{X^0}$, which must be type 2. Let C continue from the corner along h_1^0. Then C has made a right turn.

Continuing similarly, we can construct a sequence of type 2 geodesics $\{h_n^0\}$ along $\partial\overline{X_0}$. At each step, we will add to C the segment of h_n^0 lying along $\partial\overline{X^0}$. At each corner, C must make a right turn because $\overline{X_0}$ has no anticorners and lies to the right of C.

The sequence $\{h_n^0\}$ cannot continue infinitely. We cannot have $h_i^0 = h_j^0$ for $i \neq j$ because $\partial\overline{X_0}$ must be a simple closed curve and at least part of $\overline{X_0}$ lies along the lower boundary. Let h_N^0 be the last type 2 geodesic in the sequence, and let z be the endpoint of the segment of h_N^0 lying along $\partial\overline{X^0}$; let

Figure 6: Construction of $\{h_n^0\}$ and C.



w be the endpoint occuring later along C. If w lies on the upper boundary, we are done because $\overline{X}$ intersects $\overline{Y}$. If w lies on the lower boundary, then it must lie on ∂X_0; but this is impossible because the only corners of $\overline{X^0}$ on the lower boundary are at the endpoints of g_0. Therefore, w must lie on g_0.

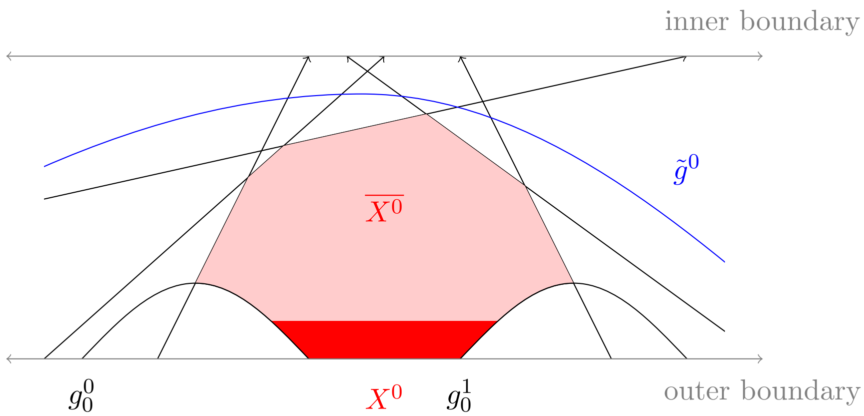
Specifically, w must lie on the copy of g_0 immediately to the right of X^0, which we will designate by g_0^1. It cannot lie to the left of h_1^0 because C makes only right turns. It cannot lie to the right of z along g_0^0 because we assumed c was the first cell in W that was not in $\overline{X^0}$. Since we showed there was at least one type 2 geodesic intersecting g_0 and since $\overline{X^0}$ is the minimal intersection of half-planes containing X^0, we know $\overline{X^0}$ cannot include any cells farther right than g_0^1. Therefore, C intersects g_0^1 at z.

Orient the curves h_n^0 with the starting point on the lower boundary and the ending point on the upper boundary. Let $\mathcal{U}$ be the set of h_n^0's such that the orientations of h_n^0 and C are the same on $h_n^0 \cap C$. Let $\mathcal{V}$ be the set of h_n^0's such that the orientations of h_n^0 and C are opposite on $h_n^0 \cap C$.

Lemma 3.4. *There is an integer K such that $\mathcal{U} = \{h_1^0, h_2^0, \ldots, h_K^0\}$ and $\mathcal{V} = \{h_{K+1}^0, h_{K+2}^0, \ldots, h_N^0\}$.*

Proof. Suppose this is not the case. We know both $\mathcal{U}$ and $\mathcal{V}$ are nonempty because $h_1^0 \in \mathcal{U}$ and $h_N^0 \in \mathcal{V}$. Thus, there must be some J such that $h_J^0 \in \mathcal{V}$ and $h_{J+1}^0 \in \mathcal{U}$. Let p be the point of intersection of h_J^0 and h_{J+1}^0. Let q_J be the endpoint of h_J^0 on the upper boundary and let q_{J+1} be the endpoint of h_{J+1}^0 on the upper boundary. Let T be the region inside $\widehat{pq}[h_J^0] \cup \widehat{pr}[H_{J+1}] \cup$

Figure 7: Construction of $\tilde{g}$ as in Lemma 3.6.



$\widehat{qr}$. Let s_J be the endpoint of h_J^0 on the lower boundary and let s_{J+1} be the endpoint of h_{J+1}^0 on the lower boundary. By construction, a segment of $\widehat{s_J p}[h_J^0]$ and a segment of $\widehat{s_{J+1} p}[h_{J+1}^0]$ lie inside T. But this is impossible because the we assumed h_J^0 and h_{J+1}^0 do not form any lenses. $\square$

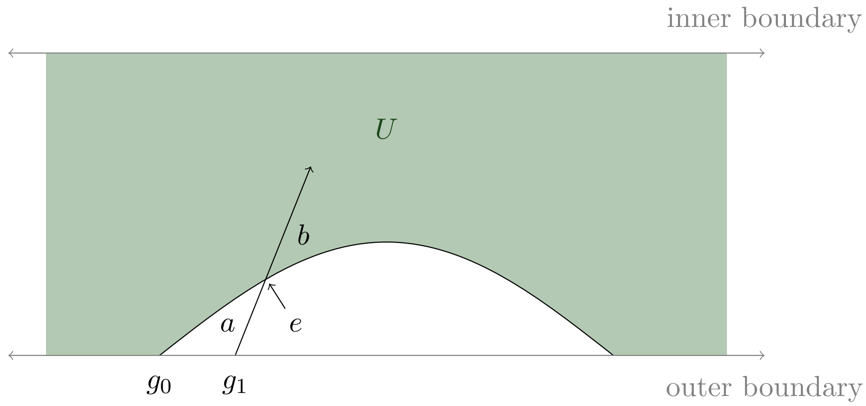
Lemma 3.5. *For each h_k^0, let s_k be the endpoint on the lower boundary. Then s_k lies to the left of s_{k-1} for $k \leq K$, and s_k lies to the right of s_{k-1} for all $k > K$.*

Proof. This follows because of the orientation we chose for h_k^0, because C makes only right turns, and because we assume h_i^0 and h_j^0 do not form a lens. $\square$

Lemma 3.6. *Suppose $\overline{X} \cup \overline{Y}$ is not connected. Then M has a lens of type 1 geodesics.*

Proof. Let c be the cell in $\overline{X^0}$ at the corner where h_K^0 and h_{K+1}^0 intersect. Since we supposed $\overline{X} \cup \overline{Y}$ is not connected, c must not be in $\overline{Y}$. Recall that $\overline{Y} = M \setminus \bigcup_{g \in \mathcal{G}_o} S(g)$. Thus, there is some $\tilde{g} \in \mathcal{G}_o$ such that $c \in S(\tilde{g})$. Specifically, c is contained in $S(\tilde{g}^0)$. Since $\tilde{g}^0$ has both its endpoints on the lower boundary of the strip, we know that h_K^0 must intersect $\tilde{g}^0$ sometime after it intersects h_{K+1}^0 and h_{K+1}^0 must intersect $\tilde{g}^0$ sometime after it intersects h_K^0.

Figure 8: Setup of Lemma 3.8.



Let p^k designate the left endpoint of $\tilde{g}^k$ and q^k its left endpoint. Since we assume $\tilde{g}^0$ only intersects each of the other curves once, we know that the arc of $\tilde{g}^0$ from p to the point where $\tilde{g}$ intersects h_K^0 lies to the left of h_K^0. Therefore, p^0 lies to the left of s_K, and by the previous lemma, p^0 lies to the left of s_1. By a similar argument, q^0 lies to the right of s_N.

If p^0 lies to the left of q^{-1}, we know that $\tilde{g}^0$ and $\tilde{g}^{-1}$ intersect, which implies that in the annulus $\tilde{g}$ forms a one-pole lens. On the other hand, if p^0 lies to the right of q^{-1}, we know that q^{-1} and p^0 lie inside $\widehat{g_0^0}$, which implies that g_0 and $\tilde{g}$ form a two-pole lens in the annulus. $\square$

Lemma 3.7. $\overline{X \cup Y} = M \setminus S(g_0)$.

Proof. $\overline{X \cup Y} = \overline{\overline{X} \cup \overline{Y}}$. From the preceding lemmas, we know $\overline{X} \cup \overline{Y}$ is connected, so $\overline{X \cup Y}$ is the minimal intersection of half-planes containing $\overline{X} \cup \overline{Y}$, which is $M \setminus S(g_0)$. $\square$

Lemma 3.8. *Let a be the cell of the medial graph in the empty boundary triangle formed by g_0 and g_1. Let b be the cell diagonally opposite to a at $g_0 \perp g_1$. Let e be the vertex of the medial graph between a and b. We can determine the conductivity of e from the Dirichlet-to-Neumann map.*

Proof. Let $U = \overline{X \cup Y}$. By the preceding lemmas, we know that $b \in U$. Setting all (co)voltages to zero on $X \cup Y$ will force the (co)voltage at b to be zero. At this point, all the (co)voltage data on the network are consistent because they are all zero.

Consider $\{a\} \cup U$ where a is taken in a single period of the graph. By a theorem of Will Johnson, $\overline{U \cup \{a\}}$ is a nice extension of $U \cup \{a\}$ and therefore setting the (co)voltage at a to some value x produces consistent data on some part of the network. Obviously, if take a over all periods, we still have consistent data. In fact, defining the (co)voltage at a determines the (co)voltage over the whole network because $U \cup \{a\}$ has cells on both sides of every geodesic. We therefore have a mixed problem with unique solution.

We know the (co)voltage on a and b by construction and on the two cells adjacent to both a and b because they are boundary cells. Therefore, we can determine the conductivity of e. □

4 Completing the Theorem

Definition 4.1. An annular planar network is called radial if all geodesics in the medial graph are type 2.

Theorem 4.2. *Suppose Γ is an annular planar network with no type 0 geodesics, no simply connected lenses, and no lenses involving type 1 geodesics. Then Γ can be reduced by geodesic elimination to a (possibly degenerate) radial network.*

Proof. If there is no type 1 geodesic, we are done. As long as there is a type 1 geodesic, the preceding lemmas show that we can find some conductivity and uncross an empty boundary triangle. At each step, we will not create any new lenses because we only used Y-Δ transformations and empty boundary triangle uncrossings.

There are only a finite number of type 1 geodesics in the network and each one has only finitely many geodesics intersecting it. Eventually, some type 1 geodesic will have only one geodesic intersecting it. This corresponds in the primal graph to a degree 1 boundary vertex with an edge connecting it to another boundary vertex. We can easily find the conductivity on that edge and delete the edge and the now disconnected vertex from the network. In the medial graph, the corresponding geodesic will disappear.

By continuing to solve empty boundary triangles, we will eventually eliminate all type 1 geodesics from the medial graph. □

Corollary 4.3. *To determine the recoverability of such a graph, we may* *delete all one-component geodesics and determine the recoverability of the remaining graph.*

Proof. The above theorem guarantees that any such network Γ can be reduced to a radial network Γ'. To prove the corollary, it suffices to show that the two-component geodesics in Γ' can be put back into their original arrangement; that is, any switches in the crossings of the two-component geodesics can be reversed. Supposing that the geodesic elimination proceeded in the manner given in the proof of the above theorem, this is obvious. We never changed the crossing set of any two-component geodesics. Thus, we can obtain the original arrangement of the two-component geodesics by reversing all the Y-Δ transformations which involved only two-component geodesics. $\square$

The above theorem and proof generalize to nonlinear conductivity functions of the type described by Will Johnson in [2]. The argument was mostly geometric and nowhere assumed the conductivities were linear.

5 Conclusion

The theorem guarantees partial recoverability for a wide range of annular networks. Although the motions (corresponding to Y-Δ transformations) will jumble the conductivities, the amount of information recovered is equivalent to the number of crossings along the geodesics removed (the number of edges in the primal graph).

The theorem simplifies the question of recoverability in a reasonably general case, but more work is needed because we still do not understand radial networks. Yet even this result required a combination of the most powerful medial graph techniques invented for circular planar graphs. It shows that, in some sense, the type 1 geodesics encapsulate the "circular planar component" in the behavior of annular networks. Type 2 geodesics and non-simply-connected lenses are new problems that will require new techniques.

References

[1] Edward B. Curtis and James A. Morrow. *Inverse Problems for Electrical Networks.* World Scientific. 2000.

[2] Will Johnson. "Nonlinear Electrical Networks."

[3] Will Johnson. "Recovery of Non-linear Conductivities for Circular Planar Graphs."

[4] Ian Zemke. "Infinite Electrical Networks: Forward and Inverse Problems."