Annular Networks: Preliminaries

David Jekel

July 15, 2013

1 Introduction

The purpose of this paper is to establish terminology, notation, and tools for describing and recovering annular networks. I assume the reader is familiar with the basic theory of electrical networks and significant results in the circular planar case. Some terms with well-established definitions will not be defined here, and the definitions presented will prioritize precision over readability.

There are known criteria for the recoverability of circular planar networks, but not for planar networks in general. Every finite planar graph with boundary can be embedded in a circular region with n holes.

2 Basic Definitions

An electrical network is a connected graph $G = (V, E)$ (with V partitioned into the boundary ∂V and the interior $\operatorname{int} V$) and a conductivity function $\gamma : E \to \{x > 0\}$. We will write either $\gamma(pq)$ or γ_{pq}. If there is no edge between p and q, we define γ_{pq} to be zero. We assume $|V|$ is finite. We assume every interior vertex has degree at least two.

A directed edge is an edge in $e \in E$ with an specified order of the vertices. When we name an edge in V, we will assume it is undirected unless otherwise specified. The symbol pq will denote the undirected edge from p to q and $p \to q$ will denote the directed edge. $p \sim q$ means that there is an edge between p and q. If $e = p \to q$, then $-e$ will denote $q \to p$.

An embedding of a graph in the plane is a function which uniquely associates coordinates in $\mathbb{C}$ to each vertex and a curve to each directed edge. (We may denote the vertex and its coordinates by the same symbol and the edge and its curve by the same symbol.) For a directed edge $p \to q$, the curve can be parametrized by an injective C^1 function $f_{pq} : [0, 1] \to \mathbb{C}$ with nonzero derivative such that $f(0) = p$ and $f(1) = q$ and $f_{pq}([0,1]) = f_{qp}([0,1])$. The curves for distinct edges may not intersect unless the edges share a vertex, and in that case, the curves may only intersect at that vertex.

An annular planar network is an electrical network whose graph can be embedded in $A = \{r < |z| < R\}$ for some positive $r < R$. The circle $\{|z| = r\}$ is called the inner boundary circle and $\{|z| = R\}$ is called the outer boundary circle.

3 Construction of the Medial Graph

The medial graph $M(G)$ is a graph constructed on G as follows. Place a vertex of M on each edge of G (but not on the endpoints of the edge). If two boundary vertices v_1 and v_2 of G are connected by an arc $\widehat{A}$ of the boundary circle such that no other boundary vertices of G lie on $\widehat{A}$, then place two boundary vertices of M along $\widehat{A}$ between v_1 and v_2.

For each interior vertex v of G, let $e_1, \ldots, e_n$ be the edges with endpoints at v, in some counterclockwise order about v. Let $w_1, \ldots, w_n$ be the corresponding vertices of M. Add an edge of M between w_1 and w_2, w_2 and w_3, $\ldots$, w_{n-1} and w_n, and finally between w_n and w_1.

For each boundary vertex v of G, let $c_1, \ldots, c_n$ be the edges and segments of the boundary circle with endpoints at v, ordered counterclockwise such that c_1 and c_n are the segments of the boundary circle. Let $w_1, \ldots, w_n$ be the corresponding vertices of M. For the segments of the boundary circle, let w_k be the closest vertex of M to v. Add an edge of M between w_1 and w_2, w_2 and w_3, $\ldots$, w_{n-1} and w_n.

Each interior vertex of the medial graph has degree 4. Thus, we can construct curves called geodesics by joining opposite edges of M at each vertex.

Suppose x and y are two points on the same boundary circle. Let $\widehat{xy}$ denote the arc of the boundary circle passing from x to y in a counterclockwise direction.

Suppose g is a geodesic and x and y are vertices of the medial graph which lie on g. Let $\widehat{xy}[g]$ be the arc of g which passes from x to y. We assume the curve is oriented with x before y.

4 Classification of Geodesics and Lenses

A lens in the medial graph is a simple closed curve which is the union of edges in the medial graph, such that all the edges belong to one or two geodesics. A lens L is called simply connected if there is a simply connected set S such that $L \subset S \subset A$.

Divide geodesics into the following categories: A type 1 geodesic has both endpoints on the same boundary circle; these geodesics are further divided into inner-to-inner or outer-to-outer based on which boundary circle their endpoints are on. A type 2 geodesic has one endpoint on each boundary circle. A type 0 geodesic has no endpoints (it is a zero-pole lens).

Let $\mathcal{G}_0$ be the family of type 0 geodesics, $\mathcal{G}_1$ the family of type 1 geodesics, $\mathcal{G}_2$ the family of type 2 geodesics. Let $\mathcal{G}_o$ be the family of outer-to-outer geodesics (type 1 geodesics with both endpoints on the outer boundary circle) and $\mathcal{G}_i$ be the family of inner-to-inner geodesics (type 1 geodesics with both endpoints on the inner boundary circle).

Suppose g is a type 1 geodesic in M, which does not intersect itself. Then g divides the annulus into two components. Let $S(g)$ be the component which does not include the hole. Let $\widehat{g}$ denote the arc of the outer circle which lies along $S(g)$. A geodesic h is said to lie inside g if $h \neq g$ and $S(h) \subset S(g)$. These terms are not defined for any other geodesics.

5 Manipulation of the Medial Graph

For each geodesic g, the crossing sequence is an ordered list of the geodesics which g crosses in the order they are crossed when g is parametrized by a piecewise smooth curve. By reversing the orientation of g, we may create an equivalent list in reverse order, but no other orderings are equivalent.

$g : h_1, h_2$ means that the geodesic g intersects geodesic h_1 and immediately afterward intersects the geodesic h_2. In other words, "h_1, h_2" appears in the crossing sequence of g. If no orientation is specified on g, $g : h_1 h_2$ is equivalent to $g : h_2, h_1$.

A motion is a change in the medial graph corresponding to a Y-Δ transformation of the primal graph. Although technically the medial graph must be redrawn after each transformation, we can identify whether two geodesics are the same before and after the transformation by their endpoints. Thus, we can describe a motion in terms of crossings in the medial graph. If $f : g, h$ and $g : f, h$ and $h : f, g$, then a motion exchanges g and h in the crossing sequence of f and similarly for the other two geodesics, which produces $f : h, g$, $g : h, f$, $h : g, f$.

An uncrossing is the removal of the intersection of two geodesics. It corresponds to an edge removal in the primal graph. If g_1 and g_2 are uncrossed in M, producing a graph M', there is no canonical way to determine which of the resulting geodesics in M' is g_1 and which is g_2.

Geodesic elimination is the process of manipulation the medial and primal graphs by

- Motions,
- Uncrossings performed at empty boundary triangles after determining the conductivity of the edge in the primal graph,
- Deleting disconnected boundary vertices.

6 Universal Cover and Multi-Valued Covoltages

For the argument which follows, it will sometimes be necessary to work on a simply connected set rather than the annulus. The universal cover of the annulus $A = \{r < |z| < R\}$ is the strip $S = \{-\log R < \mathrm{Im}(z) < -\log r\}$, which maps onto A by the function $f(z) = e^{iz}$. For any simply connected $U \subset A$, $f^{-1}(U)$ is a sequence of disjoint connected sets $V_n \subset S$, which we can index by positive and negative integers. And for each V_n, V_{n+1} is the translation of V_n to the right by 2π. For each V_n, f is homeomorphism from V_n to U.

The outer boundary of the annulus corresponds to the lower boundary of the strip and the inner boundary corresponds to the upper boundary.

When we apply f^{-1} to the vertices and edges in G, M, and $G^\dagger$, we obtain an infinite electrical network on the strip, whose vertices and edges repeat with a period of 2π. Any voltage function on the annulus maps to a periodic voltage function on the strip and any periodic voltage function on the strip maps to a voltage function on the annulus.

Because the strip is simply connected, every voltage function has a well-defined covoltage. Since the voltage is periodic, the differences in covoltage must also be periodic; however, the covoltage itself may not. When we map the strip's covoltage onto the annulus, it may become multivalued similar to the argument function of complex analysis.

In fact, we can easily determine the net change in covoltage over a period of the graph in a strip. Let $v^\dagger$ be the covoltage function. Let x be a vertex on the upper boundary of the strip and let $x + 2\pi$ be the corresponding vertex in the next period. Since the net current at a boundary vertex is the difference in the covoltages of the adjacent cells, $v^\dagger(x + 2\pi) - v^\dagger(x)$ is the sum of the currents of the upper boundary vertices over one period. This is exactly the net current flowing from the inner boundary to the outer boundary, which we may call the net outward current.[1] For any function u, this is calculated by $\chi_i \Lambda \phi$, where $\phi = u|_{\partial V}$ and χ_i is the characteristic function of the inner boundary vertices written as a vector.

Thus, if we are careful about the multiple values of the covoltages, we can use Will Johnson's method of voltages and covoltages on annular graphs to define mixed problems whose solutions will allow us to determine certain conductivities. In practice, it is often unnecessary to make covoltages multi-valued. We can find functions with net outward current zero that still allow us to determine conductivities.

Definition 6.1. For a geodesic g in the annulus, g^k will denote the inverse image of g in the strip in period k.

Lemma 6.2 (Will Johnson)**.** *The voltage-covoltage relationship is equivalent information to the Dirichlet-to-Neumann map.*

7 Medial Cell "Topology"

The "topology" invented by Will Johnson and extended by Ian Zemke describes the propagation of information through an electrical network. With their machinery we can construct mixed problems with unique solutions, from whose boundary data we can determine conductivities on certain edges. Unfortunately, these results only apply when the embedding region is simply connected; we should not apply them to the strip rather than the annulus.

Definition 7.1. Let X be a set of medial cells. Then a vertex v of the medial graph is called

- Interior if X contains all cells around v.
- Exterior if X contains no cells around v.
- A corner if X contains one cell around v.
- An anticorner if X contains three cells around v.
- A double corner if X contains two diagonally opposite cells around v.[2]

[1] The fact that the net outward current is equal to the change in covoltage over a period can also be proved by considering cycles in the dual graph which encircle the hole and applying the "Green's Theorem" in [3].

[2] Ian Zemke calls this a degenerate corner.

Definition 7.2. Let R be a simply connected region in which a graph is embedded. Let g be a geodesic of the medial graph with at least one endpoint on ∂R. Then g splits R into two components. Each of the components is called a half-plane.

Definition 7.3. A set is closed if it has no anticorners. The (geodesic) closure of X, denoted by $\overline{X}$, is the smallest closed set containing X.[3]

Theorem 7.4 (Will Johnson, Ian Zemke)**.** *Suppose X is a connected set of medial graph cells, where the medial graph is embedded in a simply connected region of the plane. Then $\overline{X}$ is the minimal intersection of half-planes containing X.*

Definition 7.5. Suppose X is a set of medial graph cells. Let ∂X denote the union of all edges in the medial graph at which some cells $c_1 \in X$ and some $c_2 \notin X$ are adjacent.

Definition 7.6. Suppose X is a set of medial graph cells. Let $\mathcal{B}(X)$ be the family of all geodesics g such that $g \cap \partial X \neq \emptyset$. Let $\mathcal{I}(X)$ be the family of all geodesics g intersecting the interior of X.

Lemma 7.7. *Suppose X is the intersection of half-planes. Then $\mathcal{B}(X) \cap \mathcal{I}(X) = \emptyset$.*

This lemma is clear from the definition. Considering the sets $\mathcal{B}(X)$ and $\mathcal{I}(X)$ allows us to interpret Will Johnson's results for a non-simply-connected region. Unless we unwrap the annulus onto the strip, we cannot define half-planes for type 2 geodesics. Thus, we cannot directly apply Theorem 7.4 to an annulus. However, we can apply the theorem to the strip and determine which geodesics may form part of the boundary of a region; this information *does* have a clear meaning on the annulus.

Theorem 7.8 (Will Johnson)**.** *Let X be a convex set with some boundary cells. Let a be a boundary cell not in X but adjacent to a cell in X. Then $\overline{X \cup \{a\}}$ is a nice extension of $X \cup \{a\}$.*

References

[1] Edward B. Curtis and James A. Morrow. *Inverse Problems for Electrical Networks.* World Scientific. 2000.

[3] The geodesic closure is NOT a topological closure.

[2] Will Johnson. "Recovery of Nonlinear Conductivities for Circular Planar Graphs."

[3] Karen Perry. "Discrete Complex Analysis."

[4] Ian Zemke. "Infinite Electrical Networks: Forward and Inverse Problems."