Definition 1 (Chromatic Number). The chromatic number, \(k(G)\) of a graph \(G\), is the minimal number of colors required so that each vertex of the graph can be given a color and no two adjacent vertices have the same color.

In this discussion, we will assume that there are NO loops, parallel edges or spikes in graphs. Finding the chromatic number of a graph is similar to finding a coloring of a map with all adjacent countries having distinct colors. We will use the following fact from Topology: every compact, connected orientable surface is a finite connected sum of tori. i.e. we can think of surfaces as spheres with finitely many handles \(g\) also known as the genus of the surface.

Definition 2 (Embedding). A graph \(G\) is embedded to a surface \(S_g\) if the graph is drawn on a surface without edges crossing. That is, for each edge \(E_j\) of \(G\) there exists an embedding: \(e_j : [0, 1] \rightarrow S_g\) such that:

1. the images of edges intersect iff edges share a vertex and
2. images of edges should only intersect at endpoints.

We’ll also assume our emeddings are such that the components of \(S_g\) are homeomorphic to disks.

Definition 3 (Chromatic Number of Surface). The chromatic number, \(K(S_g)\) of a surface \(S_g\) is the largest chromatic number of any graph \(G\) embedded in \(S_g\).

The Four color theorem states that \(K(S_0) = 4\). The Heawood Conjecture (1890) states that:

\[
K(S_g) = \left\lfloor \frac{7 + \sqrt{49 - 48g}}{2} \right\rfloor = \left\lfloor \frac{7 + \sqrt{49 - 24\chi}}{2} \right\rfloor
\]

where \(\lfloor \rfloor\) denotes the floor function and \(\chi = 2 - 2g\) is the Euler Characteristic.

Definition 4 (Criticality). A graph is critical if no subgraph has the same chromatic number.

Lemma 1. If \(G\) is a critical graph with chromatic number \(K\), then the degree of each vertex is \(\geq k - 1\).

Proof. Suppose not. Then, there would exist a vertex \(p \in G\) with degree \(< k - 1\). Now, remove \(p\) and its adjacent edges. This produces the subgraph \(G'\) where \(k(G') = k - 1\). This implies that vertices adjacent to \(p\) can be colored with \(k - 2\) colors (last remaining color is used for \(p\)). By criticality, we have that \(G\) is \(k - 1\) colored, a contradiction.

Lemma 2. If \(G\) is critical with \(\alpha_0\), number of vertices, \(\alpha_1\), number of edges and \(\chi(G) = K\), then \((K-1)\alpha_0 \leq 2\alpha_1\). For a polyhedron \(P\), define the Euler characteristic \(\chi(P) = V - E + F\) where \(V\) is the number of vertices, \(E\) is the number of edges and \(F\) is the number of faces.

Think of our embeddings of graphs into surfaces to have the following properties: vertices of graph ↔ vertices of polyhedron, edges of graph ↔ edges of polyhedron, and faces of an embedding will refer to the faces of the polyhedron. If \(P(G)\) is an embedding of \(G\) into \(S_g\), we say that \(\chi(P(G)) = V - E + F = 2 - 2g\).

Lemma 3. If \(G\) has vertices of degree \(\leq 2\) and \(P(G)\) is an embedding of \(G\), then \(|E| \leq 3|V| - 3\chi(P(G))\).

Theorem 1 (Heawood). If \(\chi(S) \leq 0\), then

\[
K(S) \leq \left\lfloor \frac{7 + \sqrt{49 - 24\chi}}{2} \right\rfloor.
\]

1
Proof. Suppose $G$ is embedded in $S$ with $K(G) = K(S)$. Without loss of generality, suppose $G$ is critical. Then from the lemmas, we see that:

$$(K - 1)V \leq 2E \leq 6V - 6\chi$$

Since $K \leq V$, we see that $K - 1 \leq 6 - \frac{6\chi}{V} \leq 6 - \frac{6\chi}{K}$. This translates to the quadratic polynomial $K^2 - 7K + 6\chi \leq 0$. Since $\chi \leq 0$ and $K > 1$, we have that $K - \frac{7 + \sqrt{49 - 24\chi}}{2} < 0$. \hfill $\Box$

We can check that for a graph $G$, $$g(G) \geq \left\lfloor \frac{E - 3V + 6}{6} \right\rfloor$$

where $g(G)$ is the smallest genus in which $G$ can be embedded. For complete graphs $K_n$, $g(K_n) \geq \left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor$.

**Theorem 2.** If $$g(K_n) = \left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor,$$

then the Heawood conjecture is true.

**Definition 5** (Triangular Embedding). An embedding $P(G)$ is triangular if all faces have 3 edges. If $P(G)$ is triangular then $g(P(G)) = g(G)$.

Note that if we can find a triangular embedding for $K_n$ (or any graph), then we’ve determined the genus of a graph. However, triangular embeddings do not always exist. For $K_n$ to have an embedding, we need $3F = 2E$. We can deduce that triangular embeddings exist for $K_n$ when $n \equiv 0, 3, 4, 7 \mod 12$.

**Definition 6** (Rotation System). A rotation system on a graph gives an embedding where a rotation system is a cyclic ordering of vertices adjacent to each vertex to a graph.

If each row $i$ of the rotational system is obtained by taking the row $i-1$, adding 1 to each entry and moding out by $n$, then the embedding is triangular. Therefore to see if a triangular embedding is possible, we only need the first row of the rotation system.