Learn to Speak Weinberg!

July 22, 2015

Notation

Ω: The null matroid, the unique matroid on the empty set.

$G \text{ ctr } S$: For a graph $G$ and edge set $S$, the graph formed by contracting all edges not in $S$. See contraction.

$G \cdot S$: The reduction of a graph $G$ to an edge set $S$, which is the smallest subgraph containing the edges of $S$.

$G \times S$: This is $(G \text{ ctr } S) \cdot S$. This means removing the isolated vertices of $G \text{ ctr } S$, which equate to connected components in $G$ containing no edge of $S$.

$\mathcal{P}(G)$: The cycle matroid of $G$, or just the set of cycles of $G$. Referred to as the polygon matroid.

$\mathcal{B}(G)$: The cut matroid of $G$, or just the set of bonds of $G$. Referred to as the bond matroid.

$J(b, e)$: The circuit formed by $e$ in $b$. For a basis $b$ and $e \notin b$, this is the unique circuit such that $e \in J(b, e) \subseteq b \cup \{e\}$.

$(e/e')S$: For $e' \in S$, $e \notin S$, the set obtained by replacing $e'$ with $e$ in $S$: $(S - \{e'\}) \cup \{e\}$.

$\alpha(S)$: Cardinality of $S$.

$\overline{S}$: Complement of $S$.

$r(\mathcal{M})$: Rank of a matroid (cardinality of each basis). May be applied to a graph as well, as detailed below.

$\mu(\mathcal{M})$: Nullity of a matroid (cardinality of each cobase). May be applied to a graph as well, as described below.

$C \times S$: For a set $C$ of circuits, this is $C \cap 2^S$, the circuits contained in $S$. 
\( \mathcal{M} \times S \): For a matroid \( \mathcal{M} = (\mathcal{C}, E) \), this is the matroid \( (\mathcal{C} \times S, S) \), the **contraction** of \( \mathcal{M} \) to \( S \). This can also be used to denote the contraction of a vector space (defined below).

\( \mathcal{C} \cdot S \): For a set \( \mathcal{C} \) of circuits, this is the collection of minimal intersections of circuits with \( S \).

\( \mathcal{M} \cdot S \): For a matroid \( \mathcal{M} = (\mathcal{C}, E) \), this is the matroid \( (\mathcal{C} \cdot S, S) \), the **reduction** of \( \mathcal{M} \) to \( S \). This can also be used to denote the reduction of a vector space (defined below).

\(||f||\): The support of a vector \( f \), the set of indices at which its entries are nonzero.

\( \mathcal{C}_V \): The set of supports of elementary vectors in the vector space \( V \) on the index set \( E \).

\( \mathcal{M}_V \): The matroid \( (\mathcal{C}_V, E) \) using the above notation. The **matroid associated with** \( V \).

\( R_f \): Given a vector \( f \), the row vector representation of it.

\( R(S) \): For a matrix \( R \) with columns indexed by \( E \) and \( S \subseteq E \), this denotes the submatrix consisting of the columns indexed by \( S \).

\( \eta(e, v) \): Given a directed graph, this takes an edge \( e \) and a vertex \( v \) and returns 0 if they aren’t incident, 1 if \( v \) is the positive end of \( e \), and \(-1\) if \( v \) is the negative end of \( e \).

\( \perp V \): The orthogonal complement of \( V \).

**Terminology**

**base**: A basis. Similarly, bases are referred to as **bases**.

**binary matroid**: A matroid associated with a binary vector space.

**binary vector space**: A vector space over \( \mathbb{F}_2 \).

**bond**: A minimal cut-set. As defined in the paper, a bond of a graph \( G \) is a set of edges \( S \) such that \( G \times S \) is a bond graph.

**bond graph**: A graph with two vertices and all of its edges between them.

**bond matroid**: The cut matroid, whose circuits are bonds/cut-sets. Denoted \( B(G) \).

**circuit formed by**: For a basis \( b \) and \( e \notin b \), the circuit formed by \( e \) in \( b \) is the unique circuit \( C \) such that \( e \in C \subseteq b \cup \{e\} \). It is denoted \( J(b, e) \).

**cobase**: The complement of a basis.

**coboundary**: For an oriented graph \( G \), this is a vector \( f \) indexed by \( E(G) \) with \( f(e) = \sum_{v \in V(G)} \eta(e, v)g(v) \) for some vector \( g \) indexed by \( V(G) \). This is the image of the transpose of the signed incidence matrix given in the definition of
1-cycles. Finally, note that these vectors correspond to voltage drops satisfying Kirchhoff’s voltage law (where \( g \) corresponds to the voltage function on vertices).

**coboundary space**: The space of coboundaries.

**coforest**: A maximal edge set containing no bonds.

**cographic matroid**: A matroid which is the cycle matroid of some graph.

**complementary orthogonal space**: Orthogonal complement.

**contraction of a graph**: Given a graph \( G \) and set of edges \( S \), the contraction \( G_{\text{ctr}}(S) \) is defined by considering the graph \( H \) formed by removing the edges in \( S \) from \( G \). Then the vertices of \( G_{\text{ctr}}(S) \) are the connected components of \( H \) and the edges are the edges of \( S \), drawn between the components containing their ends in \( G \). Intuitively, this is the result of contracting all edges except those in \( S \).

**contraction of a matroid**: For a matroid \( (\mathcal{C},E) \) and \( S \subseteq E \), \( \mathcal{C} \times S \) is the collection of circuits contained in \( S \). Then \( \mathcal{M} \times S = (\mathcal{C} \times S, S) \) is the contraction of \( \mathcal{M} \) to \( S \).

**contraction of a vector space**: For a subspace \( \mathcal{V} \) of \( F^E \) and \( S \subseteq E \), the contraction \( \mathcal{V} \times S \) is the subspace of all vectors whose entries are zero at indices not in \( S \). This definition is such that \( \mathcal{M} \times S = \mathcal{M} \times S \).

**cotree**: An edge set containing no bonds.

**1-cycle**: Given an oriented graph \( G \), a vector on \( E(G) \otimes F \) (a function \( f : E(G) \to F \)) such that for every vertex \( v \) of \( G \), \( \sum_{e \in E(G)} \eta(e, v) f(e) = 0 \). That is, at each vertex, the sum of the incident function values (respecting orientation) is 0. This can also be viewed as an element of the kernel of the \( |V(G)| \times |E(G)| \) matrix whose entries are given by the \( \eta(e_i, v_j) \)—the signed incidence matrix. Finally, note that these vectors correspond to currents satisfying Kirchhoff’s current law.

**1-cycle space**: The vector space of 1-cycles. I guess this goes without saying.

**dual matroid**: Bruno and Weinberg define the dual matroid in terms of circuits: given a matroid \( \mathcal{M} = (\mathcal{C},E) \), the dual \( \mathcal{M}^* \) has as circuits minimal sets which are orthogonal (as sets) to every member of \( \mathcal{C} \). They state without proof that this definition is equivalent to the one in terms of bases.

**elementary vector**: Given a vector space \( \mathcal{V} \) and \( f \in \mathcal{V} \); \( f \) is elementary if it is nonzero and minimal with respect to inclusion of support: there is no \( g \in \mathcal{V} \) with \( ||g|| \subset ||f|| \).

**ends**: The ends of an edge are the two vertices incident to it, which may be the same.

**forest**: A (maximal) spanning forest.

**graphic matroid**: A matroid which is the bond matroid of some graph.

**matroid associated with \( \mathcal{V} \)**: Given a vector space \( \mathcal{V} \) on an index set \( E \), this is the matroid \( \mathcal{M}_\mathcal{V} = (\mathcal{C}_\mathcal{V}, E) \), where \( \mathcal{C}_\mathcal{V} \) is the set of supports of elementary vectors in \( \mathcal{V} \).

**minor of a matroid**: Given a matroid \( \mathcal{M} \), some matroid of the form \( (\mathcal{M} \cdot S) \times T \); that is, some matroid obtained by reduction and contraction.

**nullity**: Nullity of a graph \( \mu(G) \) is the number of edges in each coforest. Nullity of a general matroid \( \mu(\mathcal{M}) \) is the cardinality of every cobase.
null matroid: The matroid with no elements or circuits. Denoted $\Omega$.

oriented graph: A directed graph. (Strictly, a graph in which each edge has a “negative” end and a “positive” end.) The orientation is given by a function $\eta(e,v)$ which is 0 if $v$ isn’t an end of $e$ and $\pm 1$ with appropriate sign if it is. In pictures of these oriented graphs, the arrow goes from the positive to the negative end.

orthogonal: For vectors and vector subspaces, this is defined normally; two sets $S$ and $T$ are orthogonal if $|S \cap T| \neq 1$.

planar matroid: A matroid which is the bond matroid of some graph and the cycle matroid of some graph; that is, both graphic and cographic.

polygon: A cycle within a graph.

polygon graph: A cycle graph.

primitive vector: An elementary vector of $\mathbb{R}^E$, all of whose entries are $\pm 1$ or 0.

rank: Rank of a graph $r(G)$ is the number of edges in each spanning forest; rank of a general matroid $r(M)$ is the cardinality of every basis.

reduction of a graph: Given a graph $G$ and a subset $S$ of its edges, the reduction $G \cdot S$ is the smallest subgraph with those edges.

reduction of a matroid: For a matroid $M = (\mathcal{C},E)$ and $S \subseteq E$, $\mathcal{C}$ is the collection of minimal intersections of circuits with $S$. Then $M \cdot S = (\mathcal{C} \cdot S,S)$ is the reduction of $M$ to $S$.

regular matrix: A matrix with rank equal to the number of rows, and such that every maximal minor is $\pm k$ or 0, for some fixed $k$.

regular matroid: The matroid associated with a regular vector space.

regular vector space: A vector space over $\mathbb{R}$ such that, for every elementary vector, there is a primitive vector (all entries $\pm 1$ or 0) with the same support.

representative matrix of $\mathcal{V}$: Given a vector space $\mathcal{V}$, a matrix whose rows form a basis of $\mathcal{V}$.

representative vector: Given a vector in function form, this is the corresponding representation as a row vector, denoted $R_f$.

standard representative matrix: For a representative matrix $R$ and cobase $\bar{b}$ of a vector space $\mathcal{V}$, the matrix $R' = R(\bar{b})^{-1} R$ is the standard representative matrix of $\mathcal{V}$ with respect to $\bar{b}$. This is set up so that $R'(\bar{b})$ is the identity matrix. (The required inverse exists because of Theorem 2.2-2, pg. 74.)

support: The support of a vector is the set of indices at which its entries are nonzero. The support of $f$ is denoted $||f||$ for some reason.

totally unimodular: A matrix is totally unimodular if every minor is $\pm 1$ or 0.

tree: A forest.

Type BI: Given the vector space $\mathcal{V} \subseteq \mathbb{F}_2^7$ with basis vectors $(1,0,0,1,0,1,1)$, $(0,1,0,1,1,0,1)$, $(0,0,1,0,1,1,1)$, the matroid $M_\mathcal{V}$ is said to be of Type BI.

Type BII: The dual of a matroid of Type BI. Among binary matroids, these two are a complete set of forbidden minors for the regularity property.

Type HI: The bond matroid of $K_5$. 
Type HII: The bond matroid of $K_{3,3}$. Among binary regular matroids, these two are a complete set of forbidden minors for the property of being a cycle matroid.

Type KI: The cycle matroid of $K_5$.

Type KII: The cycle matroid of $K_{3,3}$. Among binary regular matroids, these two are a complete set of forbidden minors for the property of being a bond matroid.

valence: Degree. (Loops are counted twice.)

vector on $E$ over $F$: A vector indexed by $E$ with entries in a field $F$. This is viewed as a function $f : E \to F$.

vector space on $E$ over $F$: A subspace of $F^E$. 