

## Chapter 2

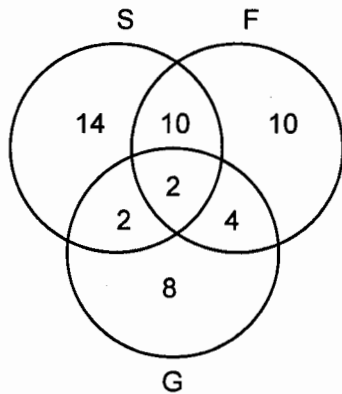
### Problems

1. (a)  $S = \{(r, r), (r, g), (r, b), (g, r), (g, g), (g, b), (b, r), b, g), (b, b)\}$   
 (b)  $S = \{(r, g), (r, b), (g, r), (g, b), (b, r), (b, g)\}$
2.  $S = \{(n, x_1, \dots, x_{n-1}), n \geq 1, x_i \neq 6, i = 1, \dots, n - 1\}$ , with the interpretation that the outcome is  $(n, x_1, \dots, x_{n-1})$  if the first 6 appears on roll  $n$ , and  $x_i$  appears on roll,  $i, i = 1, \dots, n - 1$ . The event  $(\cup_{n=1}^{\infty} E_n)^c$  is the event that 6 never appears.
3.  $EF = \{(1, 2), (1, 4), (1, 6), (2, 1), (4, 1), (6, 1)\}$ .  
 $E \cup F$  occurs if the sum is odd or if at least one of the dice lands on 1.  $FG = \{(1, 4), (4, 1)\}$ .  
 $EF^c$  is the event that neither of the dice lands on 1 and the sum is odd.  $EFG = FG$ .
4.  $A = \{1,0001,0000001, \dots\}$   $B = \{01, 00001, 00000001, \dots\}$   
 $(A \cup B)^c = \{00000 \dots, 001, 000001, \dots\}$
5. (a)  $2^5 = 32$   
 (b)  
 $W = \{(1, 1, 1, 1, 1), (1, 1, 1, 1, 0), (1, 1, 1, 0, 1), (1, 1, 0, 1, 1), (1, 1, 1, 0, 0), (1, 1, 0, 1, 0)$   
 $(1, 1, 0, 0, 1), (1, 1, 0, 0, 0), (1, 0, 1, 1, 1), (0, 1, 1, 1, 1), (1, 0, 1, 1, 0), (0, 1, 1, 1, 0), (0, 0, 1, 1, 1)$   
 $(0, 0, 1, 1, 0), (1, 0, 1, 0, 1)\}$
- (c) 8  
 (d)  $AW = \{(1, 1, 1, 0, 0), (1, 1, 0, 0, 0)\}$
6. (a)  $S = \{(1, g), (0, g), (1, f), (0, f), (1, s), (0, s)\}$   
 (b)  $A = \{(1, s), (0, s)\}$   
 (c)  $B = \{(0, g), (0, f), (0, s)\}$   
 (d)  $\{(1, s), (0, s), (1, g), (1, f)\}$
7. (a)  $6^{15}$   
 (b)  $6^{15} - 3^{15}$   
 (c)  $4^{15}$
8. (a) .8  
 (b) .3  
 (c) 0
9. Choose a customer at random. Let A denote the event that this customer carries an American Express card and V the event that he or she carries a VISA card.

$$P(A \cup V) = P(A) + P(V) - P(AV) = .24 + .61 - .11 = .74.$$

Therefore, 74 percent of the establishment's customers carry at least one of the two types of credit cards that it accepts.

10. Let  $R$  and  $N$  denote the events, respectively, that the student wears a ring and wears a necklace.
- (a)  $P(R \cup N) = 1 - .6 = .4$
- (b)  $.4 = P(R \cup N) = P(R) + P(N) - P(RN) = .2 + .3 - P(RN)$   
Thus,  $P(RN) = .1$
11. Let  $A$  be the event that a randomly chosen person is a cigarette smoker and let  $B$  be the event that she or he is a cigar smoker.
- (a)  $1 - P(A \cup B) = 1 - (.07 + .28 - .05) = .7$ . Hence, 70 percent smoke neither.
- (b)  $P(A^c B) = P(B) - P(AB) = .07 - .05 = .02$ . Hence, 2 percent smoke cigars but not cigarettes.
12. (a)  $P(S \cup F \cup G) = (28 + 26 + 16 - 12 - 4 - 6 + 2)/100 = 1/2$   
The desired probability is  $1 - 1/2 = 1/2$ .
- (b) Use the Venn diagram below to obtain the answer  $32/100$ .



- (c) since 50 students are not taking any of the courses, the probability that neither one is taking a course is  $\binom{50}{2} / \binom{100}{2} = 49/198$  and so the probability that at least one is taking a course is  $149/198$ .

- 13.
- 
- (a) 20,000  
(b) 12,000  
(c) 11,000  
(d) 68,000  
(e) 10,000

14.  $P(M) + P(W) + P(G) - P(MW) - P(MG) - P(WG) + P(MWG) = .312 + .470 + .525 - .086 - .042 - .147 + .025 = 1.057$

15. (a)  $4 \binom{13}{5} / \binom{52}{5}$   
 (b)  $13 \binom{4}{2} \binom{12}{3} \binom{4}{1} \binom{4}{1} \binom{4}{1} / \binom{52}{5}$   
 (c)  $\binom{13}{2} \binom{4}{2} \binom{4}{2} \binom{44}{1} / \binom{52}{5}$   
 (d)  $13 \binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1} / \binom{52}{5}$   
 (e)  $13 \binom{4}{4} \binom{48}{1} / \binom{52}{5}$

16. (a)  $\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6^5}$  (b)  $\frac{6 \binom{5}{2} 5 \cdot 4 \cdot 3}{6^5}$  (c)  $\frac{\binom{6}{2}^4 \binom{5}{2} \binom{3}{2}}{6^5}$

(d)  $\frac{6 \cdot 5 \cdot 4 \binom{5}{3}}{21}$  (e)  $\frac{6 \cdot 5 \binom{5}{3}}{6^5}$  (f)  $\frac{6 \cdot 5 \binom{5}{4}}{6^5}$

(g)  $\frac{6}{6^5}$

17.  $\frac{\prod_{i=1}^8 i^2}{64 \cdot 63 \cdots 58}$

18.  $\frac{2 \cdot 4 \cdot 16}{52 \cdot 51}$

19.  $4/36 + 4/36 + 1/36 + 1/36 = 5/18$

20. Let A be the event that you are dealt blackjack and let B be the event that the dealer is dealt blackjack. Then,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(AB) \\ &= \frac{4 \cdot 4 \cdot 16}{52 \cdot 51} + \frac{4 \cdot 4 \cdot 16 \cdot 3 \cdot 15}{52 \cdot 51 \cdot 50 \cdot 49} \\ &= .0983 \end{aligned}$$

where the preceding used that  $P(A) = P(B) = 2 \times \frac{4 \cdot 16}{52 \cdot 51}$ . Hence, the probability that neither is dealt blackjack is .9017.

21. (a)  $p_1 = 4/20, p_2 = 8/20, p_3 = 5/20, p_4 = 2/20, p_5 = 1/20$

(b) There are a total of  $4 \cdot 1 + 8 \cdot 2 + 5 \cdot 3 + 2 \cdot 4 + 1 \cdot 5 = 48$  children. Hence,

$$q_1 = 4/48, q_2 = 16/48, q_3 = 15/48, q_4 = 8/48, q_5 = 5/48$$

22. The ordering will be unchanged if for some  $k, 0 \leq k \leq n$ , the first  $k$  coin tosses land heads and the last  $n - k$  land tails. Hence, the desired probability is  $(n + 1)/2^n$

23. The answer is  $5/12$ , which can be seen as follows:

$$\begin{aligned} 1 &= P\{\text{first higher}\} + P\{\text{second higher}\} + p\{\text{same}\} \\ &= 2P\{\text{second higher}\} + p\{\text{same}\} \\ &= 2P\{\text{second higher}\} + 1/6 \end{aligned}$$

Another way of solving is to list all the outcomes for which the second is higher. There is 1 outcome when the second die lands on two, 2 when it lands on three, 3 when it lands on four, 4 when it lands on five, and 5 when it lands on six. Hence, the probability is  $(1 + 2 + 3 + 4 + 5)/36 = 5/12$ .

25. 
$$P(E_n) = \left(\frac{26}{36}\right)^{n-1} \frac{6}{36}, \quad \sum_{n=1}^{\infty} P(E_n) = \frac{2}{5}$$

27. Imagine that all 10 balls are withdrawn

$$P(A) = \frac{3 \cdot 9! + 7 \cdot 6 \cdot 3 \cdot 7! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 5! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 3 \cdot 3!}{10!}$$

28. 
$$P\{\text{same}\} = \frac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}}$$

$$P\{\text{different}\} = \frac{\binom{5}{1}\binom{6}{1}\binom{8}{1}}{\binom{19}{3}}$$

If sampling is with replacement

$$P\{\text{same}\} = \frac{5^3 + 6^3 + 8^3}{(19)^3}$$

$$\begin{aligned} P\{\text{different}\} &= P(RBG) + P(BRG) + P(GBR) + \dots + P(GBR) \\ &= \frac{6 \cdot 5 \cdot 6 \cdot 8}{(19)^3} \end{aligned}$$

$$29. \quad (a) \quad \frac{n(n-1) + m(m-1)}{(n+m)(n+m-1)}$$

(b) Putting all terms over the common denominator  $(n+m)^2(n+m-1)$  shows that we must prove that

$$n^2(n+m-1) + m^2(n+m-1) \geq n(n-1)(n+m) + m(m-1)(n+m)$$

which is immediate upon multiplying through and simplifying.

$$30. \quad (a) \quad \frac{\binom{7}{3}\binom{8}{3}3!}{\binom{8}{4}\binom{9}{4}4!} = 1/18$$

$$(b) \quad \frac{\binom{7}{3}\binom{8}{3}}{\binom{8}{4}\binom{9}{4}} - 1/18 = 1/6$$

$$(c) \quad \frac{\binom{7}{3}\binom{8}{4} + \binom{7}{4}\binom{8}{3}}{\binom{8}{4}\binom{9}{4}} = 1/2$$

$$31. \quad P(\{\text{complete}\}) = \frac{3 \cdot 2 \cdot 1}{3 \cdot 3 \cdot 3} = \frac{2}{9}$$

$$P(\{\text{same}\}) = \frac{3}{27} = \frac{1}{9}$$

$$32. \quad \frac{g(b+g-1)!}{(b+g)!} = \frac{g}{b+g}$$

$$33. \quad \frac{\binom{5}{2}\binom{15}{2}}{\binom{20}{4}} = \frac{70}{323}$$

$$34. \quad \frac{\binom{32}{13}}{\binom{52}{13}}$$

$$35. \quad 1 - \frac{\binom{30}{3}}{\binom{54}{3}} \approx .8363$$

36. (a)  $\binom{4}{2} / \binom{52}{2} \approx .0045,$   
 (b)  $13 \binom{4}{2} / \binom{52}{2} = 1/17 \approx .0588$
37. (a)  $\binom{7}{5} / \binom{10}{5} = 1/12 \approx .0833$   
 (b)  $\binom{7}{4} \binom{3}{1} / \binom{10}{5} + 1/12 = 1/2$
38.  $1/2 = \binom{3}{2} / \binom{n}{2}$  or  $n(n-1) = 12$  or  $n = 4.$
39.  $\frac{5 \cdot 4 \cdot 3}{5 \cdot 5 \cdot 5} = \frac{12}{25}$
40.  $P\{1\} = \frac{4}{44} = \frac{1}{64}$   
 $P\{2\} = \binom{4}{2} \left[ 4 + \binom{4}{2} + 4 \right] / 4^4 = \frac{84}{256}$   
 $P\{3\} = \binom{4}{3} \binom{3}{1} \frac{4!}{2!} / 4^4 = \frac{36}{64}$   
 $P\{4\} = \frac{4!}{4^4} = \frac{6}{64}$
41.  $1 - \frac{5^4}{6^4}$
42.  $1 - \left(\frac{35}{36}\right)^n$
43.  $\frac{2(n-1)(n-2)}{n!} = \frac{2}{n}$  in a line  
 $\frac{2n(n-2)!}{n!} = \frac{2}{n-1}$  if in a circle,  $n \geq 2$
44. (a) If  $A$  is first, then  $A$  can be in any one of 3 places and  $B$ 's place is determined, and the others can be arranged in any of  $3!$  ways. As a similar result is true, when  $B$  is first, we see that the probability in this case is  $2 \cdot 3 \cdot 3!/5! = 3/10$   
 (b)  $2 \cdot 2 \cdot 3!/5! = 1/5$   
 (c)  $2 \cdot 3!/5! = 1/10$

54.  $P(S \cup H \cup D \cup C) = P(S) + P(H) + P(D) + P(C) - P(SH) - \dots - P(SHDC)$

$$= \frac{4 \binom{39}{13} - 6 \binom{26}{13} + 4 \binom{13}{13}}{\binom{52}{13}}$$

$$= \frac{4 \binom{39}{13} - 6 \binom{26}{13} + 4}{\binom{52}{13}}$$

55. (a)  $P(S \cup H \cup D \cup C) = P(S) + \dots - P(SHDC)$

$$= \frac{4 \binom{2}{2} \binom{50}{11} - 6 \binom{2}{2} \binom{2}{2} \binom{48}{9} + 4 \binom{2}{2} \binom{46}{7} - \binom{2}{2} \binom{44}{5}}{\binom{52}{13}}$$

$$= \frac{4 \binom{50}{11} - 6 \binom{48}{9} + 4 \binom{46}{7} - \binom{44}{5}}{\binom{52}{13}}$$

$S = \{\text{having both the ace and king is spade}\}$   
etc.

(b)  $P(1 \cup 2 \cup \dots \cup 13) = \frac{13 \binom{48}{9} - \binom{13}{2} \binom{44}{5} + \binom{13}{3} \binom{40}{1}}{\binom{52}{13}}$

$k = \{\text{all 4 in denominator } k\}$

56. Player B. If Player A chooses spinner (a) then B can choose spinner (c). If A chooses (b) then B chooses (a). If A chooses (c) then B chooses (b). In each case B wins probability 5/9.

45.  $1/n$  if discard,  $\frac{(n-1)^{k-1}}{n^k}$  if do not discard

46. If  $n$  in the room,

$$P\{\text{all different}\} = \frac{12 \cdot 11 \cdot \dots \cdot (13-n)}{12 \cdot 12 \cdot \dots \cdot 12}$$

When  $n = 5$  this falls below  $1/2$ . (Its value when  $n = 5$  is .3819)

47.  $12!/(12)^{12}$

48.  $\binom{12}{4} \binom{8}{4} \frac{(20)!}{(3!)^4 (2!)^4} / (12)^{20}$

49.  $\binom{6}{3} \binom{6}{3} / \binom{12}{6}$

50.  $\binom{13}{5} \binom{39}{8} \binom{8}{8} \binom{31}{5} / \binom{52}{13} \binom{39}{13}$

51.  $\binom{n}{m} (n-1)^{n-m} / N^n$

52. (a)  $\frac{20 \cdot 18 \cdot 16 \cdot 14 \cdot 12 \cdot 10 \cdot 8 \cdot 6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}$

(b)  $\frac{\binom{10}{1} \binom{9}{6} \frac{8!}{2!} 2^6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}$

53. Let  $A_i$  be the event that couple  $i$  sit next to each other. Then

$$P(\cup_{i=1}^4 A_i) = 4 \frac{2 \cdot 7!}{8!} - 6 \frac{2^2 \cdot 6!}{8!} + 4 \frac{2^3 \cdot 5!}{8!} - \frac{2^4 \cdot 4!}{8!}$$

and the desired probability is 1 minus the preceding.



## Theoretical Exercises

5.  $F_i = E_i \cap_{j=1}^{i-1} E_j^c$

6. (a)  $EF^cG^c$

(b)  $EF^cG$

(c)  $E \cup F \cup G$

(d)  $EF \cup EG \cup FG$

(e)  $EFG$

(f)  $E^cF^cG^c$

(g)  $E^cF^cG^c \cup EF^cG^c \cup E^cFG^c \cup E^cF^cG$

(h)  $(EFG)^c$

(i)  $EFG^c \cup EF^cG \cup E^cFG$

(j)  $S$

7. (a)  $E$

(b)  $EF$

(c)  $EG \cup F$

8. The number of partitions that has  $n + 1$  and a fixed set of  $i$  of the elements  $1, 2, \dots, n$  as a subset is  $T_{n-i}$ . Hence, (where  $T_0 = 1$ ). Hence, as there are  $\binom{n}{i}$  such subsets.

$$T_{n+1} = \sum_{i=0}^n \binom{n}{i} T_{n-i} = 1 + \sum_{i=0}^{n-1} \binom{n}{i} T_{n-i} = 1 + \sum_{k=1}^n \binom{n}{k} T_k.$$

11.  $1 \geq P(E \cup F) = P(E) + P(F) - P(EF)$

12.  $P(EF^c \cup E^cF) = P(EF^c) + P(E^cF)$   
 $= P(E) - P(EF) + P(F) - P(EF)$

13.  $E = EF \cup EF^c$

$$15. \frac{\binom{M}{k} \binom{N}{r-k}}{\binom{M+N}{r}}$$

16.  $P(E_1 \dots E_n) \geq P(E_1 \dots E_{n-1}) + P(E_n) - 1$  by Bonferonni's Ineq.

$$\geq \sum_1^{n-1} P(E_i) - (n-2) + P(E_n) - 1 \text{ by induction hypothesis}$$

$$19. \frac{\binom{n}{r-1} \binom{m}{k-r} (n-r+1)}{\binom{n+m}{k-1} (n+m-k+1)}$$

21. Let  $y_1, y_2, \dots, y_k$  denote the successive runs of losses and  $x_1, \dots, x_k$  the successive runs of wins. There will be  $2k$  runs if the outcome is either of the form  $y_1, x_1, \dots, y_k, x_k$  or  $x_1, y_1, \dots, x_k, y_k$  where all  $x_i, y_i$  are positive, with  $x_1 + \dots + x_k = n, y_1 + \dots + y_k = m$ . By Proposition 6.1 there are

$2 \binom{n-1}{k-1} \binom{m-1}{k-1}$  number of outcomes and so

$$P\{2k \text{ runs}\} = 2 \frac{\binom{n-1}{k-1} \binom{m-1}{k-1}}{\binom{m+n}{n}}$$

There will be  $2k+1$  runs if the outcome is either of the form  $x_1, y_1, \dots, x_k, y_k, x_{k+1}$  or  $y_1, x_1, \dots, y_k, x_k, y_{k+1}$  where all are positive and  $\sum x_i = n, \sum y_i = m$ . By Proposition 6.1 there are

$\binom{n-1}{k} \binom{m-1}{k-1}$  outcomes of the first type and  $\binom{n-1}{k-1} \binom{m-1}{k}$  of the second.