Factoring peak polynomials

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Outline

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   - Enumeration theorems
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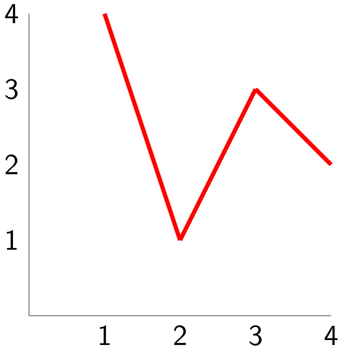
Permutations

Definition

A **permutation** $\pi = \pi_1\pi_2\ldots\pi_n$ in the symmetric group $\mathfrak{S}_n$ is a bijection from the set $\{1, 2, \ldots, n\}$ to itself.

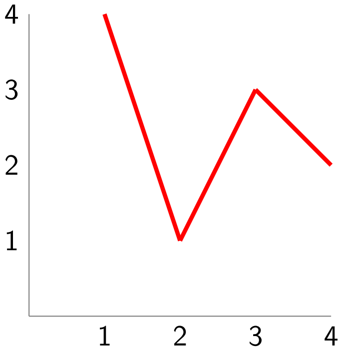


Graph of $\pi = 4\ 1\ 3\ 2$

Peak

Definition

An index i is a **peak** of a permutation π if $\pi_{i-1} < \pi_i > \pi_{i+1}$.



Graph of $\pi = 4\ 1\ 3\ 2$

Peak set

Definition

The **peak set** $P(\pi)$ of a permutation π is the set of all peaks in π.

If $\pi = 2\ 8\ 4\ 3\ 5\ 1\ 6\ 9\ 7 \in \mathfrak{S}_9$, then

$$P(\pi) =$$

Peak Set

Definition

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If $\pi = 2\ 8\ 4\ 3\ 5\ 1\ 6\ 9\ 7 \in \mathfrak{S}_9$, then

$$P(\pi) = \{2, 5, 8\}.$$

Permutations with a given peak set

Definition

Given any finite set S of positive integers, let

$$\mathcal{P}(S;n) = \{\pi \in \mathfrak{S}_n : P(\pi) = S\}.$$

Permutations in $\mathfrak{S}_3$ whose peak set is $\{2\}$:

1 2 3
1 3 2
2 1 3
2 3 1
3 1 2
3 2 1

$\mathcal{P}(\{2\};3) =$
$\{1\,3\,2,\ 2\,3\,1\}$

Empty peak set

Theorem (Billey, Burdzy, and Sagan - 2013)

For $n \geq 1$ we have

$$\#\mathcal{P}(\emptyset; n) = 2^{n-1}.$$

The peak set with a no elements is the base case for some of our inductive arguments.

Main enumeration theorem

Theorem (Billey, Burdzy, and Sagan - 2013)

If $S = \{i_1 < i_2 < \cdots < i_s\}$, $S_1 = S \setminus \{i_s\}$, *and* $S_2 = S_1 \cup \{i_s - 1\}$, *then*

$$\#\mathcal{P}(S;n) = p(S;n)2^{n-\#S-1},$$

where $p(S;n)$ *is a polynomial depending on* S *of degree* $i_s - 1$ *given by*

$$p(S;n) = \binom{n}{i_s - 1} p(S_1; i_s - 1) - 2p(S_1;n) - p(S_2;n).$$

Moreover, $p(S;i_s) = 0$.

Peak set with a constant element

Theorem (Billey, Burdzy, and Sagan - 2013)

If $S = \{m\}$, then

$$p(S; n) = \binom{n-1}{m-1} - 1.$$

The peak set with a single element is the base case for some of our inductive arguments.

Peak set with a constant element

Example: Probability that $P(\pi) = \{50\}$ if $\pi \in \mathfrak{S}_{100}$

$$\#\mathcal{P}(\{50\}; 100) = \left(\binom{100-1}{50-1} - 1 \right) 2^{100 - \#\{50\} - 1} \approx 1.536 \times 10^{58}$$

Peak set with a constant element

Example: Probability that $P(\pi) = \{50\}$ if $\pi \in \mathfrak{S}_{100}$

$$\#\mathcal{P}(\{50\};100) = \left(\binom{100-1}{50-1} - 1\right) 2^{100-\#\{50\}-1} \approx 1.536 \times 10^{58}$$

$$\frac{\#\mathcal{P}(\{50\};100)}{100!} \approx 1.713 \times 10^{-100}$$

Sage computation

We used Sage to:

- Compute $\#\mathcal{P}(S; n)$ using alternating permutations and the inclusion-exclusion principle
- Sample values to interpolate the peak set polynomial
- Factor and find the complex zeros of $p(S; n)$

Zeros of factored peak polynomials

Example: Zeros of factored peak polynomials

$$p(\{3,7\};n) = \frac{1}{80}n^2(n-3)(n-7)(n^2 - \frac{25}{3}n + \frac{62}{3})$$

$$p(\{6\};n) = \frac{1}{120}(n-6)(n^4 - 9n^3 + 31n^2 - 39n + 40)$$

$$p(\{4,6,9\};n) = \frac{5}{2016}n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-9)$$

All peaks are roots

Theorem

If $S = \{i_1 < i_2 < \cdots < i_s\}$*, then all* $i \in S$ *are zeros of* $p(S;n)$*.*

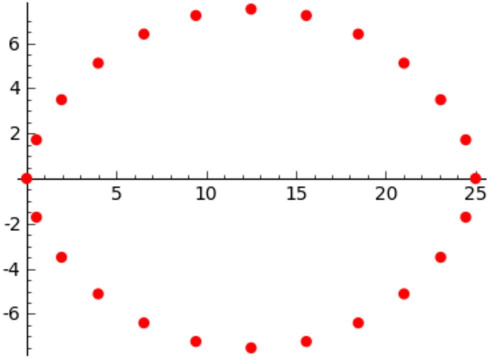
Proof sketch.

Induct on the peak sets whose maximum element is i_s. □

Complex zeros of a single peak

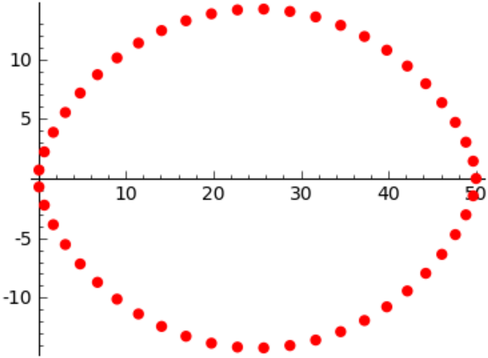
- Where are the remaining zeros of a peak polynomial?
- Recall that the degree of the polynomial is $m-1$, where $m = \max S$. We have the most unknown zeros when the peak set contains a single element.

Complex zeros of a single peak



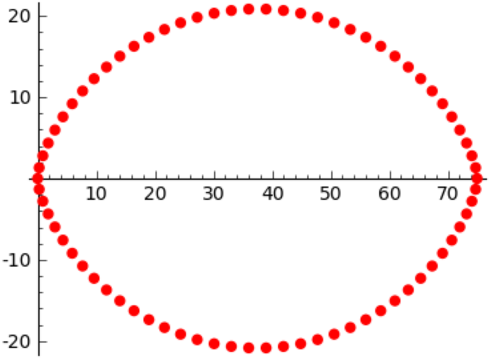
$$p(\{25\}; n) = \binom{n-1}{25-1} - 1$$

Complex zeros of a single peak



$$p(\{50\}; n) = \binom{n-1}{50-1} - 1$$

Complex zeros of a single peak



$$p(\{75\}; n) = \binom{n-1}{75-1} - 1$$

Positivity conjecture

Our motivation for studying zeros comes from the following conjecture.

Conjecture (Billey, Burdzy, and Sagan - 2013)

Let $m = \max S$ and c_k^S be the coefficient of $\binom{n-m}{k}$ in the expansion

$$p(S;n) = \sum_{k=0}^{m-1} c_k^S \binom{n-m}{k}.$$

Each coefficient c_k^S is a positive integer for all $0 < k < m$ and all admissible sets S.

Positivity conjecture

Note that we can sample values of $p(S;n)$ using the main enumeration theorem.

$$\#\mathcal{P}(S;n) = p(S;n)2^{n-\#S-1} \implies p(S;n) = \frac{\#\mathcal{P}(S;n)}{2^{n-\#S-1}}$$

Positivity conjecture

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Example: Sample value of $p(\{2,5\};6)$

We calculate $\#\mathcal{P}(\{2,5\};6)$ using a computer, so then

$$p(\{2,5\};6) = \frac{\#\mathcal{P}(\{2,5\};6)}{2^{6-2-1}} = \frac{80}{8} = 10.$$

Positivity conjecture

Example: Positivity conjecture

If $S = \{2, 5\}$ then $\deg p(S;n) = 4$, and we can interpolate $p(S;n)$ by sampling 5 points.

0=p(S;5)	10=p(S;6)	35=p(S;7)	84	168	300
10	25	49	84	132	
15	24	35	48		
9	11	13			
2	2				

Positivity conjecture

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2	2				

$$p(S;n) = 10\binom{n-5}{1} + 15\binom{n-5}{2} + 9\binom{n-5}{3} + 2\binom{n-5}{4}$$
$$= \frac{1}{12}n(n-1)(n-2)(n-5)$$

Stronger conjecture than positivity

Conjecture

If S is admissible, then $p(S;n)$ does not have any zeros whose real part is greater than $\max S$.

The conjecture above implies the truth of the positivity conjecture, because it implies that $p(S;n)$ and all of its derivatives are positive after $m = \max S$. The forward differences c_k^S are discrete analogs of the derivates of $p(S;n)$.

Odd differences

The difference between consecutive peaks of S determines the zeros of $p(S;n)$.

Example: Odd differences

$$p(\{2,7\};n) = \frac{1}{180}n(n-1)(n-2)(n-7)(n^2 - \frac{19}{2}n + 27)$$

$$p(\{3,5,8\};n) = \frac{1}{120}n(n-1)(n-2)(n-3)(n-4)(n-5)(n-8)$$

Final difference of 3

Theorem

If $S = \{i_1 < i_2 < \cdots < i_s < i_s + 3\}$, *then*

$$p(S; n) = \frac{p(S_1; i_s + 1)}{2(i_s + 1)!}(n - (i_s + 3)) \prod_{i=0}^{i_s}(n - i).$$

Note that $p(S_1; n)$ may be chaotic, but the zeros of $p(S; n)$ are well-behaved by forcing $i_s + 3$ to be a peak.

Run of adjacent peaks

Theorem

If $S = \{m, m+3, \ldots, m+3k\}$ *with* $k \geq 1$, *then*

$$p(S;n) = \frac{(m-1)(n-(m+3k))}{2(m+1)!(12^{k-1})} \prod_{i=0}^{m+3(k-1)} (n-i).$$

Example

If $S = \{3, 6, 9\}$, then

$$p(S;n) = \frac{2n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-9)}{2(4)!(12)}.$$

Summary

1. Permutations with a given peak set can be enumerated by a unique polynomial that is recursively defined.
2. We proved that all peaks in a peak set are zeros of its corresponding peak polynomial.
3. Odd gaps between adjacent peaks determines some of the zeros of the peak polynomial.
4. We know the peak polynomial for peak sets of the form $\{m, m+3, \ldots, m+3k\}$.

Questions

Questions?

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