Annular Plate Networks

David Jekel

September 9, 2013

Abstract

I consider electrical networks formed by conducting plates in an annular region of the plane. These networks are mathematically similar to electrical networks on a graph with vertices and edges which is embedded in an annulus. I describe how to remove lenses from the medial graph by $Y$-$\Delta$ transformation. By partitioning the network into subnetworks, I prove analogues of the cut-point lemma with corresponding algebraic statements. I prove recoverability for certain classes of networks.

Acknowledgements

This paper was written as part of the 2013 math REU at the UW. I want to thank the faculty, TAs, and students of the REU for their discussion and feedback, especially Jim Morrow, Ian Zemke, and Kolya Malkin.

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1 Introduction

1.1 Annular Plate Networks

Let $S_{\text{inner}}$ and $S_{\text{outer}}$ be the regions of the plane contained in piecewise-smooth simple closed curves $C_{\text{inner}}$ and $C_{\text{outer}}$ respectively, with $S_{\text{inner}} \subset S_{\text{outer}}$, and let $S = S_{\text{outer}} \setminus S_{\text{inner}}$. Let plates $P_1, \ldots, P_N$ be compact, simply connected subsets of $\overline{S}$ bounded by piecewise smooth curves such that

- The intersection of two plates is either empty or a single point,
- The point of intersection between two plates does not lie on a boundary curve,
- $\mathcal{P}_I \cap C_{\text{inner}}$ is either empty or an interval of $C_{\text{inner}}$ and $\mathcal{P}_I \cap C_{\text{outer}}$ is either empty or an interval of $C_{\text{outer}}$.

$S$ is called the region of embedding. We will call it an annular region even though it is not technically an annulus, only homeomorphic to an annulus. $C_{\text{inner}}$ and $C_{\text{outer}}$ are called the inner and outer boundary curves respectively. $S_{\text{inner}}$ is called the hole.

If two plates $P$ and $Q$ intersect at a point $x$, they are called adjacent, we write $P \sim Q$, and $x$ is called a juncture between $P$ and $Q$ and is denoted $PQ$. A directed juncture $P \rightarrow Q$ is a juncture with specified order of the adjacent plates $P$ and $Q$. The intersection of a plate and a boundary curve is called a boundary interval. If a plate intersects the boundary curve, we say it touches the boundary curve.

$\mathcal{P}$ is the collection of all plates, $\mathcal{J}$ is the collection of all directed junctures between adjacent plates, $\mathcal{I}$ is the collection of all boundary intervals. For each plate $P$, $\mathcal{I}(P)$ is the set of boundary intervals for $P$ ($\mathcal{I}(P)$ contains either zero, one, or two elements). Let $\partial \mathcal{P}$ be the collection of boundary plates (plates touching any boundary curve), and let $\text{int} \mathcal{P}$ be the collection of interior plates (plates not touching a boundary curve). Let $\mathcal{P}_{\text{inner}}$ and $\mathcal{P}_{\text{outer}}$ be the plates touching the inner and outer boundary respectively. A plate may be in both $\mathcal{P}_{\text{inner}}$ and $\mathcal{P}_{\text{outer}}$. Let $\mathcal{I}_{\text{inner}}$ and $\mathcal{I}_{\text{outer}}$ be the boundary intervals on the inner and outer boundaries respectively.

Together, $S$, $\mathcal{P}$, $\mathcal{J}$, and $\mathcal{I}$ form an annular plate network $\Gamma$. Of course, we can define plate networks on any planar region bounded by disjoint simple closed curves. In particular, we will sometimes need to consider networks in a simply connected region with one boundary curve. We will call them circular planar in accordance with standard terminology, even if the boundary curve is not strictly speaking a circle. For circular planar networks, it is reasonable
to assume each plate only intersects the boundary in one interval, although often we will allow two boundary intervals.

1.2 Conductivity Functions, Feasible Boundary Data, and Recovery

A conductivity function is a function $\gamma$ which assigns a positive number, called the conductance, to each juncture, such that $\gamma(PQ) = \gamma(QP)$. We may think of $\gamma$ as defined on undirected or directed junctures. If $P$ and $Q$ are not adjacent, we say $\gamma(PQ) = 0$.

A voltage function is a function $v : P \to \mathbb{R}$; the value assigned to each plate is a voltage. A current function is a function $c : J \cup I \to \mathbb{R}$. An electrical function $f = (v, c)$ consists of a voltage and a current function.
such that for every juncture $P \to Q$,

$$c(P \to Q) = \gamma(PQ)(v(P) - v(Q)).$$

An electrical function is called $\gamma$-harmonic if for every plate $P$,

$$\sum_{Q \sim P} c(Q \to P) + \sum_{I \in \mathcal{I}(P)} c(I) = 0.$$

For a boundary interval, positive current means current flowing into the plate $P$. If $f_1$ and $f_2$ are $\gamma$-harmonic, then so is $af_1 + bf_2$ for any $a, b \in \mathbb{R}$.

Suppose $\partial\mathcal{P} = \{P_1, \ldots, P_K\}$ and $\mathcal{I} = \mathcal{I}_1, \ldots, \mathcal{I}_L$. A vector $\mathbf{x} \in \mathbb{R}^{K+L}$ is called feasible boundary data if there exists a $\gamma$-harmonic function with $v(P_k) = x_k$ for each $k$ and $c(I_\ell) = x_{K+\ell}$ for each $\ell$. The space of all vectors which are feasible boundary data will be denoted $F$.

Suppose we are given a network $\Gamma$ with a fixed conductivity function $\gamma$. We do not know $\gamma$, and we want to determine $\gamma$ knowing only $F$. If $\gamma$ can be determined from $F$, then $\gamma$ is called recoverable. If any conductivity function $\gamma$ can be recovered, we say the network $\Gamma$ is recoverable. The problem of recovering the conductances of $\Gamma$ is called the inverse problem.

1.3 Paths and Connections

A path is a sequence of plates $P_1, \ldots, P_N$ such that $P_n \sim P_{n+1}$; it may equivalently be viewed as a sequence of junctures. A 1-connection $\alpha$ between two boundary cells $q_1$ and $q_2$ is a path $P_1, \ldots, P_N$ with $P_1 = q_1$, $P_N = q_2$, and all other $P_n$ interior cells, such that $P_i \neq P_j$ for all $i \neq j$. A path with one boundary cell $q$ is considered a connection from $q$ to $q$. A $k$-connection is a set of $k$ disjoint 1-connections.

A network is connected if there exists a path between any two plates. Here we do not assume the network is connected. However, we assume that for every plate $P$, there exists a path from $P$ to a boundary plate.

Suppose $\mathcal{U}$ and $\mathcal{V}$ are subsets of $\partial\mathcal{P}$. Let $M(\mathcal{U}, \mathcal{V})$ signify the largest $k$ such that there is a $k$-connection between $\mathcal{U}$ and $\mathcal{V}$.

1.4 Subnetworks

Suppose $U$ is an open proper subset of the region of embedding, such that $\partial U$ consists of one or two nonintersecting, piecewise-smooth simple closed curves. Suppose no junctures of $\Gamma$ lie on $\partial U$. Suppose that each plate $P$ of $\Gamma$ is either contained in $U$, it is contained in $\overline{S \setminus U}$, or $\partial U$ divides $P$ into two
or more smaller regions (called subplates). Suppose that for each subplate $Q$, $\partial U \cap Q$ is either empty or an interval of $Q$.

Then we can form a plate network $\Gamma'$ with region of embedding $U$, whose plates are the plates and subplates of $\Gamma$ contained in $\overline{U}$. $\Gamma'$ is called a subnetwork of $\Gamma$. A layer of $\Gamma$ is subnetwork $\Gamma'$ such that

- the region of embedding $U$ is bounded by two curves;
- $C_{\text{inner}}(\Gamma)$ lies inside $C_{\text{inner}}(\Gamma')$;
- either $C_{\text{inner}}(\Gamma')$ and $C_{\text{inner}}(\Gamma)$ are the same or they do not intersect;
- either $C_{\text{outer}}(\Gamma')$ and $C_{\text{outer}}(\Gamma)$ are the same or they do not intersect.

A partition of $\Gamma$ into subnetworks is a collection of subnetworks $\Gamma_1, \ldots, \Gamma_K$ with regions of embedding $S_1, \ldots, S_K$, such that $S_i \cap S_j = \emptyset$ for $i \neq j$ and $\bigcup S_k = \overline{S}$.

**Theorem 1.1.** Let $\Gamma_1, \ldots, \Gamma_K$ be a partition of $\Gamma$ with feasible data sets $F_1, \ldots, F_K$. $F$ of $\Gamma$ can be determined from $F_1, \ldots, F_K$. 
Proof. Any partition can be expressed in terms of partitions and subpartitions into two parts. Thus, by induction, it suffices to consider the case where $K = 2$.

Suppose $\Gamma$ is partitioned into $\Gamma_1$ and $\Gamma_2$. Let $C$ be the curve or union of curves $\partial S_1 \cap \partial S_2$. Let $P_1, \ldots, P_N$ be the plates of $\Gamma_1$ touching $C$ and let $Q_1, \ldots, Q_N$ be the plates of $\Gamma_2$ touching $C$. Let $I_1, \ldots, I_M$ and $J_1, \ldots, J_M$ be the sets of boundary intervals of $\Gamma_1$ and $\Gamma_2$ which are subsets of $C$. (If $P_n$ and $Q_n$ form a boundary plate of $\Gamma$, then their boundary intervals in $\Gamma_1$ and $\Gamma_2$ are not included in $\{I_m\}$ or $\{J_m\}$.)

Let vectors $x_1$ and $x_2$ represent boundary data on $\Gamma_1$ and $\Gamma_2$. Let $g_1(x_1)$ be the restriction of $x_1$ to $P_1, \ldots, P_N$ and $I_1, \ldots, I_M$ (We can think of restriction as a function). Define $g_2(x_2)$ be function which restricts $x_2$ to $Q_1, \ldots, Q_N$ and $J_1, \ldots, J_M$ and in addition changes the signs of the current entries. Define $h : F_1 \times F_2 \to \mathbb{R}^{N+M}$ by

$$h(x_1, x_2) = g_1(x_1) - g_2(x_2).$$

If $h(x_1, x_2) = 0$, the voltage on $P_n$ is equal to the voltage on $Q_n$. If $P_n$ and $Q_n$ form an interior plate of $\Gamma$, then the net current on the plate $P_n \cup Q_n$ is zero. We assumed that $x_1$ and $x_2$ were boundary data for some $\gamma$-harmonic functions $f_1$ and $f_2$ on $\Gamma_1$ and $\Gamma_2$. We can combine $f_1$ and $f_2$ into a $\gamma$-harmonic function $f$ on $\Gamma$. Of course, if $h(x_1, x_2) \neq 0$, we cannot find such a $\gamma$-harmonic function on $\Gamma$.

We can compute the boundary data for $f$ from $x_1$ and $x_2$. For plates other than $P_n$ and $Q_n$, this is obvious. Suppose $P_n$ and $Q_n$ form a boundary plate of $\Gamma$ with boundary interval $I$. It is obvious how to find the voltage of $P_n \cup Q_n$. The current on $I$ can be found from the boundary currents on $P_n$ and $Q_n$. Hence, we can define a function $w$ “restricting” $x_1$ and $x_2$ to the boundary of $\Gamma$. Then $F = w(h^{-1}(0))$. \hfill $\Box$

**Theorem 1.2.** If a subnetwork of $\Gamma$ is not recoverable, then $\Gamma$ is not recoverable.

Proof. Suppose $\Gamma$ has a nonrecoverable subnetwork. Let $\Gamma_1, \ldots, \Gamma_K$ be a partition of $\Gamma$ where $\Gamma_1$ is not recoverable. Let $g$ be a function mapping the sets $F_1, \ldots, F_K$ to the set $F$; this function exists by the previous theorem. Let $L$ be the function mapping $\gamma$ to $F$. $\Gamma$ is recoverable if and only if $L$ is injective. Let $\gamma_k$ be the restriction of $\gamma$ to the junctures of $\Gamma_k$, and let $L_k(\gamma_k)$ be the function mapping $\gamma_k$ to $F_k$. Then

$$L(\gamma) = g(L_1(\gamma_1), \ldots, L_K(\gamma_K)).$$
Since $\Gamma_1$ is not recoverable, we know $L_1$ is not injective, and so $L$ is not injective either.

1.5 Electrical Similarity and Equivalence

Two networks $\Gamma$ and $\Gamma'$ with conductivity functions $\gamma$ and $\gamma'$ are *electrically similar* if there is a one-to-one correspondence $M$ mapping each boundary plate or boundary interval of $\Gamma$ to a boundary plate or boundary interval of $\Gamma'$ such that

- $P$ is an inner/outer boundary plate of $\Gamma$ if and only if $M(P)$ is an inner/outer boundary plate of $\Gamma'$;
- $I$ is an inner/outer boundary interval of $\Gamma$ if and only if $M(I)$ is an inner/outer boundary interval of $\Gamma'$;
- $I$ is a boundary interval of $P$ if and only if $M(I)$ is a boundary interval of $M(P)$;
- $M$ preserves the counterclockwise ordering of the inner/outer boundary plates;
- The set $F$ is the same for $\Gamma, \gamma$ and $\Gamma', \gamma'$.

They are called *electrically equivalent* if in addition

- $\Gamma$ and $\Gamma'$ have the same region of embedding.
- For each boundary interval $I$, $M(I)$ and $I$ are the same curve.

The definitions of electrical similarity and equivalence for circular planar networks are similar except that there is only one boundary curve.

Electrical similarity and equivalence are transitive. As a consequence of Theorem 1.1,

**Theorem 1.3.** Let $\Gamma_1, \ldots, \Gamma_K$ and $\Gamma'_1, \ldots, \Gamma'_K$ be a partitions of $\Gamma$ and $\Gamma'$ into subnetworks. If $\Gamma_k$ and $\Gamma'_k$ are electrically equivalent for each $k$, then $\Gamma$ and $\Gamma'$ are electrically equivalent.

1.6 Comparison to Graph-Based Networks

The plate-based networks described here are similar to the graph-based electrical networks discussed by [1] and others. In a planar vertex-based network (and in particular, an annular planar network), we can construct the medial
Figure 3: A plate network and equivalent graph-based network.

graph and color the cells white and black, such that each black cell contains a vertex of the primal graph. The plates described here correspond to the black cells.

There are important differences between the two constructions:

- A plate can touch both boundary curves, but a vertex must lie on one or the other.
- Unlike a vertex, a plate can have multiple boundary currents.
- We discuss trivial connections: A plate is considered to be connected to itself.
- We do not make the usual assumption that the network is connected, only that each plate has a path to the boundary.

With these changes, we will be able to consider subnetworks which would be too small to make sense in the graph-based system.

Graph-based and plate-based networks are algebraically equivalent when a plate does not touch both boundaries. Thus, many of the results about graph-based networks for [1] and others carry over to plate-based networks.

For instance, we know that the Dirichlet problem has a nearly unique solution. That is, for given voltages on $\partial \mathcal{P}$, the voltages on the network are uniquely determined. Boundary currents for each plate are uniquely determined except when the plate touches both boundary curves, in which case, the sum of its two boundary currents is uniquely determined. Similarly, the
Neumann problem has a unique solution for plate networks, that is, for every set of boundary currents which sum to zero on each connected component of the network, there is a γ-harmonic function with those boundary currents, which is unique up to an additive constant.

As with graph-based networks, we can discuss the Kirchhoff matrix: if each plate is assigned an index, then we define a $|P| \times |P|$ matrix $K$ by

$$
\kappa_{PQ} = \begin{cases} 
-\gamma(PQ), & P \neq Q \\
\sum_{R \sim P} \gamma(PR), & P = Q.
\end{cases}
$$

The response matrix $\Lambda$ is the Dirichlet-to-Neumann map, that is, if $\phi$ is a vector representing boundary voltages, then $\Lambda \phi$ is a vector representing (sums of) boundary currents for the γ-harmonic function with boundary voltages $\phi$. $\Lambda$ is given by a Schur complement of $K$. The set $F$ is equivalent information to $\Lambda$.

Minors of $\Lambda$ are related to connections in the graph by the determinant-connection formula (Lemma 3.12 of [1]). If $\mathcal{U}$ and $\mathcal{V}$ are disjoint sets of boundary plates $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$, and $\alpha$ is a $k$-connection between $\mathcal{U}$ and $\mathcal{V}$, then $\tau_\alpha$ is the permutation of the symmetric group $S_k$ such that a 1-connection in $\alpha$ connects $P_n$ and $Q_{\tau(n)}$ for each $n$. We define $\mathcal{W}_\alpha$ as the collection of plates which are not used in any 1-connection of $\alpha$ and let $D_\alpha = \det K(\mathcal{W}_\alpha; \mathcal{W}_\alpha)$. The determinant-connection formula says that

$$
\det \Lambda(\mathcal{U}; \mathcal{V}) \cdot \det K(\text{int } \mathcal{P}; \text{int } \mathcal{P}) = (-1)^k \sum_{\tau \in S_k} \text{sgn}(\tau) \sum_{\tau_\alpha = \tau} \prod_{PQ \in \mathcal{J}_\alpha} \gamma(PQ) D_\alpha,
$$

where the second sum is taken over $k$-connections which exist between $\mathcal{U}$ and $\mathcal{V}$.

2 Geodesics and Network Modifications

2.1 Geodesics and Lenses

Suppose $\partial P$ is the boundary of a plate $P$ and $T$ is the union of the boundary intervals. The junctures of $P$ divide $\partial P \setminus T$ into smaller curves, called edges. At a juncture $y$, four edges meet (two edges from each of two plates). If these edges are $e_A, e_B, e_C,$ and $e_D$ in counterclockwise order about $y$, then $e_A$ is opposite $e_C$ and $e_B$ opposite $e_D$. Two edges are adjacent if they share a juncture.

Suppose that $e_1, e_2, \ldots, e_K$ is a sequence of edges such that $e_i \neq e_j$ for all $i \neq j$ and $e_k$ is opposite $e_{k+1}$ for all $k$. If either
Figure 4: Some geodesics.
The intersection between two geodesics is also called a crossing. There are three types of geodesics:

- Type 0 geodesics are self-loops with no endpoints on a boundary curve.

- Type 1 geodesics have both endpoints on one boundary curve.

- Type 2 geodesics have one endpoint on each boundary curve.

$G_0$, $G_1$, and $G_2$ signify the collections of type 0, type 1, or type 2 geodesics. Type 1 geodesics are further divided into type inner ($G_{inner}$) and type 1 outer ($G_{outer}$) according to where their endpoints are.

When parametrizing a geodesic, we assume the following: A type 1 or type 2 geodesic $g$ can be parametrized by a continuous function $\phi : [0, 1] \to \mathbb{C}$. Assume $\phi(0)$ and $\phi(1)$ are the endpoints of $g$ and $\phi$ is injective except at self-intersections of $g$. When parametrizing a type 0 geodesic, $\phi(0) = \phi(1)$ and $\phi$ is injective on $[0, 1)$ except at self-intersections of $g$.

Suppose $e_1, \ldots, e_K$ is a sequence of edges with $e_i \neq e_j$ for each $i \neq j$, $e_k$ and $e_{k+1}$ are adjacent for each $k$, and $e_1$ is adjacent to $e_K$. The curve $e_1 \cup \cdots \cup e_K$ is called a lens if one of the following conditions is satisfied:

- In a zero-pole lens, each $e_k$ is opposite $e_{k+1}$ and $e_1$ is opposite $e_k$.

- In a one-pole lens, each $e_k$ is opposite $e_{k+1}$, but $e_1$ and $e_k$ are not opposite. The juncture between $e_1$ and $e_K$ is the pole of the lens.
In a two-pole lens, each $e_k$ is opposite $e_{k+1}$ for $k \neq J$; $e_1$, $e_K$ and $e_J$, $e_{J+1}$ are not opposite. The junctures between $e_1$, $e_K$ and $e_J$, $e_{J+1}$ are the poles.

A lens is a closed curve formed by arcs of one or two geodesics.

2.2 \textit{Y-∆} Transformations

A \textit{wye} (or \textit{Y}) is a circular planar network with four plates, $P_0$, $P_1$, $P_2$, $P_3$ such that the boundary plates are $P_1$, $P_2$, and $P_3$, and the junctures are $P_0P_1$, $P_0P_2$, and $P_0P_3$. A \textit{delta} (or \textit{∆}) is a circular planar network with three plates $Q_1$, $Q_2$, $Q_3$; all the plates are boundary plates, and all the plates are adjacent.

Given a wye, it is always possible to find an electrically equivalent delta and vice versa. Suppose that in the wye, $\gamma(P_0P_1) = a$, $\gamma(P_0P_2) = b$, and $\gamma(P_0P_3) = c$, and that in the delta, $\gamma(Q_2Q_3) = a'$, $\gamma(Q_1Q_3) = b'$, and $\gamma(Q_1Q_2) = c'$. Then the wye and the delta are electrically equivalent (with
Figure 7: A wye and a delta.

\[ M(P_j) = Q_j \text{ for } j = 1, 2, 3 \] if and only if

\[ a' = \frac{bc}{a + b + c}, \quad b' = \frac{ac}{a + b + c}, \quad c' = \frac{ab}{a + b + c} \]

if and only if

\[ a = \frac{a'b' + b'c' + a'c'}{a'}, \quad b = \frac{a'b' + b'c' + a'c'}{b'}, \quad c = \frac{a'b' + b'c' + a'c'}{c'}. \]

Suppose \( \Gamma \) is a network with a subnetwork \( \Sigma \) which is a wye. Let \( \Gamma' \) be a network obtained from \( \Gamma \) by replacing \( \Sigma \) with a delta \( \Sigma' \). The modifications changing \( \Gamma \) to \( \Gamma' \) and \( \Gamma' \) to \( \Gamma \) are called \( Y\-\Delta \) transformations.

A \( Y\-\Delta \) transformation may produce a network which does not fit our original definition of a plate network because multiple junctures join the same two plates or a plate has a self-juncture. For the purposes of this section, we extend our definition to allow such networks.

Two networks are \( Y\-\Delta \)-equivalent if one can be transformed into the other by \( Y\-\Delta \) transformations. If \( \Gamma \) and \( \Gamma' \) are \( Y\-\Delta \) equivalent and we are given the conductivity function \( \gamma \), we can compute \( \gamma' \). As a result, \( \Gamma \) is recoverable if and only if \( \Gamma' \) is recoverable.

A \( Y\-\Delta \) transformation alters the geodesics by changing the order in which they intersect one another. In a \( Y \) or \( \Delta \) subnetwork, three geodesics meet; call them \( g_1, g_2, g_3 \). A \( Y\-\Delta \) transformation moves the crossing of \( g_1 \) and \( g_2 \) to the other side of \( g_3 \).

### 2.3 Juncture Removals and Trivial Modifications

A single-juncture network is a circular planar (sub)network with two plates, and one juncture, and one boundary interval on each juncture.
A juncture deletion removes a juncture from the network by replacing a single-juncture subnetwork with a network with two plates, one boundary interval on each plate, and no juncture. A juncture contraction replaces a single-juncture subnetwork with a network with one plate which has two boundary intervals and no junctures. Both these transformations are called juncture removals.

If two geodesics meet a juncture, then a juncture removal uncrosses them. If the geodesics $g_1$ and $g_2$ in the original network had endpoints $x_1$ and $y_1$, $x_2$ and $y_2$ respectively, then the geodesics in the modified network have endpoints $x_1$ and $y_2$, $x_2$ and $y_1$.

If a plate has a self-juncture, no current can ever flow across the juncture. Thus, changing the conductance of the juncture will not affect $F$, so the network is not recoverable. Deleting the juncture while keeping all other conductances the same will produce an electrically equivalent network.

An interior plate with only one juncture is called an interior spike. No current can flow across the juncture, so the network is not recoverable, and contracting this juncture (or contracting the spike) will produce an electrically equivalent network.

A parallel network is circular planar network in which there are two plates, both of which are boundary plates, and two junctures between the plates. A parallel network with conductances $a$ and $b$ is electrically equivalent to a network with only one juncture, with conductance $a + b$. A parallel network is not recoverable because any conductances $a'$ and $b'$ with $a' + b' = a + b$ will produce the same $F$.

A series network is a circular planar network with three plates, $P_0$, $P_1$, ...
and $P_2$, where $P_1$ and $P_2$ are boundary plates, and there are two junctures $P_0P_1$ and $P_0P_2$. A series network with conductances $a$ and $b$ is electrically equivalent to a network with only two plates and one juncture, with conductance $ab/(a+b)$. A parallel network is not recoverable because any conductances $a'$ and $b'$ with $a'b'/(a'+b') = ab/(a+b)$ produce the same $F$.

Trivial modifications are the following network transformations:

- Deleting a self-juncture.
- Contracting an interior spike.
- Replacing a parallel with a single-juncture subnetwork.
- Replacing a series with a single-juncture subnetwork.

Self-junctures and interior spikes correspond to empty one-pole lenses. Parallel and series connections correspond to empty two-pole lenses. A trivial modification removes the lens.

A lens is called removable if it can removed from the network by $Y$-$\Delta$ transformations and trivial modifications. Any network on which we can perform a trivial modification is unrecoverable (in fact, the inverse problem has infinitely many solutions). Since $Y$-$\Delta$ transformations preserve recoverability properties, we know that any network with removable lenses is unrecoverable.

2.4 Lens Removal I

Because removable lenses make a network unrecoverable, we want to determine what kinds of lenses are removable. We begin with the easiest case. A lens is called simply connected if it is contained within some simply connected subset of the region of embedding.

Theorem 2.1. Every simply connected lens is removable.

Proof. A simply connected lens is contained within some subnetwork in a simply connected region. Theorem 8.3 of [1] shows that all lenses can be removed from a circular planar network. The proof is similar but simpler than the later lens removal arguments of this paper. □

1 An empty zero-pole lens would either be an interior plate with no junctures or a junctureless hole in a plate. We assume that these configurations do not exist in the original network, and we know they cannot be produced by $Y$-$\Delta$ transformations.
Figure 9: Trivial modifications.
**Lemma 2.2.** A self-intersecting type 2 geodesic forms a simply connected lens.

*Proof.* We will use the universal cover of the annulus, which is a strip extending infinitely to the left (clockwise) and right (counterclockwise). For a geodesic $g$ or region $R$ in the annulus, we use choose one “copy” in the universal cover to be $g[0]$ or $R[0]$, and we index the other “copies” from left to right by the integers as $g[n]$ or $R[n]$.

Suppose $g$ is a self-intersecting type 2 geodesic with parametrization $\phi$ such that $\phi(0)$ is on the inner boundary and $\phi(1)$ is on the outer boundary. Let $\phi_0$ be the corresponding parametrization of $g[0]$ in the universal cover. Choose the smallest $t_1$ such that $\phi([0,t_1])$ intersects itself and the $t_0 \in (0,t_1)$ such that $\phi(t_0) = \phi(t_1)$. Let $A = \phi([0,t_0])$ and $B = \phi([t_0,t_1])$. Let $A[0]$ and $B[0]$ be the corresponding arcs of $g[0]$ in the universal cover.

If the self-loop $\phi([t_0,t_1])$ does not encircle the hole, it is a simply connected lens. Suppose $\phi([t_0,t_1])$ encircles the hole, and assume without loss of generality that it is counterclockwise. In the universal cover, $A[0]$, $B[0]$, and $A[1]$ together with the inner boundary enclose a region $S[0]$. For small positive $\epsilon$, $\phi_0(t_1 + \epsilon) \in S[1]$. To reach the outer boundary, $\phi_0$ must exit $S[1]$, but it will form a simply connected lens unless it exits along $A[2]$, entering $S[2]$. Continuing inductively, we see $\phi_0$ must enter $S[n]$ for all positive $n$, which is a contradiction. 

**Lemma 2.3.** If a type 1 inner and a type 1 outer geodesic intersect, they form a simply connected lens.

*Proof.* Suppose $g_1 \in G_{\text{inner}}$ with parametrization $\phi$ and $g_2 \in G_{\text{outer}}$ with parametrization $\psi$ intersect. Let $t^*$ be the first time $\psi(t)$ intersects $g_1$ and let $u^*$ be the value such that $\phi(u^*) = \psi(t^*)$.

Since $\psi([0,t^*])$ does not intersect $g_2$, we can assume without affecting the lenses formed by $g_1$ and $g_2$ that $\psi([0,t^*])$ does not intersect itself. If
\(\phi([0, u^*])\) intersects itself, we can join \(\phi([0, u^*])\) and \(\phi([0, t^*])\) into a self-intersecting curve from the inner boundary to the outer boundary, which must have a simply connected lens by the previous lemma. No pole of this lens can lie on \(\psi([0, t^*])\) by assumption, so \(g_1\) forms a simply connected lens. Therefore, suppose \(\phi([0, u^*])\) does not intersect itself.

If \(g_1\) is not self-intersecting, the proof is easy, so assume \(g_1\) is self-intersecting. Choose the smallest \(u_1\) such that \(\phi([0, u_1])\) is self-intersecting.

Assume the loop winds counterclockwise; the other case is similar. Let \(u_0\) be the number in \([0, u_0]\) with \(\phi(u_0) = \phi(u_1)\). Let \(A = \phi_1([0, u_0])\) and \(B = \phi_2([u_0, u_1])\). In the universal cover, \(A[0], B[0],\) and \(A[1]\) enclose a region \(S[0]\). For small positive \(\epsilon\), \(\psi_0(t^* + \epsilon)\). As in the previous lemma, \(\psi_0\) must eventually hit the outer boundary, but cannot exit \(S[n]\) without forming a simply connected lens except by passing across \(A[n + 1]\) into \(S[n + 1]\).

**Lemma 2.4.** If two type 2 geodesics \(g\) and \(h\) intersect without forming a simply connected lens, then \(g\) always crosses \(h\) in the same direction (always counterclockwise or always clockwise).

**Proof.** Suppose \(g[0]\) crosses \(h[0]\) counterclockwise; the other case is similar. Let \(S[n]\) be the fundamental domain between \(h[n]\) and \(h[n + 1]\). After entering \(S[0]\), \(g[0]\) cannot exit along \(h[0]\) without forming a simply connected lens. Thus, it must either reach the outer boundary from \(S[0]\) or enter \(S[1]\). By the same argument, if \(g[0]\) enters \(S[n]\), it cannot exit clockwise across \(h[n]\). Eventually, \(g[0]\) reaches the outer boundary, and it has never crossed any \(h[n]\) clockwise.

**Definition 2.5.** Let \(i(g)\) be the number of self-intersections of a geodesic or curve \(g\). Let \(i(g_1, g_2)\) for \(g_1 \neq g_2\) be the number of intersections between \(g_1\) and \(g_2\).

**Definition 2.6.** A type 0 geodesic can be parametrized in two directions and has a well-defined winding number around the hole for each one. Let \(w(g)\) be the nonnegative winding number.

**Lemma 2.7.** If a type 0 geodesic \(g\) does not form a simply connected lens, then \(w(g) = i(g) + 1\).

**Proof.** Let \(x\) be a point on \(g\) such that two curves \(C_1\) and \(C_1'\) connect the inner boundary to \(x\) without intersecting \(g\) or each other except at \(x\). Let \(y \neq x\) be a point on \(g\) such that two curves \(C_2\) and \(C_2'\) connect the outer boundary to \(x\) without intersecting \(g\) or each other except at \(y\). Parametrize \(g\) by a function \(\phi\) with \(\phi(0) = \phi(1) = x\). Let \(t_0 = \phi^{-1}(y)\) and let \(h = \)
\[ \phi([0, t_0]) \text{ and } h' = \phi([t_0, 1]). \]  
\[ C_1, C_1', h, \text{ and } h' \text{ meet at } x. \]  
Assume without loss of generality that \( C_1 \) is opposite \( h \) and \( C_1' \) is opposite \( h' \) at \( x \). Assume also that at \( y \), \( C_2 \) is opposite \( h \) and \( C_2' \) is opposite \( h' \).

Let \( C = C_1 \cup h \cup C_2 \) and \( C' = C_1' \cup h' \cup C_2' \). If \( C \) and \( C' \) form a simply connected lens, then so does \( g \). This is obvious if the poles of the lens are not \( x \) or \( y \). If \( x \) or \( y \) is the pole of a lens of \( C \) and \( C' \) and the other pole is not \( y \) or \( x \), then there is a simply connected one-pole lens in \( g \). If \( x \) and \( y \) are the poles of a simply connected lens, then \( g \) forms a simply connected zero-pole lens.

Suppose \( g \) does not form a simply connected lens; then neither do \( C \) and \( C' \). By Lemma 2.2, \( C \) and \( C' \) do not have self-intersections, so \( i(g) = i(C, C') - 2 \). By Lemma 2.4, one of \( C \) and \( C' \) always crosses the other counterclockwise, so \( w(g) = i(C, C') - 1 \). \( \square \)
Before removing other types of lenses, we must discuss empty boundary triangles and stubs.

2.5 Empty Boundary Triangles and Stubs

An empty boundary triangle is a triangle formed by two edges of plates and an interval of a boundary curve. For an empty boundary triangle, there are two possibilities:

1. The triangle is a plate. In this case, the plate is called a boundary spike.

2. The triangle is adjacent to two plates. Then, the juncture at the vertex of the triangle is called a boundary juncture.

A type 1 geodesic is called empty if it does not intersect any other geodesics. A stub is a boundary plate which touches only one boundary curve and is not adjacent to any other plates; its edge is an empty geodesic. Since we assumed that a plate intersects the each boundary curve in at most one interval, there cannot be an empty geodesic which is not the edge of a stub.

Here we prove the existence of an empty boundary triangles or stubs in certain families of geodesics, which is an essential step for the rest of this paper’s arguments.

Lemma 2.8. Suppose $g$ be a type 1 inner geodesic which does not form a simply connected lens. Suppose that at a juncture point $y$, two opposite edges $e_1$ and $e_2$ are in $g$ and another edge $e_3$ touches the inner boundary. Then $e_3$ forms a triangle with some arc of $g$ and some arc of the inner boundary.

Proof. The juncture point $y$ splits $g$ into two segments (possibly intersecting). Choose a curve $C$ which begins at the outer boundary and ends at a non-juncture point $z$ on $g$ such that $C$ only intersects $g$ once. Parametrize $g$ by $\phi$ such that $\phi(t_0) = y$ and $\phi(t_1) = z$ with $t_0 < t_1$. Then $C \cup \phi([0,t_1])$ forms a curve from the inner boundary to the outer boundary. By Lemma 2.2, the curve cannot intersect itself without forming a simply connected lens, but we know no pole of such a lens can lie on $C$. Hence, $\phi([0,t_1])$ does not intersect itself, so $\phi([0,t_0])$ and $e_3$ form a triangle with some arc of the boundary.

Definition 2.9. A family of geodesics $F$ is connected if

- For any two points $x$ on $g \in F$ and $y$ on $h \in F$, there is a path from $x$ to $y$ along arcs of geodesics in $F$. 

• If $g$ and $h$ intersect and $g \in \mathcal{F}$, then $h \in \mathcal{F}$.

**Theorem 2.10.** Suppose $\mathcal{F}$ is a family of type 1 inner and type 2 geodesics with no simply connected lenses and at least one intersection or at least one type 1 inner geodesic. There is an empty boundary triangle or empty geodesic on the inner boundary.

**Proof.** First, consider the case where the family of geodesics is connected.

Let $g_0$ be a geodesic which intersects some other geodesic. By hypothesis, $g_0$ has one endpoint $x_0$ on the inner boundary. Orient $g_0$ so that the positive direction moves from $x$ to $g_0$’s other endpoint. Let $y_0$ be the first juncture along $g_0$, let $g_1$ be the other geodesic at $y_0$, and let $\hat{x}_0y_0$ be the open arc of $g_0$ from $x_0$ to $y_0$. If $g_1$ is type 2, it cannot intersect itself, and so it must form a triangle on the inner boundary with $g_0$. If $g_1$ is type 1, then by the previous lemma, it forms a triangle with $g_0$.

In either case, let $T_0$ be the triangle, let $x_1$ be an endpoint of $g_1$ on the inner boundary which is the vertex of the triangle, and let $x_1y_0$ be the open arc of $g_1$ from $x_1$ to $y_0$.

If $T_0$ is not empty, let $y_1$ be the first intersection point along $g_1$, and let $g_2$ be the other geodesic intersecting $g_1$. There is an arc $s_2$ of $g_2$ which lies inside $T_0$ and has both endpoints on the boundary of $T_1$. Since $s_2$ cannot intersect $x_0y_0$ and it cannot intersect $x_1y_0$ more than once without forming a simply connected lens, $s_2$ must have its other endpoint on $R_0$. Letting $x_2y_1 = s_2$, we have a triangle $T_1$ formed by $x_2y_1$, $x_1y_1$, and an arc of the boundary curve $R_1$, and $T_1 \subseteq T_0$.

If $T_1$ is not an empty boundary triangle, repeat the above construction to find $T_2, T_3, \ldots$. There are only finitely many junctures, so eventually $T_n$ will be an empty boundary triangle.

Now consider the case where there are multiple connected families of geodesics. Since one them must have an intersection or type 1 inner geodesic, one of them has a boundary triangle which is empty with respect to other geodesics in that family. If the triangle is not completely empty, then it contains some other connected family of geodesics (which must all be type 1). In that case, we can repeat the argument. We will eventually reach an empty boundary triangle or empty geodesic.

**Corollary 2.11.** Suppose $\mathcal{F}$ is a lensless family of geodesics in a simply connected region with at least one crossing. Suppose $R$ is an arc of the boundary curve such that every geodesic has an endpoint on $R$. Then there is an empty boundary triangle or empty geodesic on $R$.
Proof. Suppose $g$ and $h$ intersect. Then an arc of $g$, an arc of $h$, and an arc of $R$ form a triangle. By the previous argument, this triangle must contain an empty boundary triangle or empty geodesic.

**Corollary 2.12.** Suppose $F$ is a family of type 1 and type 2 geodesics with no simply connected lenses at least one type 1 inner geodesic. There is an empty boundary triangle or empty geodesic on the inner boundary.

**Proof.** By Lemma 2.3, we know the type 1 inner and type 1 outer geodesics do not intersect. We can construct a curve which partitions the network into two layers, one of which contains all the type 1 inner geodesics. Then apply the lemma to the subnetwork.

### 2.6 Lens Removal II

**Definition 2.13.** For a point $x$, let $w(x)$ be the sum of the winding numbers about $x$ of all type 0 geodesics, parametrized counterclockwise.

**Lemma 2.14.** Suppose a type 2 geodesic $g$ is parametrized by $\phi$ with $\phi(0)$ on the inner boundary and $\phi(1)$ on the outer boundary. If there are no simply connected lenses, then $w(\phi(t))$ is weakly decreasing.

**Proof.** Consider the case with only one type 0 geodesic $h$. Let $t_0$ and $t_1$ be the first and last times $\phi$ crosses $h$; let $x = \phi(t_0)$ and $y = \phi(t_1)$. Then $h$ can be broken into two curves $A$ and $B$ which each begin at $x$ and end at $y$, parametrized by $\psi_A$ and $\psi_B$. By the argument for Lemma 2.7, we know $A$ and $B$ are not self-intersecting and by the argument of Lemma 2.4, we know that one of them, say $\psi_A$, always crosses $g$ counterclockwise and the other, $\psi_B$, always crosses $g$ clockwise. Orienting $h$ counterclockwise gives the same orientation as $\psi_A$ and the opposite orientation from $\psi_B$, which implies $h$ always crosses $g$ counterclockwise. Hence, $h$ always crosses $g$ counterclockwise, which implies $w(\phi(t))$ is weakly decreasing.

If there are several type 0 geodesics $h_1, \ldots, h_N$, then $w(\phi(t))$ is the sum of weakly decreasing functions $w_n(\phi(t))$, where $w_n(x)$ is the winding number of $h_n$ about $x$.

**Definition 2.15.** Let $A$ be a continuous oriented curve consisting of oriented arcs $A_1, \ldots, A_N$ of type 0 geodesics $h_1, \ldots, h_N$. We say $A$ is a counterclockwise (respectively clockwise) curve if the orientation of each $A_n$ matches the counterclockwise (respectively clockwise) orientation of $h_n$.

**Lemma 2.16.** If a type 1 and type 0 geodesic intersect, there is a removable lens.
Proof. Assume there are no simply connected lenses. Suppose a type 1 inner geodesic intersects a type 0 (the other case is similar). The type 0 geodesics $h_0, \ldots, h_N$ divide the region of embedding into simply connected or annular subregions $S_1, \ldots, S_K$. Order the subregions by weakly decreasing $w(S_k)$ ("innermost to outermost"). $S_0$ is the region touching the inner boundary, and $S_K$ is the region touching the outer boundary.

For $0 < k < K$, I claim that $\partial S_k$ oriented positively can be partitioned into a counterclockwise curve and a clockwise curve. Consider uncrossing all the type 0 geodesics without changing the orientation of any arc. At each intersection $x$, four edges $e_1, e_2, e_3, e_4$ meet. If $e_1$ comes before $e_2$ in counterclockwise order along one geodesic, and $e_3$ comes before $e_4$ on the other, then uncross to join $e_1$ with $e_4$ and $e_2$ with $e_3$. When all crossings are removed, the type 0 geodesics are nonintersecting simple closed curves which each wind around the hole once. In the modified network, the claim is clearly true, (although the subregions have changed), and the claim remains true when we reverse each uncrossing.

Let $g_1, \ldots, g_M$ be the type 1 inner geodesics which intersect type 0 geodesics. Let $J$ be the first positive number such that $\bigcup_{k=0}^J S_k$ fully contains some $g_m$. We will show that all crossings of type 1 and type 2 geodesics can be removed from $S_1, S_2, \ldots, S_{J-1}$.

If $1 < J$, divide $\partial S_1$ into a clockwise curve $A$ and a counterclockwise curve $B$. All geodesics which enter $S_1$ across $A$ must exit across $B$. Consider a subnetwork $\Sigma_1$ whose region of embedding lies inside $S_1$ and which contains all the junctures inside $S_1$. The boundary of $\Sigma_1$ consists of two curves $A_1'$ and $B_1'$ which cross the same geodesics as $A_1$ and $B_1$ respectively. By either Lemma 2.10 or Corollary 2.11, $\Sigma_1$ has an empty boundary triangle along $A_1'$ (it cannot have a stub because there are no "type 1" geodesics in $\Sigma_1$). This implies that two geodesics in $\Gamma$ form an empty triangle with $A_1$. $A_1$ must consist of a single geodesic arc by construction of $S_1$. Thus, the crossing at the vertex of the triangle can be moved out of $S_1$ by a $Y$-$\Delta$ transformation.

Continue the process until all crossings are removed from $S_1$. Then consider $S_2, S_3,$ and $S_4$. By the same argument, two geodesics $g_1$ and $g_2$ which cross within $S_2$ form an empty triangle with $A_2$. There may be a crossing of some $h_{n_1}$ and $h_{n_2}$ on the side of the triangle along $A_2$. In that case, because there are no crossings within $S_1$, the crossing of $h_{n_1}$ and $h_{n_2}$ can be freely moved across $g_1$ or $g_2$, so that the $A_2$-side of the triangle formed by $g_1, g_2,$ and $A_2$ contains no crossings. Move the crossing of $g_1$ and $g_2$ out of $S_2$ and into either $S_0$ or $S_1$. If the crossing is in $S_1$, move it into $S_0$ by the procedure of the previous paragraph.

Continue inductively until $S_1, \ldots, S_{J-1}$ contain no crossings. Let $\Sigma_J$ be
Figure 12: The regions $S_1, \ldots, S_K$ in Lemma 2.16. Darker color indicates higher $w(S_k)$.

the subnetwork inside $S_J$. By Lemma 2.12, $\Sigma_J$ will always have an empty boundary triangle or empty geodesic on $A'_J$. Move crossings out of $\Sigma_J$ as in the previous paragraph until there is an empty geodesic $g_0$ of $\Sigma_J$ with endpoints on $A'_J$, and the $A_J$-side of the “biangle” formed by $g_0$ and $A_J$ contains no crossings. The lens can be removed by a trivial modification.  

Lemma 2.17. If two type 0 geodesics intersect, there is a removable lens.

Proof. Let $\mathcal{F}_1, \ldots, \mathcal{F}_K$ be the connected families of type 0 geodesics (where “connected families” is defined by considering only type 0 geodesics, not type 1 or 2). Let $\mathcal{F}_J$ be the innermost family with more than one geodesic. Consider a layer $\Sigma$ which contains $\mathcal{F}_J$ and no other $\mathcal{F}_k$.

Let $h$ be a geodesic of $\mathcal{F}_J$ such that there exists a curve $C$ from some point of $h$ to the inner boundary of $\Sigma$ which does not cross any other type 0
geodesic. Construct a curve $D$ which begins at the inner boundary, crosses $h$ once, crosses back across $h$, then returns to the inner boundary, such that $D$ does not intersect a type 0 geodesic anywhere else and $D$ partitions $\Sigma$ into two subnetworks. Then one of the subnetworks $\Sigma'$ is annular and has a type 1 geodesic $h'$ which is an arc of $h$. In $\Sigma'$, $h'$ is a type 1 geodesic which intersects a type 0 geodesic, so $\Sigma'$ has a removable lens.

**Theorem 2.18.** A network with no removable lenses can be partitioned into three layers such that

- the first layer contains all type 1 inner geodesics;
- the second layer contains all type 0 geodesics;
- the third layer contains all type 1 outer geodesics.

**Proof.** It follows from the previous lemmas.

2.7 Layered Form

**Definition 2.19.** A network is in layered form if it can be partitioned into four layers such that

- the first layer contains all type 1 inner geodesics;
- the second layer contains all crossings between type 2 geodesics;
- the third layer contains all type 0 geodesics;
- the fourth layer contains all type 1 outer geodesics.

**Theorem 2.20.** A network with no removable lenses can be put into layered form by $Y$-$\Delta$ transformations.

**Proof.** By the previous theorem, we only have to show that all crossings of type 2 geodesics can be moved into one layer. Assume there are at least two type 2 geodesics (there must be an even number).

Consider moving them out of the layer with type 1 inner geodesics. For each type 1 geodesic in the universal cover, let $R(g[n])$ be the bounded region enclosed by $g[n]$ and one of the boundary curves. Choose a $g$ such that no geodesic is fully contained in $R(g[0])$. This is possible because otherwise we could construct an infinite sequence of geodesics $g_n$ with $R(g_n[0]) \not\subseteq R(g_{n-1}[0])$. If $R(g[0])$ contains no crossings of type 2 geodesics, then $g[0]$ is...
Figure 13: Geodesics of a network in layered form.
irrelevant for the proof, and we can work on a subnetwork which is outside of \( R(g[n]) \) for all \( n \).

So suppose \( R(g[0]) \) contains some crossings. Consider a subnetwork \( \Sigma \) inside \( R(g[0]) \) which contains all the junctures in the region. In \( \Sigma \), all geodesics have one endpoint on the upper boundary and one on \( C \). By Corollary 2.11, there is an empty boundary triangle of \( \Sigma \) along \( C \), which means that in \( \Gamma \) some geodesics form a triangle with \( g \). By a \( Y-\Delta \), we can move the crossing out of \( R(g[0]) \). The crossing will not enter \( R(g[n]) \) for any integer \( n \) and type 1 geodesic \( g \). The \( Y-\Delta \) transformation in the universal cover corresponds to one in the annulus, and so we can apply the same transformation at each period in the universal cover. By repeating this argument, we can move all crossings out of \( R(g[0]) \), which includes crossings of type 2 geodesics.

Repeat the last two paragraphs until crossings of type 2 geodesics are removed from \( R(g[0]) \) for all \( g \in \mathcal{G}_{\text{inner}} \). Then apply the same argument to the outer boundary. To move the crossings of type 2 geodesics inside the layer of type 0 geodesics, use the same argument as in Lemma 2.16.

\[ \square \]

3 The Relationship between the Two Boundaries

The cut-point lemma of [1] is a geometric statement relating connections and geodesics, but it has clear algebraic implications by way of the determinant-connection formula. Ian Zemke in fact uses linear algebra to prove a cut-point lemma for infinite graphs [5]. Here I develop an analogue of the cut-point lemma in which the partition of the boundary separates the inner and outer boundaries. Both geometric and algebraic statements are proved using partitions into elementary networks.

3.1 Solution Spaces

Let \( P_1, \ldots, P_N \) be the plates of \( \mathcal{P}_{\text{inner}} \) in counterclockwise order and let \( I_1, \ldots, I_N \) be the corresponding inner boundary intervals. For each \( \gamma \)-harmonic function \( f \), let \( f_{\text{inner}} \) be the restriction of \( f \) to \( \mathcal{P}_{\text{inner}} \) and \( I_{\text{inner}} \) (\( f_{\text{inner}} \) assigns a voltage to each \( P_n \) and a current to each \( I_n \)). Let \( f_{\text{inner}} \) be written as a vector in \( x \in \mathbb{R}^{2|\mathcal{P}|} \) where \( x_1, x_2, \ldots, x_N \) represent the voltages on \( P_1, \ldots, P_N \), and \( x_{N+1}, x_{N+2}, \ldots, x_{2N} \) represent the currents on \( i_1, \ldots, i_N \).

A vector \( x \in \mathbb{R}^{2N} \) is called feasible inner boundary data if there exists a \( \gamma \)-harmonic function \( f \) with \( f_{\text{inner}} = x \). Let \( F_{\text{inner}} \) be the set of vectors which are feasible inner boundary data, and let \( F_{\text{outer}} \), the set of feasible outer boundary data, be defined similarly. A vector \( x \in F_{\text{inner}} \) and \( y \in F_{\text{outer}} \) are compatible if there exists a \( \gamma \)-harmonic function with \( f_{\text{inner}} = x \) and
\( f_{\text{outer}} = y \). An \( x \in F_{\text{inner}} \) is called zero-compatible if it is compatible with \( 0 \in F_{\text{outer}} \), and a similar definition holds for \( y \in F_{\text{outer}} \). Let \( Z_{\text{inner}} \) be the set of zero-compatible vectors in \( F_{\text{inner}} \) and let \( Z_{\text{outer}} \) be the set of zero-compatible vectors in \( F_{\text{outer}} \).

Obviously, \( F_{\text{inner}}, F_{\text{outer}}, Z_{\text{inner}}, \) and \( Z_{\text{outer}} \) can be determined from \( F \). All these sets are examples of what I will call solution spaces for the network, and I will discuss other solution spaces later. An immediate, purely algebraic fact about these solution spaces is

**Theorem 3.1.** \( \dim F_{\text{inner}} - \dim Z_{\text{inner}} = \dim F_{\text{outer}} - \dim Z_{\text{outer}} \).

**Proof.** Let \( N = |\mathcal{P}_{\text{inner}}| \) and \( M = |\mathcal{P}_{\text{outer}}| \). \( F_{\text{inner}} \) and \( Z_{\text{inner}} \) are linear subspaces of \( \mathbb{R}^{2N} \) and \( F_{\text{outer}} \) and \( Z_{\text{outer}} \) are linear subspaces of \( \mathbb{R}^{2M} \). Let \( Z_{\text{inner}}^\bot \) be the orthogonal complement of \( Z_{\text{inner}} \) in \( \mathbb{R}^{2N} \) with respect to the standard basis and inner product and let \( Z_{\text{outer}}^\bot \) be the orthogonal complement of \( Z_{\text{outer}} \) in \( \mathbb{R}^{2M} \). Let \( U = Z_{\text{inner}}^\bot \cap F_{\text{inner}} \) and \( V = Z_{\text{outer}}^\bot \cap F_{\text{outer}} \).

There is a one-to-one correspondence between vectors in \( U \) and vectors in \( V \). For suppose \( x \in U \). Then there is a \( \gamma \)-harmonic function \( f \) such that \( f_{\text{inner}} = x \). Then \( u = f_{\text{outer}} \) is in \( F_{\text{outer}} \) and can be written uniquely as the sum of some \( v \in Z_{\text{outer}} \) and some \( y \in Z_{\text{outer}}^\bot \). Since \( u \) and \( v \) are in \( F_{\text{outer}} \), so is \( y \), and so \( y \in V \). To show \( y \) is unique, suppose \( y_1 \) and \( y_2 \) are both in \( V \) and compatible with \( x \). Then \( y_1 - y_2 \) is compatible with zero, and it is in \( V \) because \( V \) is a linear subspace of \( \mathbb{R}^{2N} \). Thus, \( y_1 - y_2 \) is in both \( Z_{\text{outer}} \) and \( Z_{\text{outer}}^\bot \). Hence, \( y_1 = y_2 \).

A similar argument shows that for each \( y \in V \), there is a unique compatible \( x \in U \). Thus, there is a bijection between \( U \) and \( V \). This bijection is obviously linear because if \( x_1, x_2 \in U \) have compatible vectors \( y_1, y_2 \in B \), then \( a x_1 + b x_2 \in U \) is compatible with \( a y_1 + b y_2 \in V \) because the sum of two \( \gamma \)-harmonic functions is \( \gamma \)-harmonic. Therefore, \( \dim U = \dim V \). But \( \dim U = \dim F_{\text{inner}} - \dim Z_{\text{inner}} \) and \( \dim V = \dim F_{\text{outer}} - \dim Z_{\text{outer}} \). Hence, \( \dim F_{\text{inner}} - \dim Z_{\text{inner}} = \dim F_{\text{outer}} - \dim Z_{\text{outer}} \). \( \square \)

Statements about solution spaces like \( F_{\text{inner}}, Z_{\text{inner}}, F_{\text{outer}}, \) and \( Z_{\text{outer}} \) have interpretations in terms of the response matrix. For example, if no plate touches both boundaries, then \( \mathcal{P}_{\text{inner}} \) and \( \mathcal{P}_{\text{outer}} \) are a partition of \( \partial \mathcal{P} \) and we can write \( \Lambda \) in block form as

\[
\Lambda = \begin{pmatrix}
\Lambda_{II} & \Lambda_{IO} \\
\Lambda_{OI} & \Lambda_{OO}
\end{pmatrix},
\]

where the first row/column deals with the inner boundary and the second row/column deals with the outer boundary. Then
Proposition 3.2.

\[ \dim Z_{\text{inner}} = \dim \ker \Lambda_{OI} , \quad \dim F_{\text{inner}} = |P_{\text{inner}}| + \rank \Lambda_{OI}. \]

Proof. A vector \( \mathbf{x} = (\mathbf{x}_v, \mathbf{x}_c) \) is in \( Z_{\text{inner}} \) if and only if

\[
\begin{pmatrix}
\Lambda_{II} & \Lambda_{IO} \\
\Lambda_{OI} & \Lambda_{OO}
\end{pmatrix}
\begin{pmatrix}
\mathbf{x}_v \\
\mathbf{0}
\end{pmatrix}
= \begin{pmatrix}
\mathbf{x}_c \\
\mathbf{0}
\end{pmatrix},
\]

which is true if and only if \( \mathbf{x}_v \in \ker \Lambda_{OI} \) and \( \mathbf{x}_c = \Lambda_{II} \mathbf{x}_v \). Hence, there is a bijective linear transformation from \( \mathbf{x} \in Z_{\text{inner}} \) and \( \mathbf{x}_v \in \ker \Lambda_{OI} \), so that \( \dim Z_{\text{inner}} = \dim \ker \Lambda_{OI} \).

Let \( U \) be the set of all \( \mathbf{y} \) such that

\[
\begin{pmatrix}
\Lambda_{II} & \Lambda_{IO} \\
\Lambda_{OI} & \Lambda_{OO}
\end{pmatrix}
\begin{pmatrix}
\mathbf{y}_v \\
\mathbf{0}
\end{pmatrix}
= \begin{pmatrix}
\mathbf{y}_c \\
\ast
\end{pmatrix}.
\]

Then \( U \) is a linear subspace of \( \mathbb{R}^{2N} \), where \( N = |P_{\text{inner}}| \). For any \( \mathbf{y}_v \), there is a unique \( \mathbf{y}_c \) which satisfies the above equation, namely \( \mathbf{y}_c = \Lambda_{II} \mathbf{y}_v \). Hence, \( \dim U = N \). Let \( V \) be the set of all \( \mathbf{z} = (\mathbf{0}, \mathbf{z}_c) \) such that

\[
\begin{pmatrix}
\Lambda_{II} & \Lambda_{IO} \\
\Lambda_{OI} & \Lambda_{OO}
\end{pmatrix}
\begin{pmatrix}
\mathbf{0} \\
\mathbf{w}
\end{pmatrix}
= \begin{pmatrix}
\mathbf{z}_c \\
\ast
\end{pmatrix}
\]

for some \( \mathbf{w} \). \( V \) is isomorphic to the image space of \( \Lambda_{OI} \), so \( \dim V = \rank \Lambda_{OI} \).

Every vector in \( F_{\text{inner}} \) can be uniquely written as \( \mathbf{y} + \mathbf{z} \) where \( \mathbf{y} \in U \) and \( \mathbf{z} \in V \). Hence, \( \dim F_{\text{inner}} = |P_{II}| + \rank \Lambda_{OI} \). \( \square \)

3.2 Partition into Elementary Networks

To get deeper results about solution spaces and connections, we can partition the network into specific types of subnetworks. An **elementary network** is any of the following four types:

1. An **elementary boundary-juncture network** is a network in which
   
   - every plate touches both boundaries,
   - there is exactly one juncture.

2. An **elementary spike** network is a network in which
   
   - every plate touches the inner boundary except one plate \( P \)
   - every plate touches the outer boundary except one plate \( Q \)
   
   - every plate touches both boundaries,
   - there is exactly one juncture.
• there is a juncture point between $P$ and $Q$
• there are no other junctures.

3. An **elementary inner-stub network** is a network in which
• every plate touches the inner boundary
• every plate except one (a stub) touches the outer boundary
• there are no junctures.

4. An **elementary outer-stub network** is a network in which
• every plate touches the outer boundary
• every plate except one (a stub) touches the inner boundary
• there are no junctures.

5. A **trivial network** is a network in which
• Every plate touches both boundaries.
• There are no junctures.

6. A **zigzag network** is a network in which
• There are $2N$ plates, each of which touches exactly one boundary curve.
• The inner boundary plates are $P_1, \ldots, P_N$ in counterclockwise order, the outer boundary plates are $Q_1, \ldots, Q_N$.
• The junctures in the network are $P_nQ_n$ and $P_nQ_{n+1}$ for $n = 1, \ldots, N$ with indices reduced modulo $N$.

Any network can be partitioned into elementary networks of the first four types, but for our purposes, the most important fact is

**Lemma 3.3.** Suppose $\Gamma$ has no removable lenses and all geodesics are type 1 inner or type 2. Then $\Gamma$ can be partitioned into elementary boundary-juncture, spike, and inner-stub networks. The number of inner-stub networks is equal to the number of type 1 geodesics.

**Proof.** Let $\Sigma_0 = \Gamma$. We will define subnetworks $\Sigma_0, \Sigma_1, \ldots$ inductively. Each $\Sigma_m$ will be partitioned into an elementary network $\Gamma_{m+1}$ and a layer $\Sigma_{m+1}$, where $\Gamma_{m+1}$ is next to the inner boundary of $\Sigma_m$ and $\Sigma_{m+1}$ is next to the outer boundary.
Figure 14: Elementary networks.

Boundary-juncture

Spike

Inner-stub

Outer-stub

Trivial network

Zigzag
If $\Sigma_m$ is not the trivial network, there are three cases. Let $S_m$ be the region of embedding for $\Sigma_m$. The cases are not mutually exclusive, but they should be considered in the order given here (if more than one case is satisfied, follow the directions on the case listed first):

1. There is a stub on the inner boundary. Let $P$ be the stub. Let $C$ be the inner boundary of $S_m \setminus P$. Choose $C'$ on the outside of $C$ and so close to $C$ that no junctures lie between $C$ and $C'$ and $C'$ does not intersect any edges not intersected by $C$. Then $C'$ partitions $\Sigma_m$ into $\Gamma_{m+1}$ and $\Sigma_{m+1}$, where $\Gamma_{m+1}$ is an inner-stub network.

2. There is a boundary-juncture on the inner boundary. Let $T$ be the inner boundary triangle at the juncture, and let $C$ be the inner boundary of $S_m \setminus T$. Construct $C'$, $\Gamma_{m+1}$, and $\Sigma_{m+1}$ from $C$ as in the previous case. $\Gamma_{m+1}$ is an elementary boundary-juncture network.

3. There is a spike on the inner boundary. Let $P$ be the spike and $Q$ be the adjacent plate. We assume since $P$ is a spike that it does not touch the outer boundary. Since we assumed there was no inner-boundary juncture, we know $Q$ does not touch the inner boundary. Let $C$ be the inner boundary of $S_m \setminus P$ and construct $\Gamma_{m+1}$ and $\Sigma_{m+1}$. $\Gamma_{m+1}$ is a spike network.

These are the only three cases because by Theorem 2.10, a nontrivial network with only type 1 inner and type 2 geodesics has an empty boundary triangle or stub and because if $\Sigma_m$ has only type 1 inner and type 2 geodesics, then the same is true of $\Sigma_{m+1}$. The construction will continue until $\Sigma_M$ is the trivial network. Then $\Gamma_1, \ldots, \Gamma_{M-1}, \Sigma_{M-1}$ are the desired partition.

The number of geodesic endpoints on each boundary is twice the number of plates. If $K$ is the number of inner-stub networks, then there are $K$ more plates on the inner boundary than the outer boundary. Since all the geodesics are type 1 inner or type 2, the number of type 1 inner geodesics is $K$, which is the same as the number of inner-stub networks.

### 3.3 Geodesics and Connections

**Lemma 3.4.** Suppose $\Gamma$ has no removable lenses, and all geodesics are type 1 inner or type 2. Then $2 \cdot M(P_{\text{inner}}, P_{\text{outer}}) = |G_2|$. The same is true if all geodesics are type 1 outer and type 2.
Figure 15: A network partitioned into elementary boundary-juncture, spike, and inner-stub networks.
Proof. Consider the case where all geodesics are type 1 inner or type 2. The other case is symmetrical.

\( \Gamma \) can be partitioned into elementary boundary-juncture, spike, and inner-stub networks \( \Gamma_1, \ldots, \Gamma_M \). Let \( P_{1,1}, P_{2,1}, \ldots, P_{N,1} \) be the outer boundary plates. Define sequences \( \{P_{1,k}\}, \ldots, \{P_{N,k}\} \) inductively as follows:

For each \( n, k \), let \( \Gamma_{n,k} \) be the innermost subnetwork of the partition which includes a subplate of \( P_{n,k} \). Then \( \Gamma_{n,k} \) must be a spike, boundary-juncture, or inner stub network. If \( P_{n,k} \) touches the inner boundary of \( \Gamma_{n,k} \), then \( P_{n,k} \) touches the inner boundary, and we let \( P_{n,k} \) be the last plate of its sequence. Otherwise \( \Gamma_{n,k} \) is a spike network and \( P_{n,k} \) has a juncture in \( \Gamma_{n,k} \). We let \( P_{n,k+1} \) be the plate which meets \( P_{n,k} \) at this juncture.

The sequences \( \{P_{1,k}\}, \ldots, \{P_{N,k}\} \) define \( N \) disjoint paths from the outer to the inner boundary. They form an \( N \)-connection using all the plates of the outer boundary, so there cannot be any larger \( k \)-connection. \( N \) is half the number of type 2 geodesics.

Lemma 3.5. Suppose \( \Gamma \) has no removable lenses, and all geodesics are type 2 or type 0. Then \( 2 \cdot M \langle P_{\text{inner}}, P_{\text{outer}} \rangle = |G_2| \).

Proof. As in the proof of Lemma 2.16, we can uncross the type 0 geodesics, so as to preserve the orientation of each arc of a type 0 geodesic. We are left with a network \( \Gamma' \) in which all the type 0 geodesics are simple closed curves winding once about the hole. Removing junctions cannot create any connections, only break them. Hence, if we can show that \( \Gamma' \) has a \( k \)-connection between all vertices on the inner boundary and all vertices on the outer boundary, the proof will be complete.

For each type 0 geodesic \( g \), draw a closed curve along the inside of \( g \) so close to \( g \) that there are no junctions between \( g \) and the curve. Draw another closed curve along the outside of \( g \). We can create such curves next to every type 0 geodesic in such a way that the curves do not intersect. The curves partition \( \Gamma' \) into \( \Gamma_1, \ldots, \Gamma_M \), where each \( \Gamma_m \) is either a zigzag or has only type 2 geodesics.

It is obvious that a zigzag has a \( k \)-connection between all inner and all outer boundary plates. If \( \Gamma_m \) is not a zigzag, then the previous lemma shows that it has such a \( k \)-connection. By similar reasoning as in the previous lemma, we join the \( k \)-connections of the subnetworks into a \( k \)-connection of \( \Gamma' \) with the desired size.

Theorem 3.6. If \( \Gamma \) has no removable lenses, \( 2 \cdot M \langle P_{\text{inner}}, P_{\text{outer}} \rangle = |G_2| \).

Proof. By Theorem 2.18, \( \Gamma \) can be partitioned into three layers \( \Gamma_1, \Gamma_2, \) and \( \Gamma_3 \), where \( \Gamma_1 \) contains the type 1 inner geodesics, \( \Gamma_2 \) contains the type 0,
and $\Gamma_3$ contains the type 1 outer. By the preceding lemmas, $\Gamma_1$ has a $k$-
connection from some plates of the inner boundary to all plates of the outer
boundary, $\Gamma_2$ has a $k$-connection from the whole inner boundary to the whole
outer boundary, and $\Gamma_3$ has a $k$-connection from the whole inner boundary
to some subset of the outer boundary. The theorem follows.

3.4 Geodesics and Solution Spaces

Lemma 3.7. If $\Gamma$ is an elementary boundary-juncture or spike network with
$N$ boundary intervals on each boundary, then $\dim F_{\text{inner}} = \dim F_{\text{outer}} = 2N$
and $\dim Z_{\text{inner}} = \dim Z_{\text{outer}} = 0$.

Proof. Consider a boundary-juncture network with plates $P_1, \ldots, P_N$ and
juncture between $P_1$ and $P_2$. To show any data is feasible on the inner
boundary, suppose we are given inner boundary voltages and currents, and
we will find a $\gamma$-harmonic function with that boundary data. The voltages
are all determined. If $I_n$ and $J_n$ are the inner and outer boundary intervals
for $P_n$, then we let $c(J_n) = -c(I_n)$ for $n \neq 1, 2$. We let $c(P_1 \rightarrow P_2) = 
\gamma(P_1P_2)(v(P_1) - v(P_2))$, and $c(J_1) = -c(I_1) + c(P_1 \rightarrow P_2)$ and $c(J_2) = 
-c(I_2) + c(P_2 \rightarrow P_1)$.

On the other hand, if all the outer boundary data is zero, then no current
can flow across the juncture, and all the boundary currents are zero. Thus,
the only zero-compatible inner boundary data is 0. The arguments for $F_{\text{outer}}$
and $Z_{\text{outer}}$ are symmetrical.

The argument for a spike network is similar and is left to the reader.

Lemma 3.8. For an elementary inner-stub network with $N + 1$ plates,
$\dim F_{\text{outer}} = 2N$, $\dim Z_{\text{outer}} = 0$, $\dim F_{\text{inner}} = 2N + 1$, and $\dim Z_{\text{inner}} = 1$.

Proof. Obviously, any data on the outer boundary is feasible and if voltages
and currents are zero on the inner boundary, they must be zero on the outer
boundary. Any data on the inner boundary is feasible so long as the current
on the stub is zero. If voltages and currents on the outer boundary are zero,
then all voltages and currents on the inner boundary must be zero except
the voltage on the stub.

Lemma 3.9. Suppose $\Gamma$ has no removable lenses, and all geodesics are type
1 inner or type 2. Then

\begin{align*}
\dim F_{\text{outer}} & = |G_2|, & \dim Z_{\text{outer}} & = 0, \\
\dim F_{\text{inner}} & = |G_2| + |G_{\text{inner}}|, & \dim Z_{\text{inner}} & = |G_{\text{inner}}|.
\end{align*}
Proof. By Lemma 3.3, $\Gamma$ can be partitioned into elementary boundary juncture, boundary spike, and inner stub networks $\Gamma_1, \ldots, \Gamma_M$, ordered from outermost to innermost. Let $\Sigma_m$ be the subnetwork consisting of $\Gamma_1, \ldots, \Gamma_m$. We show that the theorem is true for each $\Sigma_m$ by induction. The base case follows from the previous lemmas. The induction step is broken into four claims:

1. $\dim F_{\text{outer}} = |G_2|$. Any outer boundary data which was feasible for $\Sigma_m$ is still feasible for $\Sigma_{m+1}$; it does not matter what the inner boundary data of $\Sigma_m$ is because any outer boundary data is feasible for $\Gamma_{m+1}$.

2. $\dim Z_{\text{outer}} = 0$. Suppose that a $\gamma$-harmonic function on $\Sigma_{m+1}$ has inner boundary data zero. Then the outer boundary data on $\Gamma_{m+1}$ must be zero. This implies that the inner boundary data on $\Sigma_m$ is zero, and so by inductive hypothesis, the inner boundary data on $\Sigma_m$ must be zero.

3. $\dim F_{\text{inner}} = |G_2| + |G_{\text{inner}}|$. If $\Gamma_{m+1}$ is a boundary-juncture or spike network, then there is a unique compatible vector of outer boundary data for each vector of inner boundary data, so there is a linear isomorphism between $F_{\text{inner}}$ and $F_{\text{outer}}$ of $\Gamma_{m+1}$. This implies that there is a linear isomorphism between $F_{\text{inner}}$ of $\Sigma_{m+1}$ and $F_{\text{inner}}$ of $\Sigma_m$, so $\dim F_{\text{inner}}$ is the same for both networks.

If $\Gamma_{m+1}$ is an inner stub network, then the set of feasible voltages and currents on the inner boundary plates and intervals of $\Sigma_{m+1}$ other than the stub is exactly the same as $F_{\text{inner}}$ of $\Sigma_m$. Any voltage is feasible on the stub, and it will not affect the rest of the network; however, the current on the stub must be zero. Thus, $\dim F_{\text{inner}}$ of $\Sigma_{m+1}$ is one more than $\dim F_{\text{inner}}$ of $\Sigma_m$. $\Sigma_{m+1}$ also has one more type 1 inner geodesic than $\Sigma_m$.

4. $\dim Z_{\text{inner}} = |G_{\text{inner}}|$. The argument is the same as the previous claim.

\[ \square \]

**Theorem 3.10.** If $\Gamma$ has no removable lenses or type 0 geodesics, then

\[
\begin{align*}
\dim F_{\text{inner}} &= |G_2| + |G_{\text{inner}}|, & \dim Z_{\text{inner}} &= |G_{\text{inner}}|, \\
\dim F_{\text{outer}} &= |G_2| + |G_{\text{outer}}|, & \dim Z_{\text{outer}} &= |G_{\text{outer}}|.
\end{align*}
\]
Proof. $\Gamma$ can be partitioned into a network $\Gamma_1$ with only type 1 inner and type 2 geodesics and $\Gamma_2$ with only type 1 outer and type 2. We apply the previous lemma to these networks (we can switch roles of the inner and outer boundaries and the lemma is still true). Since any data is feasible on the inner boundary of $\Gamma_2$, $F_{\text{inner}}$ is the same for $\Gamma$ as for $\Gamma_1$. Since the only zero-compatible data on the inner boundary of $\Gamma_2$ is 0, $Z_{\text{inner}}$ is the same for $\Gamma$ as for $\Gamma_1$. The argument for the other claims is symmetrical.

**Theorem 3.11.** Suppose $\Gamma$ has no removable lenses, but has type 0 geodesics. If $\frac{1}{2} |\mathcal{G}_2|$ is odd, Theorem 3.10 holds.

Proof. We will use the determinant-connection formula. We know that no plate touches both boundaries because there is a type 0 geodesic. By Theorem 3.6, the maximum size $k$-connection between the inner and outer boundaries is $K = \frac{1}{2} |\mathcal{G}_2|$. Let $\mathcal{U}$ and $\mathcal{V}$ be sets of plates on the inner and outer boundaries respectively such that there is a $K$-connection between $\mathcal{U}$ and $\mathcal{V}$. If $\alpha$ is any $K$-connection between $\mathcal{U}$ and $\mathcal{V}$, then $\tau_{\alpha}$ must be a cyclic permutation of the form $\tau_{\alpha}(n) \equiv n + J \mod K$ for some integer $J$. Otherwise, the paths in $\alpha$ would intersect as a consequence of the Jordan curve theorem. Since $K$ is odd, all such permutations are even.

Thus, by the determinant-connection formula, $\det \Lambda(\mathcal{U}; \mathcal{V})$ is strictly negative. This implies $\text{rank } \Lambda_{OI} = K$ (the rank cannot be any larger because no larger $k$-connections exist). By applying Proposition 3.2 and by counting geodesics and boundary plates, we see

$$\dim F_{\text{inner}} = |P_{\text{inner}}| + \text{rank } \Lambda_{OI} = \frac{1}{2} |\mathcal{G}_2| + |\mathcal{G}_{\text{inner}}| + \frac{1}{2} |\mathcal{G}_2|,$$

$$\dim Z_{\text{inner}} = |P_{\text{inner}}| - \text{rank } \Lambda_{OI} = \frac{1}{2} |\mathcal{G}_2| + |\mathcal{G}_{\text{inner}}| - \frac{1}{2} |\mathcal{G}_2|,$$

and the corresponding statements for $\Lambda_{IO}$ and the outer boundary.

**Theorem 3.12.** Suppose $\frac{1}{2} |\mathcal{G}_2|$ is even. If Theorem 3.10 does not hold, then

$$\dim F_{\text{inner}} = |\mathcal{G}_2| + |\mathcal{G}_{\text{inner}}| - 1, \quad \dim Z_{\text{inner}} = |\mathcal{G}_{\text{inner}}| + 1,$$

$$\dim F_{\text{outer}} = |\mathcal{G}_2| + |\mathcal{G}_{\text{outer}}| - 1, \quad \dim Z_{\text{outer}} = |\mathcal{G}_{\text{outer}}| + 1.$$

Proof. There exists a $K-1$-connection from the inner to the outer boundary, and $K-1$ is odd, so by the argument in the previous theorem, $\text{rank } \Lambda_{OI} = K-1$, and the statements about dimensions follow from Proposition 3.2.

The preceding theorem shows that for annular networks the algebraic versions of the cut-point lemma sometimes require stronger hypotheses than the corresponding geometric statements. Dimensions of solution spaces are
not always what we would expect based on the connection properties. However, in this case, the matrix is “almost” invertible:

**Proposition 3.13.** Suppose the network has no removable lenses. Suppose $K = \frac{1}{2}[G_2]$ is even and there is at least one type 0 geodesic. Suppose there exists a $K$-connection between $U \subset P_{\text{inner}}$ and $V \subset P_{\text{outer}}$. Then every $K - 1 \times K - 1$ minor of $\Lambda(U; V)$ is strictly negative.

**Proof.** Since $K - 1$ is odd, it suffices to show that every $K - 1$ connection from a subset of $U$ to a subset of $V$ exists. Since $Y$-$\Delta$ transformations do not affect connections, assume the network is in layered form. As in Lemma 3.5, uncross the type 0 geodesics until they have no self-intersections; this cannot create any new connections. It now suffices to show that all $K - 1$-connections exist in a single zigzag.

Suppose the plates on the inner boundary are $P_1, \ldots, P_K$ and on the outer boundary $Q_1, \ldots, Q_K$ such that these are junctures between $P_n$ and $Q_n$ and $P_n$ and $Q_{n+1}$ with indices reduced modulo $K$. Assume without loss of generality that the inner-boundary plates in the desired connection are $P_1, \ldots, P_{K-1}$. Suppose we want to connect them with $Q_n$ for $n \neq J$. For $n < J$, connect $P_n$ and $Q_n$. For $n \geq J$, connect $P_n$ and $Q_{n+1}$. \hfill \Box

A similar proof will show that, in general, if the sum of the winding numbers of the type 0 geodesics is $N$, and if $m$ is an odd integer with $K - N \leq m \leq K$, then every $m \times m$ minor of $\Lambda(U; V)$ is strictly negative. This is true whether $K$ is odd or even.

### 4 Cuts of One Boundary

#### 4.1 Definitions

Let $I_1, \ldots, I_N$ be the boundary intervals of a network. A cut $R$ of the inner boundary is an arc of $C_{\text{inner}}$ whose endpoints are not the endpoints of any $I_n$. We will denote by $R^C$ the union of $C_{\text{outer}}$ and the arc in $C_{\text{inner}}$ which is complementary to $R$ in $C_{\text{inner}}$. (For circular planar networks, $R^C$ is simply the complementary arc of the boundary curve.) We assume that $R$ contains at least one endpoint of a geodesic, and does not contain all geodesic endpoints on the inner boundary.

The boundary intervals of $R$ (denoted $I_R$) include any $I_n$ which is a subset of $R$, and the plates of $R$ (denoted $P_R$) include all plates with these boundary intervals. The endpoint of $R$ may fall within some boundary interval $J$ corresponding to a plate $P$. The endpoint divides $J$ into two intervals
$J_A$ and $J_B$ with $J_A \subset R$ and $J_B \not\subset R$. In that case, $J_A$ is considered a boundary interval of $R$ and $J_B$ is a boundary interval of $R^C$. $P$ is considered a plate both of $R$ and of $R^C$.

The solution spaces of $R$ are as follows. Let $P_1, \ldots, P_K$ be the plates of a cut $R$. Let $J_1, \ldots, J_K$ be the corresponding boundary intervals of the cut. Let $x$ be a vector in $\mathbb{R}^{2K}$. We say $x \in F_R$ if there exists a $\gamma$-harmonic function with voltages $x_1, \ldots, x_K$ on $P_1, \ldots, P_K$ and currents $x_{K+1}, \ldots, x_{2K}$ on $J_1, \ldots, J_K$. Suppose $J_k$ is at one of the endpoints of $R$ and that some boundary interval $I$ of $\Gamma$ was split into $J_k$ and $J^* \subset R^C$ by an endpoint of $R$. Then we consider $c(I) = c(J_k) + c(J^*)$. An $x \in F_R$ may have any current on $J_k$ because we can always choose $c(J^*)$ to make $c(I)$ correct for a $\gamma$-harmonic function on the whole network.

Define the maximum connection $M(R)$ as $M(P_R, P_{R^C})$. Define the family of reentrant geodesics $R_R$ as the collection of all geodesics with both endpoints in $R$. A reentrant geodesic $g$ forms a closed curve $C$ with some interval of $R$. $C$ can be oriented and its winding number about the hole can be computed. If the winding number is nonzero, $g$ is called reentrant around the hole.

4.2 Circular Planar Case

We can prove the cut-point lemma for circular planar graphs by “changing the region of embedding into an annulus” and using the results of the previous section. Here we assume that each boundary plate of the given circular planar graph has only one boundary interval.

Consider a cut $R$ of the boundary of a circular planar network in a region $S$. Let $C_1$ be an arc of $R$ which is slightly shorter at the endpoints, but contains all the same geodesic endpoints. Let $x_1$ and $y_1$ be the endpoints of $C_1$. Let $C_2$ be a similar arc of $R^C$ with endpoints $x_2$ and $y_2$. Construct a curve $C'_1$ which connects $x_1$ and $y_1$ and remains outside $\overline{S}$ and a curve $C'_2$ which connects $x_2$ and $y_2$ which connects $x_2$ and $y_2$, such that $C_1 \cup C'_1$ and $C_2 \cup C'_2$ are nonintersecting simple closed curves which form the boundary of an annular region $S'$, and such that $C_1 \cup C'_1$ is inside $C_2 \cup C'_2$.

Let $\Gamma'$ be the plate network in $S'$. Do not change the shape of any plates, even if a plate contains an endpoint of $R$. For each plate of $\Gamma$ which contains an endpoint of $R$, $\Gamma'$ will have an extra “type 2” geodesic. Otherwise, the geodesics will not change. $F_R$ of $\Gamma$ is exactly $F_{inner}$ of $\Gamma'$, $Z_R$ is $Z_{inner}$, $M(R)$ is $M(P_{inner}, P_{outer})$, and $R_R$ is $G_{inner}$.

By applying the results of the previous section to $\Gamma'$, we have

$$M(P_{inner}, P_{outer}) = |G_2|, \quad \dim F_{inner} = |G_2| + |G_{inner}|, \quad \dim Z_{inner} = |G_{inner}|.$$
The first two are not convenient formulae because we modified the number of “type 2” geodesics. But we can express $|G_2|$ as

$$|G_2| = 2|P_{\text{inner}}| - 2|R_{\text{inner}}| = 2|R_R| - 2|Z_R|,$$

which yields

**Theorem 4.1** (Cut-Point Lemma). Let $\Gamma$ be a circular planar network and let $R$ be a cut of the boundary curve. Then

$$M(R) = |P_R| - |R_R|,$$

$$\dim F_R = 2|P_R| - |R_R|,$$

$$\dim Z_R = |R_R|. $$

We could have proved this directly by partitioning the simply connected region into “elementary layers;” the proof given in [1] by uncrossing empty boundary triangles can be interpreted as constructing such a partition.

### 4.3 Annular Case

In the annular case, the connections and solution spaces of a cut are not as easy to describe. However, there are certain cases where the same formulas hold:

**Lemma 4.2.** Suppose $\Gamma$ has only type 2 and type 1 inner geodesics, no removable lenses, and no self-intersecting geodesics, and that the type 2 geodesics do not intersect each other. Let $R$ be a cut of the inner boundary with no reentrant geodesics such that at least one type 2 geodesic does not have an endpoint in $R$. Then $M(R) = |P_R|$, $\dim F_R = 2|P_R|$, and $\dim Z_R = 0$.

**Proof.** Let $A$ and $B$ be curves which begin at the clockwise and counterclockwise endpoints of $R$ and ends at some points on the outer boundary, such that

- $A$ and $B$ do not intersect themselves.
- $A$ and $B$ do not intersect each other.
- Neither one contains a juncture.
- Neither one intersects any type 2 geodesic. This is possible because the type 2 geodesics do not intersect each other.
• Neither one intersects the same geodesic twice. This is possible because of our assumptions about lenses.

(See Figure 16.)

Let $U$ be the region of the annulus which is counterclockwise of $A$ and clockwise of $B$, and let $V$ be the region which is counterclockwise of $B$ and clockwise of $A$. Let $\Gamma_U$ and $\Gamma_V$ be the subnetworks in these regions. Since we assumed there was a type 2 geodesic with no endpoint on $R$, we know that no plate of $\Gamma_V$ touches both $A$ and $B$.

No geodesic of $\Gamma_U$ has both endpoints on $R$ because no geodesic of $\Gamma$ does. On the other hand, all geodesics of $\Gamma_V$ have an endpoint on the outer boundary, on $A$, or on $B$ by construction. A geodesic with an endpoint on the outer boundary must have been a type 2 geodesic of $\Gamma$, so it must not intersect $A$ or $B$. If a geodesic had one endpoint on $A$ and one on $B$, it would been a type 1 geodesic of $\Gamma$, and would have both endpoints on $R$, which is impossible.

Let $C$ and $C'$ be the arcs of the outer boundary along $\Gamma_U$ and $\Gamma_V$ respectively. By Theorem 4.1, there is a $k$-connection $\alpha$ of $\Gamma_U$ from $R$ to $C \cup A \cup B$ which uses all boundary plates of $R$. Similarly, there is a $k$-connection $\alpha'$ of $\Gamma_V$ from $R'$ to $C' \cup A \cup B$ which uses all boundary plates of $C' \cup A \cup B$.

We construct a connection $\beta$ in $\Gamma$ from $R$ to $R^C$ which uses all the plates of $R$ in the following way. Let $P_1, \ldots, P_N$ be the plates along $R$ in $\Gamma_U$. For each $P_n$, let $\alpha_n$ be the path in $\alpha$ connecting $P_n$ to some plate $Q_n$ of $\Gamma_V$. We know $Q_n$ touches $C \cup A \cup B$. There are two cases:

• If $Q_n$ touches $C$, let $\beta_n = \alpha_n$ (except that if $Q_n$ is a subplate in $\Gamma$, we replace it with the whole plate).

• If $Q_n$ touches $A$ or $B$, but not $C$, there is a plate $Q'_n$ of $\Gamma_V$ such that $Q_n \cup Q'_n$ forms one plate of $\Gamma$. Since $Q_n$ does not touch $C$, neither does $Q'_n$. Let $\alpha'_n$ is the path of $\alpha'$ connecting $Q'_n$ to some plate touching $R'$, and join $\alpha_n$ and $\alpha'_n$ to form $\beta_n$.

The paths $\beta_1, \ldots, \beta_N$ are distinct and form an $N$-connection from $R$ to $R^C$. This proves the first assertion in the theorem.

For the other two claims, repeat the above argument using the theorems about solution spaces instead of the theorems about connections.

\[\square\]

**Theorem 4.3.** Suppose $\Gamma$ has no removable lenses and no self-intersecting type 1 inner geodesics. Let $R$ be a cut of the inner boundary with no geodesics
Figure 16: Curves in the proof of Lemma 4.2.
reentrant around the hole, such that at least one type 2 geodesic does not have an endpoint on \( R \). Then \( M(R) = |\mathcal{P}_R| - |\mathcal{R}_R| \).

**Proof.** We can assume that \( \Gamma \) is in layered form without changing the connections or solution spaces.

Let \( C \) be a curve which begins at the clockwise endpoint of \( R \) and proceeds to the counterclockwise endpoint without intersecting itself, intersecting the same geodesic twice, or including any junctures, such that the region \( T \) bounded by \( R \cup C \) is a simply connected subset of the annulus. Let \( \Gamma_T \) be the subnetwork in this region. Notice \( T \) contains all the reentrant geodesics of \( R \), and no geodesic of \( \Gamma_T \) is reentrant for \( C \).

Divide the rest of \( \Gamma \) into two layers \( \Gamma_U \) and \( \Gamma_V \) (in regions \( U \) and \( V \)) such that \( \Gamma_U \) contains all type 1 inner geodesics and \( \Gamma_V \) contains all type 0 and type 1 outer geodesics and crossings between type 2 geodesics.

By Theorem 3.6, there is a \( k \)-connection between the plates touching the inner boundary of \( \Gamma_V \) and some subset of the plates on the outer boundary. By the previous lemma, there is a \( k \)-connection between all the plates of \( \Gamma_U \) along \( C \) and some of the plates of \( \Gamma_U \) in \( C^C \). Thus, we have a connection in \( \Gamma_{U \cup V} \) from the plates touching \( C \) to the some of the plates in \( C^C \).

By Theorem 4.1, the maximum \( k \)-connection in \( \Gamma_T \) from the plates along \( R \) to the plates touching \( C \) is \( 2|\mathcal{P}_R| - |\mathcal{R}_R| \). We can join the paths in a maximal \( k \)-connections with the paths of the connections in the previous paragraph. Thus, the maximum \( k \)-connection in \( \Gamma \) from the plates along \( R \) to the plates along \( R^C \) is the same size.

**Theorem 4.4.** Let \( \Gamma \) and \( R \) be in the previous theorem. Suppose that Theorem 3.10 holds. Then
\[
\dim F_R = 2|\mathcal{P}_R| - |\mathcal{R}_R|,
\]
\[
\dim Z_R = |\mathcal{R}_R|.
\]

**Proof.** Let \( \Gamma_T, \Gamma_U, \) and \( \Gamma_V \) be as in the previous proof. For \( \Gamma_V \), any data is feasible for the plates on the inner boundary because we assumed Theorem 3.10 holds. For \( \Gamma_U \), any data is feasible for the plates touching \( C \). Hence, for \( \Gamma_{U \cup V} \), any data is feasible for the plates touching \( C \). For \( \Gamma_T \), \( \dim F_R = 2|\mathcal{P}_R| - |\mathcal{R}_R| \) by Theorem 4.1. Hence, the same is true for \( \Gamma \).

### 4.4 Partial Recovery by Removal of Type 1 Geodesics

We can recover boundary junctures and spikes in networks with type 1 geodesics using the algebraic “cut-point lemma.”
Figure 17: Proof of Theorem 4.3. The black dashed curve divides $U$ and $V$. 
Lemma 4.5. Suppose $\Gamma$ has no removable lenses or self-intersecting type 1 inner geodesics and suppose Theorem 3.10 holds. Let $P$ be a spike on the inner boundary with juncture $PQ$ connecting it to another plate $Q$. Suppose that $g$, one of the geodesics touching $P$, is type 1 and is not part of a two-pole lens. Suppose there is a type 2 geodesic which does not intersect $g$. Then the conductance of $PQ$ is recoverable.

Proof. The geodesics $g$ splits the region of embedding into two subregions; let $T$ be the simply connected subregion. Let $A_T$ be the arc of the inner boundary curve which is part of $\partial T$. Let $R \supset A_T$ be an arc which contains the same geodesic endpoints as $A_T$ such that the endpoints of $R$ are not endpoints of a geodesic. Let $C$ be a curve with endpoints at the endpoints of $R$ which remains close enough to $g$ that no junctures lie between $C$ and $g$. Let $C'$ be a curve with endpoints at the endpoints of $R$ which remains between $g$ and $C$ except that $C'$ crosses $g$ twice along the boundary of $P$. $C'$ enters $P$ and then immediately exits $P$.

Let $U$ be the region bounded by $R$ and $C'$, let $V$ be the region bounded by $C$ and $C'$, and let $W$ be the annular region bounded by $R_C$ and $C$. Let $\Gamma_U$, $\Gamma_V$, and $\Gamma_W$ be the subnetworks in these regions.

Consider the boundary value problem for $\Gamma$ where all voltages and currents on $R_C$ are 0 and the voltage of $P$ is 1. I claim this problem has a solution and that the voltage of $Q$ is 0.

For $\Gamma_W$, $C$ is a cut of the inner boundary with no reentrant geodesics. Hence, by Theorem 4.4 the only zero-compatible data on $C$ is zero. In particular, the boundary data for our problem force the voltage of $Q$ to be zero. $\Gamma_{U \cup V}$ is the same as $\Gamma_W$ except with a stub $P'$ on the inner boundary. Thus, to have zero-compatible data on $C'$, we need all voltages and currents zero except that $P'$ can have whatever voltage we want. Thus, we can set the voltage of $P'$ to 1.

Finally, notice that for $\Gamma_U$, $C'$ is a cut of the boundary with no reentrant geodesics, so any boundary data is feasible, and in particular, we can have zero voltage and current everywhere except voltage 1 on $P'$. Thus, for $\Gamma$, there is zero-compatible data on $R$ with voltage 1 on $P'$, which is the same thing as voltage 1 on $P$.

Therefore, there is a boundary value problem which forces the voltage of $Q$ to be zero and the voltage of $P$ to be 1. Knowing $F$, we can compute the current on the boundary interval of $P$, and it is equal to the conductance of $PQ$.

Lemma 4.6. Suppose $\Gamma$ has no removable lenses or self-intersecting type 1 inner geodesics and suppose Theorem 3.10 holds. $PQ$ be a juncture between
inner boundary plates $P$ and $Q$. Suppose that $g$, one of the geodesics at $PQ$, is type 1 and is not part of a two-pole lens. Suppose there is type 2 geodesic which does not intersect $g$. Then the conductance of $PQ$ is recoverable.

**Proof.** Assume without loss of generality that $P$ is inside $g$ and $Q$ is outside $g$. Construct $R$, $C$, $C'$, $U$, $V$, and $W$ as in the previous proof. Consider the boundary value problem with all voltages and currents zero on $RC$ and voltage 1 on $P$. By the same argument, this problem has a solution, and all voltages and currents on $\Gamma_U$ are zero. In particular, the current is zero on all junctures of $Q$ other than $PQ$. Knowing $F$, we can find the boundary current on $Q$, and it is negative the conductance of $PQ$. □

**Theorem 4.7.** Let $\Gamma$ be a network in layered form with no removable lenses and no lenses involving type 1 geodesics. Suppose Theorem 3.10 holds. Suppose that for every type 1 geodesic there is type 2 geodesic which does not intersect it. Then all conductances in the layers with type 1 geodesics can be recovered.

**Proof.** Consider the type 1 geodesics on the inner boundary first. Let $\Gamma_{\text{inner}}$ be the layer which contains all the type 1 inner geodesics. By Theorem 2.10, $\Gamma_{\text{inner}}$ has an empty boundary triangle or stub. Remove all stubs from the network. Then there is an empty boundary triangle, and one of the geodesics must be type 1. By the previous two lemmas, the conductance of the juncture at this triangle is recoverable.

Uncross the triangle and update the solution spaces (or response matrix) to match the modified network. The modified network will still satisfy the hypotheses of this theorem. Repeat the above argument and keep removing stubs and uncrossing empty boundary triangles until all conductances in the layer are recovered. The conductances in the layer with type 1 outer geodesics can be recovered in the same way. □

This theorem has implications for networks which are not in layered form because any network with no removable lenses can be put into layered form by $Y$-$\Delta$ transformations. Knowing the conductances of a $Y$-$\Delta$-equivalent network is (theoretically) just as good as knowing the conductances of the original network. However, at this point we have only partially recovered the network. After removing the type 1 geodesics, we still have to recover the conductances of the remaining network, and that is the focus of the next section.
5 Radial Networks

A radial network is an annular network with only type 2 geodesics. Radial networks with no removable lenses have special properties, as we would expect from the theorems of §3. They can be partitioned into elementary boundary-juncture and spike networks (Lemma 3.3). There is a $k$-connection from all the plates on the inner boundary to all plates on the outer boundary (Theorem 3.6). Any data is feasible on the inner boundary or the outer boundary, and the only zero-compatible data is zero (Theorem 3.10). Thus, complete voltage and current data on one boundary curve determines a unique $\gamma$-harmonic function on the network.

In this section, I explore the geometric and electrical properties of radial networks in greater depth and prove recoverability for a certain class of networks.

5.1 Structure

In a radial network, there is a canonical way to classify the junctures. Orient all the geodesics so the positive direction moves from the inner to the outer boundary. At each juncture point $y$, designate the four edges as

- a counterclockwise inner edge $e_1$,
- a counterclockwise outer edge $e_2$,
- a clockwise inner edge $e_3$,
- a clockwise outer edge $e_4$,

where $e_1$ and $e_4$ appear in that order on one of the geodesics, $e_2$ and $e_3$ appear in that order on the other geodesic, and the counterclockwise ordering of the four edges about the point $y$ is $e_1, e_2, e_4, e_3$.

There are two possibilities:

1. $e_1$ and $e_2$ are edges of the same plate and $e_3$ and $e_4$ are edges of the same plate. In this case, $y$ is called a counterclockwise-clockwise juncture.

2. $e_1$ and $e_3$ are edges of the same plate and so are $e_2$ and $e_4$. Then $y$ is called an inward-outward juncture.

By Lemma 3.3, a radial network with no removable lenses can be partitioned into boundary-juncture and boundary-spike networks. In any partition, the counterclockwise-clockwise junctures are part of an elementary
boundary-juncture network, and the inward-outward edges are part of an
elementary spike network. As a result,

**Theorem 5.1.** In a radial network with no removable lenses, there is ex-
actly one $k$-connection between $P_{\text{inner}}$ and $P_{\text{outer}}$, and the junctures in the
connection are exactly the inward-outward junctures.

**Proof.** Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_M$ be a partition of $\Gamma$ where $\Gamma_0$ is a trivial network
and $\Gamma_1, \ldots, \Gamma_M$ are elementary boundary-juncture and spike networks listed
from innermost to outermost. Let $\Sigma_m$ be the subnetwork consisting of
$\Gamma_0, \ldots, \Gamma_m$. We prove the theorem by induction for each $\Sigma_m$. The base
case is trivial.

Suppose the theorem is true of $\Sigma_m$. If $\Gamma_{m+1}$ is a spike network, then
its juncture is inward-outward. There is exactly one $k$-connection from the
inner to the outer boundary of $\Sigma_m$. To find the $k$-connection for $\Sigma_{m+1}$, sim-
ply add the spike of $\Gamma_{m+1}$ to the appropriate path. If $\Gamma_{m+1}$ is a boundary-
juncture network, then its juncture is counterclockwise-clockwise. The junc-
ture cannot be used in an interboundary $k$-connection in $\Sigma_{m+1}$ because it
is a boundary juncture. Thus, the set of interboundary $k$-connections for
$\Sigma_{m+1}$ is the same as that of $\Sigma_m$. $\square$

Each path in this single connection will be called a ray (by analogy with the
“circles and rays” networks of [2]). We will index the rays $1, \ldots, N$
in some counterclockwise order. The collection of plates along the $n$th ray
will be called $P_n$, and the plates of $P_n$ will be $P_{n,1}, P_{n,2}, \ldots, P_{n,K_n}$, ordered
from innermost to outermost. In the following sections, we assume that
the indexing of the inner and outer boundary plates is consistent with the
indexing of the rays. We label the inner boundary intervals $I_1, \ldots, I_N$, and
the outer boundary intervals $J_1, \ldots, J_N$.

### 5.2 Pseudo-Geodesics and Dominant Geodesics

Let $y_0$ be a geodesic endpoint on the inner boundary. Let $e_1$ be a plate edge
with endpoint $y_0$. Let $y_1$ the other endpoint of $e_1$. For $k = 1, 2, \ldots$, let $e_{k+1}$
be the counterclockwise outer edge at $y_k$, and let $y_{k+1}$ be the other endpoint
of $e_{k+1}$. Continue inductively until $y_K$ is on the outer boundary. The
curve formed by $e_1, \ldots, e_K$ is a **counterclockwise outward** pseudo-geodesic.
A clockwise outward pseudo-geodesic is defined in a symmetrical way. For **counterclockwise** and **clockwise inward** pseudo-geodesics, we perform the
same process but begin at the outer boundary and choose counterclockwise
or clockwise **inner** edges.
Figure 18: Geodesics in the universal cover. A counterclockwise outward pseudo-geodesic is shown in blue. $g_2$ and $g_8$ are counterclockwise-dominant; $g_1$, $g_4$, $g_5$, and $g_6$ are clockwise-dominant.

A geodesic is called *counterclockwise-dominant* if it is identical to a counterclockwise outward pseudo-geodesic. Equivalently, $g$ is counterclockwise-dominant if for every geodesic $h$ which crosses $g$, $g$ crosses $h$ counterclockwise and $h$ crosses $g$ clockwise. Similarly, a geodesic is *clockwise-dominant* if it is identical to a clockwise outward pseudo-geodesic.

The slant of a geodesic or pseudo-geodesic is defined as follows. Let the geodesic endpoints on the upper and lower boundaries of the universal cover be $y_j$ and $z_j$ for all integers $j$. Index them from left to right, and such that $y_j$ and $z_j$ are on the same ray in the universal cover and the same side of the ray. Let $g$ be a (pseudo-)geodesic with endpoints $y_i$ and $z_j$. The slant of $g$ is $j - i$.

The outward counterclockwise pseudo-geodesic beginning at $y$ is the path of maximal slant out of all paths consisting of positively oriented edges which begin at $y$ and end at the outer boundary.

As we will see in the next section, pseudo-geodesics and dominant geodesics are important for analyzing information propagation and recoverability. To do so, we need the following results:

**Lemma 5.2.** Let $h$ be a counterclockwise outward pseudo-geodesic. Let $g_1, \ldots, g_K$ be the geodesics which share edges with $h$, listed in outward order along $h$. Then $g_K$ is counterclockwise-dominant; $g_K$ is either the same as $g_1$ or the first counterclockwise-dominant geodesic intersected by $g_1$. 

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Proof. Consider the geodesics in the universal cover. The lower endpoint of $h$ must lie to the left of the lower endpoint of each $g_k$. If $h$ crosses any geodesic $g$, then $h$ must cross $g$ counterclockwise or $g$ must intersect $h$ in an edge. Suppose that some geodesic $g^*$ intersects $g_K$ clockwise. Then the endpoint of $g^*$ is to the left of the endpoint of $g_K$, so $g^*$ must not intersect $h$ in an edge. Thus, $g^*$ must cross $h$ counterclockwise, which is impossible. Therefore, $g_K$ is counterclockwise-dominant.

Suppose $g^*$ is a counterclockwise-dominant geodesic such that $g_1$ intersects $g^*$ before $g_K$. Then $g^*$ crosses $h$ counterclockwise. The argument of the previous paragraph shows this is impossible.

**Corollary 5.3.** The endpoints and slants of pseudo-geodesics are unaffected by $Y$-$\Delta$ transformations.

Proof. The endpoint of $h$ is determined by $g_K$, the first counterclockwise-dominant geodesic which is intersected by $g_1$. No two counterclockwise-dominant geodesics intersect because if they did, then one would cross the other counterclockwise. Also, $g_1$ must cross any counterclockwise-dominant geodesic in the clockwise direction. Thus, $Y$-$\Delta$ transformations cannot change the order in which $g_1$ intersects these counterclockwise-dominant geodesics. Therefore, they do not affect which geodesic is $g_K$.

A symmetrical argument holds for the other types of pseudo-geodesics.

**Lemma 5.4.** Let $g_1, \ldots, g_{2N}$ be the geodesics of $\Gamma$, ordered counterclockwise by their endpoints on the outer boundary. If $\Gamma$ is not the trivial network, there exists a $j$ such that $g_j$ is counterclockwise-dominant and intersects $g_{j-1}$ (indices reduced modulo $2N$).

Proof. Suppose that no such geodesic exists. Let $g_J$ be a geodesic of maximal slant. If a geodesic $h$ intersected $g_J$ counterclockwise, then $h$ would have greater slant. Thus, $g_J$ is counterclockwise-dominant. Since $g_{J-1}$ does not intersect $g_J$, the slant of $g_{J-1}$ is greater than or equal to the slant of $g_J$, so $g_{J-1}$ must also have maximal slant. Proceeding by induction, we see that all the geodesics have maximal slant and are counterclockwise-dominant, which is impossible unless $\Gamma$ is the trivial network.

**Lemma 5.5.** A nontrivial radial network with no removable lenses is $Y$-$\Delta$ equivalent to one which has an empty boundary triangle on the outer boundary, such that one of the geodesics forming the triangle is counterclockwise-dominant.
Proof. Let \( g_j \) be a counterclockwise-dominant geodesic which intersects \( g_{j-1} \). Let \( y \) be their point of intersection closest to the outer boundary. Then there is a triangle \( T \) with a vertex at \( y \) formed by \( g_j, g_{j-1} \), and an arc of the outer boundary, such that \( T \) lies within a simply connected region of the network. Any geodesic which intersects \( T \) must enter \( T \) along \( g_{j-1} \) and exit along \( g_j \). By \( Y-\Delta \) transformations, we can remove all crossings out of \( T \). After that, we can empty \( T \) of all geodesics. Then \( T \) is an empty boundary triangle. \( \square \)

5.3 Principal Electrical Functions

For any inner boundary plate \( P_{j,1} \), there is a unique \( \gamma \)-harmonic function \( f \) with \( v(P_{j,1}) = 1 \) and all other voltages and all currents on the inner boundary equal to zero. We will call \( f \) the principal electrical function for \( P_{j,1} \).

Let \( y_A \) and \( y_B \) the the clockwise and counterclockwise endpoints of \( I_j \). Let \( h_A \) be the outward clockwise pseudo-geodesic with endpoint \( y_A \) and let \( h_B \) be the outward counterclockwise pseudo-geodesic with endpoint \( y_B \). Let \( R \) be the arc of the inner boundary complementary to \( I_j \). Let \( S \) be the region of embedding.

For \( P_{j,1} \), the zone of no propagation \( U \) is the component of \( S \setminus (h_A \cup h_B) \) which touches the inner boundary and does not intersect \( P_{j,1} \). The zone of propagation \( V \) is \( S \setminus U \). Let \( Q_A \) and \( Q_B \) be the outer boundary plates along \( h_A \) and \( h_B \). See Figure 19 (\( W \) is explained later).

The zone of propagation is called simple if it is simply connected and \( Q_A \neq Q_B \). This means that if either \( Q_A \) or \( Q_B \) is in \( U \), then there must be at least one geodesic endpoint outside of \( V \).

Theorem 5.6. Let \( f \) be the principal electrical function for \( P_{j,1} \). Suppose the zone of propagation is simple and that the rays are indexed with \( Q_B = P_{1,K_1} \). If \( P_{j,1} \) has at least two junctures, then

- If \( P_{n,k} \) intersects \( U \), then \( v(P_{n,k}) = 0 \).
- If \( P_{n,k} \subset \overline{V} \), then \((-1)^{n+j}v(P_{n,k}) > 0\).
- If the juncture \( P_{n,k}P_{n,k+1} \in U \), then \( c(P_{n,k} \to P_{n,k+1}) = 0 \).
- If \( P_{n,k}P_{n,k+1} \in V \), then \((-1)^{n+j}c(P_{n,k} \to P_{n,k+1}) < 0 \).
- If \( J_n \subset \overline{U} \), then \( c(J_n) = 0 \).
- If \( J_n \) intersects \( \overline{V} \), then \((-1)^{n+j+1}c(J_n) < 0 \).
Figure 19: Zones of propagation for an inner boundary plate.
Otherwise, we make the following exceptions:

- If $P_{j,1}$ touches both boundaries and has no junctures, then $c(J_1) = 0$.
- If $P_{j,1}$ is a boundary spike, then $c(P_{j,1} \rightarrow P_{j,2}) = 0$.
- If $P_{j,1}$ is a boundary spike, and $P_{j,2}$ is on the outer boundary and has no junctures besides $P_{j,1}P_{j,2}$, then $c(J_1) = 0$.

Roughly speaking, this means that $f$ is zero in the zone of no propagation, and in the zone of propagation, the voltages are positive and increasing or negative and decreasing on each ray, in an alternating pattern.
Proof. Let $\Gamma_0, \ldots, \Gamma_M$ be a partition of $\Gamma$ where $\Gamma_0$ is a trivial network and $\Gamma_1, \ldots, \Gamma_M$ are elementary boundary-juncture and spike networks. Let $\Sigma_m$ consist of $\Gamma_0, \ldots, \Gamma_m$. Let $J_{n,m}$ be the outer boundary interval for $\Sigma_m$ on the $n$th ray. Notice that the zone of no propagation for $\Sigma_m$ is exactly the part of $U$ which intersects $\Sigma_m$’s region of embedding, and the same is true for the zone of propagation.

We show that the theorem is true for $\Sigma_m$ by induction. The base case is trivial. Next, we show that if the theorem holds for $\Sigma_m$, it holds for $\Sigma_{m+1}$.

Suppose $\Gamma_{m+1}$ is a spike network with juncture $P_{n,k}P_{n,k+1}$. There are several cases:

1. $P_{n,k} = P_{j,1}$ and $P_{j,1}$ is a spike. In this case, $c(J_{n,m}) = 0$, so $c(P_{n,k} \to P_{n,k+1}) = 0$ and $v(P_{n,k}) = 1 > 0$.

2. $P_{n,k} = P_{j,2}$ and $P_{j,1}$ is a spike. We know that $P_{j,2}$ has junctures with other plates besides $P_{j,1}$ and $P_{j,3}$ because otherwise it would form a series connection, which corresponds to a removable lens. Thus, by inductive hypothesis, $c(J_{n,m}) < 0$. For the rest of the argument, see the next case.

3. $P_{n,k}P_{n,k+1} \in V$. Then $(-1)^{n+j}c(P_{n,k} \to P_{n,k+1}) = (-1)^{n+j+1}c(J_{n,m})$, which is negative by hypothesis because $J_{n,m}$ intersects $\overline{V}$. Also by hypothesis, $(-1)^{n+j}v(P_{n,k}) \geq 0$, and $(-1)^{n+j}v(P_{n,k+1}) > (-1)^{n+j}v(P_{n,k})$ because of the sign of $c(P_{n,k} \to P_{n,k+1})$.

4. $P_{n,k}P_{n,k+1} \in U$. In this case, $J_{n,m} \subset U$, so its current is 0 by inductive hypothesis. This implies $c(P_{n,k} \to P_{n,k+1}) = 0$. Since $P_{n,k}$ intersects $U$, its voltage is zero, so $v(P_{n,k+1}) = 0$.

Suppose $\Gamma_{m+1}$ is a boundary juncture network with juncture $P_{n,k}P_{n+1,k'}$ with $1 \leq n \leq N - 1$. There are two cases:

1. In the case where $P_{n,k}P_{n+1,k'} \in U$, both plates intersect $U$, so their voltages are zero. Thus, the outer boundary data of $\Sigma_{m+1}$ is the same as that of $\Sigma_m$.

2. Suppose $P_{n,k}P_{n+1,k'} \in V$. The outer boundary voltages for $\Sigma_{m+1}$ are the same as for $\Sigma_m$. By inductive hypothesis, $(-1)^{n+j}v(P_{n,k}) \geq 0$ and $(-1)^{n+j+1}v(P_{n+1,k'}) \geq 0$. At least one of the plates is in the zone of propagation, so at least one of the inequalities is strict. This implies $(-1)^{n+j}c(P_{n,k} \to P_{n+1,k'}) > 0$. Both boundary intervals $J_{n,m+1}$ and $J_{n+1,m+1}$ intersect $\overline{V}$. Because of the sign of $c(P_{n,k} \to P_{n+1,k'})$,

$$(-1)^{j+n+1}c(J_{m+1}) > (-1)^{j+n+1}c(J_{m,m}),$$

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which is nonnegative by hypothesis. Similarly,

\[ (-1)^{j+n}c(J_{n+1,m+1}) > (-1)^{j+n}c(J_{n+1,m}) \geq 0. \]

Finally, if \( \Gamma_{m+1} \) is a boundary-juncture network with juncture \( P_{N,k}P_{1,k'} \), then the juncture is in \( U \) because of the zone of propagation is simple and because of how we indexed the rays. This completes the induction argument and hence the proof. \( \square \)

The principal electrical function for an inner boundary interval \( I_j \) is defined similarly. It is the \( \gamma \)-harmonic function \( f \) with \( c(I_j) = 1 \) and all other voltages and all currents on the inner boundary equal to zero. We let \( y_A \) and \( y_B \) be the clockwise and counterclockwise endpoints of \( I_j \). Counterintuitively, \( h_A \) is the clockwise outward pseudo-geodesic with endpoint at \( y_B \) and \( h_B \) is the counterclockwise outward pseudo-geodesic with endpoint at \( y_A \).

If \( P_{j,1} \) touches both boundaries, then \( h_A \) and \( h_B \) do not intersect, and we define the zone of no propagation as \( S \) and the zone of propagation as \( J_j \). Otherwise, \( h_A \) and \( h_B \) intersect at \( P_{j,1}P_{j,2} \). We let \( h'_A \) and \( h'_B \) be the arcs of \( h_A \) and \( h_B \) starting at \( P_{j,1}P_{j,2} \) and ending at the outer boundary. Then \( U \), the zone of no propagation, is the component of \( S \setminus (h'_A \cup h'_B) \) which touches the inner boundary, and \( V = S \setminus U \). (The picture is the same as Figure 19 except that \( h_A \) and \( h_B \) are crossed at their inner endpoints.)

Then we have the following theorem. The proof is essentially the same as for Theorem 5.6, so the details are left to the reader:

**Theorem 5.7.** Let \( f \) be the principal electrical function for \( I_j \). Suppose the zone of propagation is simple and that the rays are indexed with \( Q_B = P_{1,K_1} \). Then

- If \( P_{n,k} \) intersects \( U \), then \( v(P_{n,k}) = 0 \).
- If \( P_{n,k} \subset V \), then \( (-1)^{n+j}v(P_{n,k}) < 0 \).
- If the juncture \( P_{n,k}P_{n,k+1} \in U \), then \( c(P_{n,k} \rightarrow P_{n,k+1}) = 0 \).
- If \( P_{n,k}P_{n,k+1} \in V \), then \( (-1)^{n+j}c(P_{n,k} \rightarrow P_{n,k+1}) > 0 \).
- If \( J_n \subset \overline{U} \), then \( c(J_n) = 0 \).
- If \( J_n \) intersects \( V \), then \( (-1)^{n+j+1}c(J_n) > 0 \).

The last two theorems hold in more generality when \( N \) is even:
**Corollary 5.8.** If $N$ is even, Theorems 5.6 and 5.7 hold even if the zone of propagation is not simple.

**Proof.** Examine the proof of Theorem 5.6. The only case that required the assumption that the region of propagation was simple was the last case where $\Gamma_{m+1}$ was a boundary-juncture network with juncture $P_{N,k}P_{1,k'}$. If we remove that assumption, $P_{N,k}P_{1,k'}$ may be in the zone of propagation. If $N$ is odd, $v(P_{N,k})$ and $v(P_{1,k'})$ have the same sign, so we do not know the sign of $c(P_{N,k} \rightarrow P_{1,k'})$. However, if $N$ is even, the two voltages have opposite signs, and we can apply the argument given in subcase 2 of the case when $\Gamma_{m+1}$ is a boundary-juncture network.

For the odd case, the theorems do not hold in general, but there is a subregion of the network for which they do hold. We define the zone of simple propagation $W$ as follows. If the zone of propagation is simple, $W = V$. Otherwise, if the zone of propagation is simply connected but not simple, then $Q_A = Q_B$ and $W$ is defined to be $V$ minus the outer boundary interval of $Q_A$. Otherwise, if the zone is for a plate, let $h''_A$ and $h''_B$ be the arcs $h_A$ and $h_B$ from their first point of intersection to the outer boundary; if the zone is for a boundary interval, let $h''_A$ and $h''_B$ be the arcs of $h_A$ and $h_B$ from their second point of intersect to the outer boundary. Let $X$ be the component of $S \setminus (h''_A \cup h''_B)$ which touches the inner boundary, and let $W = X \setminus U$.

**Corollary 5.9.** Theorems 5.6 and 5.7 hold in the zone of no propagation and the zone of simple propagation.

**Proof.** If the whole zone of propagation is simple, then $W = V$, so we are done. Otherwise, consider a subnetwork of $\Gamma$ which intersects all the plates and contains all the junctures in the zone of no propagation and the zone of simple propagation, but no other plates and junctures. Apply the theorems to this subnetwork.

### 5.4 Factorization of the Interboundary Map

For each $y \in \mathbb{R}^{2N}$ representing data on one boundary, let $\tilde{y}$ be the vector with the signs of the current entries changed. That is, if $y = y_1, \ldots y_N, y_{N+1}, \ldots, y_{2N}$, then $\tilde{y} = y_1, \ldots, y_N, -y_{N+1}, \ldots, -y_{2N}$. The current entries for $y$ represent current entering the network; the current entries for $\tilde{y}$ represent current exiting the network.

Let $x \in \mathbb{R}^{2N}$ be a vector representing voltage and current data on the inner boundary of a radial network $\Gamma$ with no removable lenses. There is a
unique \( y \) representing data on the outer boundary which is compatible with \( x \). There is an invertible linear transformation \( \Xi \) mapping \( x \) to \( \tilde{y} \), which we will call the interboundary map. By direct computation,

**Proposition 5.10.** In a radial network with no removable lenses, if no plate touches both boundaries, then

\[
\Xi = \begin{pmatrix} -\Lambda_{II}^{-1}\Lambda_{II} & \Lambda_{IO}^{-1} \\ -\Lambda_{OI} + \Lambda_{OO} \Lambda_{II}^{-1} \Lambda_{ii} & -\Lambda_{OO} \Lambda_{IO}^{-1} \end{pmatrix}.
\]

Each column of \( \Xi \) represents outer boundary data for one of the principal electrical functions. Thus, the theorems of the previous sections provide sign conditions on the entries of \( \Xi \). \( \Xi \) behaves nicely with respect to partitions:

**Theorem 5.11.** Let \( \Gamma_1, \ldots, \Gamma_M \) be a partition of \( \Gamma \) into elementary boundary-juncture and spike networks, with boundary plates indexed according to the rays of \( \Gamma \). Then \( \Xi = \Xi_M \Xi_{M-1} \cdots \Xi_1 \).

**Proof.** Let \( \Gamma_1, \ldots, \Gamma_M \) be a partition of \( \Gamma \) into elementary boundary-juncture and spike networks. Let \( \Sigma_m \) consist of \( \Gamma_0, \ldots, \Gamma_m \). We prove the theorem for each \( \Sigma_m \) by induction. The base case \( \Sigma_1 \) is trivial.

Suppose the claim is true for \( \Sigma_m \), and I will show it is true for \( \Sigma_{m+1} \). Let \( Q_{n,m} \) and \( J_{n,m} \) be the outer boundary plate and interval for \( \Sigma_m \) on the \( n \)th ray, and let \( R_{n,m} \) and \( I_{n,m} \) be the inner boundary plate and interval for \( \Gamma_m \).

Let \( x \) be a vector of inner boundary data for \( \Sigma_{m+1} \), which determines a unique \( \gamma \)-harmonic function \( f \). Then \( \Xi_m \Xi_{m-1} \cdots \Xi_1 x \) gives the data on the outer boundary of \( \Sigma_m \): voltages for \( Q_{1,m}, \ldots, Q_{N,m} \) and minus the current on \( I_{1,m}, \ldots, I_{N,m} \). Since \( Q_{n,m} \) and \( R_{n,m+1} \) are subplates of the same plate in \( \Sigma_{m+1} \), we need \( \nu(R_{n,m+1}) = \nu(Q_{n,m}) \) and \( c(I_{n,m+1}) = -c(J_{n,m}) \). Thus, the data on the inner boundary of \( \Gamma_{m+1} \) is \( \Xi_m \Xi_{m-1} \cdots \Xi_1 x \), so the data on the outer boundary of \( \Sigma_{m+1} \) is \( \Xi_{m+1} \Xi_m \Xi_{m-1} \cdots \Xi_1 x \).

We can think of \( \Xi \) as a matrix with rows and columns indexed by integers \( 1, \ldots, 2N \). Alternatively, we can index the rows and columns by the boundary plates and the boundary intervals, where the plates correspond to the voltage data and the intervals to the current data. That is, if the inner and outer boundary plates are \( R_n = P_{n,1} \) and \( Q_n = P_{n,K_n} \), then for
For $k, \ell = 1, \ldots, N$, we write

$$
\xi_{\ell,k} = \xi(Q_\ell, R_k)
$$

$$
\xi_{\ell,N+k} = \xi(Q_\ell, I_k)
$$

$$
\xi_{N+\ell,k} = \xi(J_\ell, R_k)
$$

$$
\xi_{N+\ell,N+k} = \xi(J_\ell, I_k).
$$

Partition into elementary networks corresponds to factorization of $\Xi$. Each factor is a nearly-elementary matrix. For instance, consider an elementary spike network with inner boundary plates where $R_n = Q_n$ for $n \neq 1$ and $R_1$ and $Q_1$ are connected by a juncture with conductance $\gamma$. For $n \neq 1$, we need $v(Q_n) = v(R_n)$. Since the currents on $R_n$ must add up to zero, $c(J_n) = -c(I_n)$ for $n \neq 1$. Also, $c(I_1) = c(R_1 \rightarrow Q_1) = -c(J_1)$ and, since $c(R_1 \rightarrow Q_1) = \gamma(v(R_1) - v(Q_1))$,

$$
v(Q_1) = v(R_1) - \frac{1}{\gamma}c(I_1).
$$

Thus, $\xi(Q_n, R_n) = 1$ and $\xi(J_n, I_n) = 1$ for all $n$, $\xi(Q_1, I_1) = -1/\gamma$, and all other entries of $\Xi$ are zero. For example, when $N = 3$,

$$
\Xi = \begin{pmatrix}
1 & 0 & 0 & -1/\gamma & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Now consider an elementary boundary-juncture network where the juncture connects $P_1$ and $P_2$. Since $R_n = Q_n$ for all $n$, we need $v(R_n) = v(Q_n)$. For $n \neq 1, 2$, $c(J_n) = -c(I_n)$. Since $c(I_1) + c(R_2 \rightarrow R_1) + c(J_1) = 0$ and $c(R_2 \rightarrow R_1) = \gamma(v(R_2) - v(R_1))$ for any $\gamma$-harmonic function, we have

$$
-c(J_1) = c(I_1) + \gamma v(R_2) - \gamma v(R_1).
$$

Similarly,

$$
-c(J_2) = c(I_2) + \gamma v(R_1) - \gamma v(R_2).
$$

Thus, $\xi(Q_n, R_n) = 1$ and $\xi(J_n, I_n) = 1$ for all $n$, $\xi(J_1, R_2) = \xi(J_2, R_1) = \gamma$, $\xi(J_1, R_1) = \xi(J_2, R_2) = -\gamma$, and all other entries of $\xi$ are zero. For example,
when \( N = 3 \),

\[
\Xi = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
-\gamma & \gamma & 0 & 1 & 0 \\
\gamma & -\gamma & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

If the juncture is at a different place, the matrix has a similar form, only the rows and columns are permuted by shifting the indices some amount modulo \( N \).

Each “elementary” matrix has determinant 1, so \( \det \Xi = 1 \).

When we modify \( \Gamma \) by contracting a spike or deleting a boundary juncture, \( \Xi \) is easy to update. Suppose that \( \Gamma \) is partitioned into elementary networks \( \Gamma_1, \ldots, \Gamma_M \) where \( \Gamma_M \) is an elementary spike network. Suppose that \( \Gamma' \) is obtained from \( \Gamma \) by contracting the spike. Then \( \Xi = \Xi_M \Xi_{M-1} \cdots \Xi_1 \), and \( \Xi_M \). Contracting the spike, or transforming \( \Gamma \) into \( \Gamma' \), is equivalent to replacing \( \Gamma \) with the subnetwork consisting of \( \Gamma_1, \ldots, \Gamma_{M-1} \). Hence, the interboundary map for \( \Gamma' \) is

\[
\Xi' = \Xi_{M-1} \Xi_{M-2} \cdots \Xi_1 = \Xi_M^{-1} \Xi.
\]

If \( Q \) is the spike, \( PQ \) is its juncture, and \( I \) is the boundary interval, then \( \Xi_M \) is an elementary matrix which corresponds to the row operation of subtracting \( 1/\gamma(PQ) \) times row \( I \) to row \( Q \). \( \Xi_M^{-1} \) corresponds to the reverse row operation. Contracting a spike on the inner boundary is similar, except that we right-multiply by the appropriate matrix (perform a column operation).

Deleting a boundary edge is similar. If \( \Gamma_M \) is an elementary boundary-juncture network, then \( \Xi_M \) has four off-diagonal entries, which correspond to four row operations. Thus, the inverse matrix also corresponds to four row operations.

### 5.5 Recovery

We can recover certain boundary edges and spikes using the interboundary map and the principal electrical functions. Consider a boundary plate \( P_{3,1} \). Let \( h_A \) and \( h_B \) be the corresponding pseudo-geodesics, and let \( Q_A \) and \( Q_B \) be as in §5.3, and let \( f \) be the principal electrical function for \( P_{3,1} \). Suppose that \( Q_B = P_{i,K_i} \) is a boundary spike. If the zone of propagation is simple, then \( P_{i,K_i-1} \) intersects the zone of no propagation, so its voltage is zero. But we know by Theorem 5.6 that the voltage and current of \( P_{i,K_i} \) are nonzero. Thus, we can determine the conductance of the spike.
The outer boundary data for \( f \) is represented by the \( j \)th column of \( \Xi \). Thus,

\[
\gamma(P_{i,K_i} P_{i+1,K_{i+1}}) = c(P_{i,K_i} \rightarrow P_{i+1,K_{i+1}}) \frac{-c(J_i)}{-v(P_{i,K_i})} = -\frac{\xi_{N+i,j}}{\xi_{i,j}}.
\]

The conductance of the spike is the quotient of two entries of \( \Xi \).

The same reasoning applies if \( f \) is the principal electrical function for \( I_j \) and \( Q_B = P_{i,K_i} \) is a boundary spike. If the zone of propagation is simple, then

\[
\gamma(P_{i,K_i} P_{i+1,K_{i+1}}) = -\frac{\xi_{i,j}}{\xi_{i+1,j}}.
\]

In fact, this method will work in a slightly more general case. If the zone of propagation is not simple, but if \( P_{i,K_i} \) is the only plate not in the zone of simple propagation, then \( P_{i,K_i} \) will still have voltage zero. If \( P_{i,K_i} \) has nonzero voltage (which will happen if \( N \) is even), then the above formulae are still valid.

Now consider the case of a boundary juncture. Let \( f \) be the principal electrical function for \( P_{j,1} \). Suppose \( P_{i,K_i} P_{i+1,K_{i+1}} \) is a boundary juncture and \( Q_B = P_{i,K_i} \), and that the zone of propagation is simple. Then \( P_{i,K_i} \) and all junctures of \( P_{i+1,K_{i+1}} \) except \( P_{i,K_i} \) are in the zone of no propagation. Thus, \( v(P_{i,K_i}) = 0 \) and \( c(J_i) = c(P_{i,K_i} \rightarrow P_{i+1,K_{i+1}}) \neq 0 \). Thus,

\[
\gamma(P_{i,K_i} P_{i+1,K_{i+1}}) = \frac{c(J_i)}{-v(P_{i+1,K_{i+1}})} = \frac{\xi_{N+i,j}}{\xi_{i+1,j}}.
\]

Similarly, if \( f \) is the principal electrical function for an interval \( I_j \) instead of a plate \( P_{j,1} \), then

\[
\gamma(P_{i,K_i} P_{i+1,K_{i+1}}) = \frac{\xi_{N+i,N+j}}{\xi_{i+1,N+j}}.
\]

Unfortunately, this method does not work if the zone of propagation is not simple. Even the “simplicity” requirement is stronger for the boundary juncture than for the spike. If \( Q_B \) is on the clockwise side of \( h_B \) as in the case with the spike, then it merely requires that the outer endpoint of \( h_B \) is clockwise from the outer endpoint of \( h_A \) (that is, clockwise as we pass through the zone of no propagation). However, if \( Q_B \) is on the counterclockwise side of \( h_B \), the endpoint of \( h_B \) must be clockwise from the outer endpoint of \( h_A \) and there must be at least one geodesic endpoint in between them.

To discuss recoverability globally, we use the lemmas of §5.2.
Theorem 5.12. Let $\Gamma$ be a radial network with no removable lenses. Suppose that for each inner boundary plate and boundary interval, the zone of propagation is simple. Then $\Gamma$ is recoverable.

Proof. To determine the conductances of $\Gamma$, it is sufficient to determine the conductances of a $Y-\Delta$ equivalent network. We recover conductances in the following way:

1. Use $Y-\Delta$ transformations to change $\Gamma$ into a network with an empty boundary triangle at which one of the geodesics, $g$, is counterclockwise-dominant. This is possible by Lemma 5.5. By Corollary 5.3, the endpoints of pseudo-geodesics are not affected by $Y-\Delta$ transformations, so the zones of propagation are still simple.

2. Let $P_{j,1}$ be the plate at the inner endpoint of $g$. If $P_{j,1}$ is on the clockwise side of $g$, we can use the principal electrical function of $P_{j,1}$ to recover the conductance of spike or boundary juncture at the empty boundary triangle. If $P_{j,1}$ is on the counterclockwise side of $g$, then we can use the principal electrical function of $I_j$.

3. Uncross the empty boundary triangle. Perform one or four row operations on $\Xi$ to find the interboundary map of the modified network. In the modified network the zones of propagation are still simple. This is because uncrossing an empty boundary triangle can only decrease the slant of counterclockwise outward pseudo-geodesics and increase the slant of clockwise outward pseudo-geodesics.

4. Repeat the process. The modified network still satisfies the hypotheses of the theorem.

Eventually, all junctures have been removed and all conductances recovered. $\Gamma$ has been transformed into a trivial network, and $\Xi$ has been row-reduced to the identity matrix.

Of course, there is an analogous theorem with the roles of the inner and outer boundaries reversed.

6 Open Problems

6.1 Recoverability

The recoverability results of this paper are incomplete in several ways. First and most importantly, I have not dealt with networks which have type 0
geodesics. For some results on a particular class of networks with type 0 geodesics, see [4]. For networks with type 0 geodesics, the inverse problem may have *finitely many solutions*, something which does not happen in circular planar networks. In other words, the network is *discretely unrecoverable*.

I conjecture that

- If a type 0 geodesic intersects itself, then the inverse problem has infinitely many solutions.
- If the network is discretely unrecoverable, then there are type 0 geodesics.

Second, Theorem 4.7 and 5.12 are incomplete as recoverability criteria. There are recoverable networks in which type 1 geodesics form lenses. There are recoverable radial networks which cannot be recovered by the method described in this paper. However, I believe that networks which “flagrantly” violate the hypotheses of these theorems are not recoverable.

Third, the recovery methods of this paper relied heavily on $Y$-$\Delta$ transformations. More numerically efficient methods should be explored.

### 6.2 Nonlinear Conductances

We should consider a generalization to nonlinear conductance functions of the type described by Will Johnson in [3]. Like Johnson’s geodesic closures, partition into subnetworks provides a way to analyze how boundary-value information propagates through the network. And partition into subnetworks does not rely on covoltages (which are problematic for non-simply-connected regions).

Solution spaces are well-defined for nonlinear conductances. Certain theorems of this paper state that any data is feasible and the only zero-compatible data is 0 for a certain boundary curve or cut (such as Lemmas 3.9 and 4.2). I believe these theorems will hold for nonlinear conductances with essentially the same proof. In other cases (Theorems 3.10 and 4.1), it is difficult to define the dimension of a solution space, although we can at least say that it has a continuous injective parametrization in a specified number of variables.

I believe Theorem 4.7 (partial recovery by removal of type 1 geodesics) will hold with essentially the same proof. However, widespread application of the theorem requires the ability to perform $Y$-$\Delta$ transformations while computing conductances which make the new network electrically equivalent. To my knowledge, this has not yet been done for nonlinear conductances. The same issue may interfere with the proof of Theorem 5.12 (recoverability of certain radial networks).
The theorems on the principal electrical functions for radial networks will still hold. However, “factorization” of the interboundary map will require care.

It is anyone’s guess what might happen when we put nonlinear conductances on discretely unrecoverable networks.

6.3 More Complicated Regions of Embedding

Consider a generalization to planar regions with an arbitrary number of boundary curves. Such networks will require all the lens removal techniques for annular networks and possibly more, although I suspect the most dramatic contrast is between networks in simply connected regions and networks in non-simply-connected regions. There are probably analogues of Theorem 2.20 (layered form) and Theorem 4.7.

We might also consider networks on a surface. [5] has considered infinite networks in a half-plane, but perhaps annular techniques can be applied to networks in an infinite, non-simply-connected region.
References


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