On the theorem of Hayman and Wu

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Preliminary Version

Abstract

We show that the Hayman-Wu constant is strictly smaller than 4π. A main tool is an analysis of the hyperbolic geodesic curvature of straight lines in simply connected domains.

1 Introduction

Let $\Omega \subset \mathbb{C}$ be a simply connected domain, L be a line or a circle, and $f : \Omega \to \mathbb{D}$ a conformal map of Ω onto the unit disc $\mathbb{D}$. The Hayman-Wu theorem [5] asserts the existence of a universal constant C such that

$$\ell(f(L)) \leq C,$$

where ℓ denotes one-dimensional Hausdorff (=length) measure. We denote the smallest such constant, often called the Hayman-Wu constant, by Ø in memory of Knut Øyma [7],[8].

Many authors have contributed to problems related to the Hayman-Wu theorem. For the sake of brevity we will only mention the results that are directly concerned with the constant Ø. The first estimate $Ø \leq 2\times 10^{35}$ appeared in the paper Hayman and Wu [5]. Garnett, Gehring and Jones [3] obtained a shorter proof, but did not use it to estimate Ø. Fernández, Heinonen and Martio [2] showed that $Ø \leq 4\pi^2$. They offered a conjecture for the value of Ø, but this conjecture was disproved by Øyma [8] who showed $Ø \geq \pi^2$ by means of an example (rather a family of examples: For every $\varepsilon > 0$ there is a domain and associated conformal map such that $\ell > \pi^2 - \varepsilon$). Previously Øyma [7] had shocked the community by

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proving $\emptyset \leq 4\pi$ on just two pages. More precisely, he showed that $\ell(f(L)) < 4\pi$ for every tripel (Ω, L, f), and he conjectured

$$\emptyset = \pi^2.$$

In oral communication, he asked the more modest question to prove $\emptyset < 4\pi$, that is to show $\ell(f(L)) \leq 4\pi - \varepsilon$ for some universal $\varepsilon > 0$ and every tripel (Ω, L, f).

Theorem 1.1 *The Hayman-Wu constant satisfies*

$$\emptyset < 4\pi.$$

Our proof of Theorem 1.1 combines the method of Øyma with an analysis of the geodesic curvature of $f(L)$. The idea of using the geodesic curvature was introduced by Fernández and Granados [4] in their proof that the Hayman-Wu constant for convex domains is 2π.

If $\gamma \subset \mathbb{D}$ is a smooth (C^2, say) curve through 0, then the absolute value of the geodesic curvature of γ at 0 in the hyperbolic metric of constant curvature -1 can be defined by

$$k_h = k_h(\gamma, 0) = \frac{1}{2} k_e, \tag{1}$$

where k_e denotes the absolute value of the euclidean curvature of γ at 0. The factor 1/2 accounts for the fact that our hyperbolic metric has curvature -1 rather than -4. At points $z \neq 0$ in γ, the geodesic curvature is then defined through conformal invariance: $k_h(\gamma, 0) = k_h(T \circ \gamma, T(0))$ for every automorphism T of $\mathbb{D}$. See [6] and [4] for more information about geodesic curvature.

Theorem 1.2 *Let Ω be a simply connected domain bounded by an analytic curve, and let $f : \Omega \to \mathbb{D}$ be a conformal map. If L is a straight line, then $f(L)$ can be parametrized as follows: There is an open set $A \subset \mathbb{T}$, consisting of finitely many arcs, and a smooth map $\phi : A \to f(L)$, such that*

$$\frac{1 - |\phi(\zeta)|^2}{|\zeta - \phi(\zeta)|^2} \geq [2 + k_h(f(L), \phi(\zeta))] \frac{|\phi'(\zeta)|}{1 - |\phi(\zeta)|^2} \tag{2}$$

for all $\zeta \in A$. Furthermore, $f(L) \setminus \phi(A)$ consists of finitely many points only.

Notice that (2) immediately implies $|\phi'| \leq 2$, which in turn implies $\ell(f(L)) \leq 2\ell(A) \leq 4\pi$. By exhausting a given simply connected domain with analytically bounded domains, we reobtain Øymas result $\emptyset \leq 4\pi$.

A crucial element in some of the proofs of the Hayman-Wu theorem is some control over the change of the density λ_Ω of the hyperbolic metric of a domain Ω when passing to another domain $\tilde{\Omega}$ (for instance by symmetrization). In this direction, we obtained in joint work with J. Fernández the following estimate.

Theorem 1.3 *Let Ω_1, Ω_2 and Ω_3 be simply connected planar domains, such that*

$$\Omega_1 \cap \Omega_2 \subset \Omega_3.$$

Then

$$\lambda_{\Omega_3}(x) \leq \lambda_{\Omega_1}(x) + \lambda_{\Omega_2}(x)$$

for all $x \in \Omega_3$.

As a simple consequence we obtain that the method of [2] actually gives $\emptyset \leq 2\pi^2$ rather than $\emptyset \leq 4\pi^2$. This is explained in section 3.

Section 2 contains the proofs of the Theorems. Some consequences, remarks and open questions are discussed in section 3.

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2 Proofs

Notation : Throughout the rest of the paper, λ_Ω will denote the density of the hyperbolic metric of the planar domain Ω, in particular $\lambda_{\mathbb{D}}(z) = 2/(1-|z|^2)$ and $\lambda_{\mathbb{H}}(z) = 1/\operatorname{Im}(z)$ The hyperbolic distance will be denoted d_Ω. Given a simply connected planar domain $\Omega \neq \mathbb{C}$, a curve $\gamma \subset \Omega$ and a point $x \in \gamma$, the hyperbolic curvature $k_h(\gamma, x, \Omega)$ of γ at x is defined by means of (1) and conformal invariance. For sets $A \subset \mathbb{C}$, we write $\tilde{A} = \{\overline{z} : z \in A\}$ for the reflection of A in $\mathbb{R}$. In what follows we normalize the line L to be the real axis.

The proof of Theorem 1.2 is based on Øymas construction [7] and the following lemma.

Lemma 2.1 *Let $\Omega \subset \mathbb{C}$ be simply connected, $x_0 \in \mathbb{R} \cap \Omega$ and consider the component Ω' of $\Omega \cap \tilde{\Omega}$ that contains x_0. Then*

$$\lambda_{\Omega'}(x_0) \geq \frac{k_h(\mathbb{R}, x_0, \Omega) + 2}{2} \lambda_\Omega(x_0). \tag{3}$$

The estimate is sharp: For every $0 \leq k \leq 2$ there is a domain Ω and a point $x_0 \in \mathbb{R} \cap \Omega$ such that equality holds in (3) and $k_h(\mathbb{R}, x_0) = k$. Inequality (3) contains the idea behind the proof of Theorem 1.1: Locally, either $\mathbb{R}$ is close to a geodesic (curvature near zero), or the desity of the hyperbolic metric increases by a definite amount when passing from Ω to Ω'. However, the proof of Theorem 1.1 is easier if we use the following:

Lemma 2.2 *In the situation of Lemma 2.1, if* $\frac{\lambda_{\Omega'}(x_0)}{\lambda_\Omega(x_0)}$ *is close to 1 then* $\mathbb{R}$ *is close to a geodesic near* x_0*. More precisely, for every* $M > 0$ *and* $\varepsilon > 0$ *there is* $\delta > 0$ *such that whenever* $\frac{\lambda_{\Omega'}(x_0)}{\lambda_\Omega(x_0)} < 1 + \delta$*, the following holds: If* $\sigma(t)$ *is the hyperbolic geodesic through* x_0 *tangent to* $\mathbb{R}$*, parametrized by hyperbolic arc length and such that* $\sigma(x_0) = x_0$*, then*

$$d_\Omega(\sigma(t), t) \leq \varepsilon$$

for all $-M \leq t \leq M$*.*

Proof of Lemmas 2.1 and 2.2: Since Ω' is simply connected, we may consider the conformal maps $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D} \to \Omega'$, normalized by $f(0) = g(0) = x_0$ and $f'(0) > 0, g'(0) > 0$.

Then

$$\phi(z) = f^{-1}(g(z)) = a_1 z + a_2 z^2 + ...$$

is a univalent function from $\mathbb{D}$ into $\mathbb{D}$ with $a_1 > 0$. Because Ω' is symmetric with respect to $\mathbb{R}$, we have $\Omega' \cap \mathbb{R} = g((-1,1))$. Thus

$$k_h(\mathbb{R}, x_0, \Omega) = k_h(\phi((-1,1)), 0, \mathbb{D}) = \frac{1}{2} k_e(\phi((-1,1)), 0, \mathbb{D}) = \frac{|\operatorname{Im} a_2|}{a_1^2}.$$

Since ϕ/a_1 is a normalized univalent function bounded by $1/a_1$, we have

$$|\frac{a_2}{a_1}| \leq 2(1 - a_1)$$

by Pick's theorem. It follows that

$$k_h(\mathbb{R}, x_0, \Omega) \leq 2\frac{1 - a_1}{a_1}.$$

Lemma 2.1 now follows from

$$\frac{\lambda_{\Omega'}(x_0)}{\lambda_\Omega(x_0)} = \frac{f'(0)}{g'(0)} = \frac{1}{a_1}.$$

Lemma 2.2 follows since ϕ is close to the identity if a_1 is close to 1. □

Proof of Theorem 1.2: Following Øyma [7], we consider the (finitely many) components Ω_n of $\Omega \cap \tilde{\Omega}$. Each of them is simply connected, symmetric with respect to $\mathbb{R}$, and bounded by piecewise analytic arcs. Let Ω' be one of these components, and let g be a conformal map from Ω' onto the upper half plane $\mathbb{H}$ such that $\mathbb{R} \cap \Omega'$ maps to the imaginary axis. The map

$$\mathbb{R} \ni x \mapsto i|x| \tag{4}$$

from $\partial\mathbb{H}$ onto $i\mathbb{R}_+$ corresponds via g to a map from $\partial\Omega'$ onto $\mathbb{R}\cap\Omega'$. This map is two-to-one. Restricting it to those open arcs of $\partial\Omega'$ which are also contained in $\partial\Omega$, we obtain a one-to-one parametrization of $\mathbb{R}\cap\Omega'$ by arcs of $\partial\Omega$. In case that Ω was already symmetric with respect to $\mathbb{R}$, just restrict to $\partial\Omega'\cap\mathbb{H}$. Since the domains Ω_n are disjoint, the parametrizations coming from different components use disjoint arcs of $\partial\Omega$. We thus obtain a smooth map

$$\psi : B \to (\mathbb{R}\cap\Omega)\setminus P$$

defined on a finite union B of open arcs on $\partial\Omega$, where P is the finite set of points corresponding to the corners of Ω_n'. Now set $A = f(B)$ and $\phi = f\circ\psi\circ f^{-1}$, where $f:\Omega\to\mathbb{D}$ is the given conformal map. In order to prove (2), consider two nearby points $\zeta,\zeta'\in A$ together with the component Ω' of $\Omega\cap\tilde{\Omega}$ that contains $z = f^{-1}(\phi(\zeta))$. If ζ' is sufficiently close to ζ, we find that $x = f^{-1}(\zeta)\in\partial\Omega'$ and $x' = f^{-1}(\zeta')\in\partial\Omega'$ are close to each other and that $z' = f^{-1}(\phi(\zeta))$ belongs to Ω'. Denote $C(x,x')$ the (shorter) arc of $\partial\Omega'$ between x and x'. Using (4) and conformal invariance, we obtain

$$\omega(C(x,x'),z,\Omega') = \frac{1}{2\pi}d_{\Omega'}(z,z')(1+o(1))$$

as $\zeta'\to\zeta$. Since $\Omega'\subset\Omega$, we obtain

$$\omega(C(x,x'),z,\Omega) \geq \frac{1}{2\pi}d_{\Omega'}(z,z')(1+o(1)) = \frac{1}{2\pi}\frac{\lambda_{\Omega'}(z)}{\lambda_{\Omega}(z)}d_{\Omega}(z,z')(1+o(1)).$$

Applying f and letting $\zeta'\to\zeta$ yields

$$\frac{1-|\phi(\zeta)|^2}{|\zeta-\phi(\zeta)|^2} \geq \frac{\lambda_{\Omega'}(f^{-1}(z))}{\lambda_{\Omega}(f^{-1}(z))}\frac{2|\phi'(\zeta)|}{1-|\phi(\zeta)|^2} \tag{5}$$

The theorem now follows from Lemma 2.1. □

Proof of Theorem 1.1: Let $\Omega\subset\mathbb{C}$ be simply connected and $f:\Omega\to\mathbb{D}$ conformal. Replacing Ω by $f^{-1}(\{z:|z|<r\})$ if necessary, we may assume that $\partial\Omega$ is an analytic curve.

Denote by $p(z) = z/|z|$ the projection onto the unit circle. The following observation is easily proved using the fact that geodesics meet the unit circle radially: There are constants $M>0$ and $\varepsilon_1>0$ such that every arc $\gamma\subset\mathbb{D}$ of hyperbolic length M that has hyperbolic Hausdorff distance $\leq\varepsilon_1$ from some geodesic arc has a subarc $\hat{\gamma}$ of hyperbolic length 1 such that

$$\ell(p(\hat{\gamma})) \leq \frac{1}{8}\ell(\hat{\gamma}). \tag{6}$$

Let δ_1 be the constant from Lemma 2.2 associated with these constants M and ε_1.

Let $\phi : A \to f(\mathbb{R})$ be the parametrization from Theorem 1.2. Subdivide $f(\mathbb{R})$ into pairwise disjoint arcs Γ_n of hyperbolic length M. We claim that there is a universal constant ε_0 such that

$$\ell(\Gamma_n) \leq (2-\varepsilon_0)\ell(\phi^{-1}(\Gamma_n)) \tag{7}$$

for all n. Then the theorem immediately follows by summing over n and using the disjointness of the $\phi^{-1}(\Gamma_n)$.

To prove (7), fix n and let us first assume that

$$\frac{\lambda_{\Omega'}(f^{-1}(z))}{\lambda_{\Omega}(f^{-1}(z))} \geq 1+\delta_1$$

for all $z \in \Gamma_n$, where again Ω' denotes the component of $\Omega \cap \tilde{\Omega}$ that contains $f^{-1}(z)$. Then (5) implies $|\phi'(z)| \leq 2/(1+\delta_1)$ and (7) follows for this n.

Otherwise we apply Lemma 2.2 and obtain a subarc $\hat{\gamma}$ of hyperbolic length 1 satisfying (6). We may assume that $\hat{\gamma} \subset \{|z| \geq \frac{1}{2}\}$, since otherwise $|\phi'| < 2-\varepsilon_2$ on $\phi^{-1}(\hat{\gamma})$ by (5) and (7) would easily follow using the fact that $\ell(\hat{\gamma})$ and $\ell(\Gamma_n)$ are comparable with universal constants depending only on M.

Now $\ell(\phi^{-1}(\hat{\gamma})) \geq \frac{1}{2}\ell(\hat{\gamma})$ since $|\phi'| \leq 2$. Hence there is a set $B \subset \phi^{-1}(\hat{\gamma})$ with $\ell(B) \geq \frac{1}{4}\ell(\hat{\gamma})$ such that $\operatorname{dist}(\zeta, \mathrm{p}(\hat{\gamma})) \geq \frac{1}{32}\ell(\hat{\gamma})$ for all $\zeta \in B$. For these ζ we have

$$1-|\phi(\zeta)| \leq (1-\varepsilon_3)|\zeta - \phi(\zeta)|$$

and obtain $|\phi'| \leq 2(1-\varepsilon_3)^2$. Again (7) follows since $\ell(\hat{\gamma})$ and $\ell(\Gamma_n)$ are comparable. The proof is complete. □

Proof of Theorem 1.3: We write $A^c = \hat{\mathbb{C}} \setminus A$ for sets $A \subset \hat{\mathbb{C}}$. Fix a point $x \in \Omega_1 \cap \Omega_2$ and denote by f_k (k=1,2,3) conformal maps from $\mathbb{D}$ onto Ω_k with $f_k(0) = x$. Then

$$\lambda_{\Omega_k}(x) = \frac{1}{|f_k'(0)|}. \tag{8}$$

Since $1/(f_k(1/z)-x)$ is conformal in $\Delta = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and fixes ∞, we obtain

$$\lambda_{\Omega_k}(x) = \operatorname{cap}\left(\frac{1}{\partial\Omega_\mathrm{k} - \mathrm{x}}\right),$$

where the right-hand side is the logarithmic capacity of the image of $\partial\Omega_k$ under $z \mapsto 1/(z-x)$. Now

$$\Omega_1 \cap \Omega_2 \subset \Omega_3$$

implies

$$(\Omega_3 - x)^c \subset (\Omega_1 - x)^c \cup (\Omega_2 - x)^c,$$

which yields

$$\frac{1}{(\Omega_3 - x)^c} \subset \frac{1}{(\Omega_1 - x)^c} \cup \frac{1}{(\Omega_2 - x)^c}.$$

The theorem follows from $\operatorname{cap} \mathrm{A} = \operatorname{cap} \partial \mathrm{A}$, (8) and the subadditivity of the capacity of connected unions ([9], chapter 22), noticing that $\frac{1}{(\Omega_1 - x)^c} \cup \frac{1}{(\Omega_2 - x)^c}$ is connected (through 0). $\square$

Remark: The proof shows that the conclusion of the theorem is true under the slightly weaker assumption that $\partial\Omega_3 \subset \Omega_1^c \cup \Omega_2^c$.

3 Remarks and Questions

The Hayman-Wu theorem if usually formulated as $\ell(f(L)) \leq C$ whenever L is a *circle or a line*. One is tempted to argue that there is no difference between circles and lines by composing with an automorphism of $\hat{\mathbb{C}}$. However, most proofs of the Hayman-Wu theorem make essential use of the fact that $\Omega \subset \mathbb{C}$ rather than $\hat{\mathbb{C}}$, and then the case of L bounding a disc D that is compactly contained in Ω requires an extra argument: In this case it is known that D is hyperbolically convex, and according to a theorem of Brown-Flinn [1] the boundary of any hyperbolically convex subset of the *unit disc* has length $\leq \pi^2$. Hence $\ell(f(L)) \leq \pi^2$ in this case.

Let us restrict our attention to simply-connected planar domains Ω and circles L whose center x_0 is contained in Ω. It is no loss of generality to assume that $L = \mathbb{T}$ and $x_0 = 0$. Let us further consider only those conformal maps $f : \Omega \to \mathbb{D}$ that fix 0, and denote by $\emptyset_1$ the smallest universal bound for $\ell(f(L))$. Obviously $\emptyset_1 \leq \emptyset$, and Øyma's example [8] is easily seen to give $\emptyset_1 \geq \pi^2$.

Conjecture 1: $\emptyset_1 = \emptyset$.

It might be easier to determine $\emptyset_1$ than $\emptyset$, for the following reason:

Proposition 3.1 *For every pair* (Ω, f) *as above and every* $M > 0$ *there is* $(\hat{\Omega}, \hat{f})$ *with*

$$\ell(f(\mathbb{T})) < \ell(\hat{f}(\mathbb{T}))$$

and such that the hyperbolic distance of $\hat{f}(\mathbb{T})$ *from 0 is at least* M.

In other words, there is no extremal configuration, and in determining $\varnothing_1$ we may assume that the circle is hyperbolically as far out as we want.

Proof of Proposition 3.1: Set $\phi = f^{-1}$ and consider the n-th root transform $\psi(z) = \phi(z^n)^{1/n}$. Then

$$\psi^{-1}(\mathbb{T}) = \{z \in \mathbb{D} : z^n \in f(\mathbb{T})\}$$

and we obtain

$$\ell(\psi^{-1}(\mathbb{T})) = n \int_{f(\mathbb{T})} |\frac{z^{\frac{1-n}{n}}}{n}||dz| > \ell(f(\mathbb{T})).$$

Choosing n large enough we see that $\psi^{-1}(\mathbb{T})$ gets as close to $\mathbb{T}$ as we please. The proposition follows with $\hat{f} = \psi^{-1}$. □

We now turn to the approach of Fernández, Heinonen and Martio [2]. Using the notation $\tilde{z} = x + i|y|$ for $z = x + iy$ and $\tilde{E} = \{\tilde{z} : z \in E\}$, they associate with any simply connected planar domain Ω the following simply connected domain $\hat{\Omega}$: Setting $a = \inf\{z : z \in \Omega \cap \mathbb{R}\}, b = \sup\{z : z \in \Omega \cap \mathbb{R}\}$, and $E = (-\infty, a) \cup \partial\Omega \cup (b, \infty)$, then $\hat{\Omega}$ is the component of $\mathbb{C} \setminus \tilde{E}$ that contains the lower half plane. If $z_0 \in \Omega$ is in the lower half plane, if f resp. $\hat{f}$ are the conformal maps of Ω resp. $\hat{\Omega}$ onto $\mathbb{D}$ sending z_0 to 0, if g resp. $\hat{g}$ denotes the (positive) Greens functions with pole at z_0, and if $x \in \Omega \cap \mathbb{R}$, then

$$|f'(x)| = \frac{2|f'(x)|}{1 - |f(x)|^2}(1 - |f(x)|^2) = \lambda_\Omega(x)(1 - e^{-2g(x)}).$$

Baernstein [2] proved $g(x) \leq \hat{g}(x)$, hence

$$|f'(x)| \leq \frac{\lambda_\Omega(x)}{\lambda_{\hat{\Omega}}(x)}|\hat{f}'(x)|. \tag{9}$$

From the Koebe one-quarter theorem it easily follows that $\lambda_\Omega(x)/\lambda_{\hat{\Omega}}(x) \leq 4$, so that $|f'(x)| \leq 4|\hat{f}'(x)|$ on $\mathbb{R}$. Now the result $\ell(f(\mathbb{R})) \leq 4\pi^2$ of [2] follows at once from the aforementioned result of Brown-Flinn. The use of the Koebe one-quarter theorem is not optimal: Indeed, from Theorem 1.3 (apply the remark after the proof of Theorem 1.3 to $\Omega_1 = \hat{\Omega}$, $\Omega_2 = \overline{\hat{\Omega}}$ and $\Omega_3 = \Omega$) we conclude $\lambda_\Omega(x) \leq 2\lambda_{\hat{\Omega}}(x)$, obtain $|f'(x)| \leq 2|\hat{f}'(x)|$ and finally have $\ell(f(\mathbb{R})) \leq 2\pi^2$. The inequality $|f'(x)| \leq 2|\hat{f}'(x)|$ is best possible, equality is attained only (up to linear transformations) for $\Omega = \mathbb{C} \setminus ((i, i\infty) \cup (-i, -i\infty))$ and $x = x_0 = 0$. Notice that this domain is symmetric about $\mathbb{R}$. The inequality $g \leq \hat{g}$ is optimal, too, but here equality occurs in the unsymmetric case $\Omega = \hat{\Omega}$. Hence the estimate $|f'(x)| \leq 2|\hat{f}'(x)|$ should improve if x is far from x_0. We formulate this as

Conjecture 2: For every $\varepsilon > 0$ there is $\delta > 0$ (independent of Ω) such that

$$|f'(x)| \leq (1+\varepsilon)|\hat{f}'(x)|$$

whenever $x_0 \in \Omega \cap \mathbb{H}^-$ and $d_\Omega(x, x_0) > \delta^{-1}$.

From Conjecture 2 it follows that $\emptyset_1 = \pi^2$: Indeed, by Proposition 3.1 we may assume that $\mathbb{R}$ is hyperbolically far from x_0, and the above reasoning gives $\ell(f(\mathbb{R})) \leq (1+\varepsilon)\pi^2$ for every $\varepsilon > 0$.

A first draft of this paper contained the following conjecture, which would have constituted an alternative approach to proving $\emptyset = \pi^2$:

Conjecture 3: If $\gamma \subset \overline{\mathbb{D}}$ is a smooth curve with both endpoints on $\mathbb{T}$ and otherwise contained in $\mathbb{D}$, if A is an open subset of $\mathbb{T}$ and if $\phi : A \to \gamma$ is a parametrization of γ minus finitely many points satisfying

$$\frac{1 - |\phi(\zeta)|^2}{|\zeta - \phi(\zeta)|^2} \geq [2 + k_h(\gamma, \phi(\zeta))] \frac{|\phi'(\zeta)|}{1 - |\phi(\zeta)|^2}$$

on A, then

$$\ell(\gamma) \leq \frac{\pi}{2} \ell(A).$$

By means of Theorem 1.2, this would immediately prove $\emptyset = \pi^2$. However, in joint work with Ana Granados we found counterexamples to this conjecture. These examples, together with additional material, will be presented in a forthcoming joint paper.

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