The Loewner differential equation and slit mappings

D. E. Marshall* and S. Rohde†
Department of Mathematics
University of Washington

Abstract

We show that the Loewner equation generates slits if the driving term is Hölder continuous with exponent 1/2 and small norm, and that this is best possible.

1 Introduction and results

In his study of extremal problems for univalent functions, K. Löwner [10] (who later changed his name into C. Loewner) introduced the differential equation named after him. It was a key ingredient in the proof of the Bieberbach conjecture by de Branges [2]. It was used by L. Carleson and N. Makarov in their investigation of a process similar to DLA [3]. Recently, O. Schramm found a description of the scaling limits of some stochastic processes in terms of the Loewner equation [17] (assuming the validity of some conjectures such as existence of the limits). This led him to the definition of a new stochastic process, the "Stochastic Loewner Evolution" SLE. The SLE has been further explored in the work of Lawler, Schramm and Werner (see [12] and the references therein) and led to a proof of Mandelbrot's conjecture about the Hausdorff dimension of the Brownian frontier. The SLE has also played a crucial role in S. Smirnov's [18] recent work on percolation.

Somewhat surprisingly, the geometry of the solutions to the Loewner equation is not very well understood. This paper addresses the regularity of solutions in the deterministic setting.

Let Ω_t, $t_0 \leq t \leq t_1$, be a continuously increasing sequence of simply connected planar domains, and let $z_0 \in \cap_t \Omega_t$. Then there are conformal maps $f_t : \mathbb{D} \to \Omega_t$ of the unit disc with $f_t(0) = z_0$ and $f_t'(0) > 0$. The continuity of the domain sequence can be expressed by saying that the map $t \mapsto f_t$ is continuous in the topology of locally uniform convergence. Since $f_t'(0)$ is increasing in t, we may assume (by reparametrizing Ω_t if necessary) that $f_t'(0) = e^t$ for $t_0 \leq t \leq t_1$.

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The family $f_t, t_0 \leq t \leq t_1$, is called a (normalized) *Loewner chain*. The Loewner differential equation

$$\partial_t f(z,t) = z\ p(z,t)\ \partial_z f(z,t) \tag{1.1}$$

describes the evolution of a (normalized) Loewner chain $f_t(z)$, where $p = p(z,t)$ is analytic in z and has positive real part (see [14]). A particularly important case is the case of *slit domains*

$$\Omega_t = \Omega \setminus \gamma[t, t_1], \tag{1.2}$$

where $\Omega \subseteq \mathbb{C}$ is simply connected and γ parametrizes a simple arc that is contained in Ω except for one endpoint $\gamma(t_1) \in \partial\Omega$. In this case, p has a particularly simple form and (1.1) becomes

$$\partial_t f = z \frac{\lambda(t) + z}{\lambda(t) - z} \partial_z f, \tag{1.3}$$

where $\lambda(t) = f_t^{-1}(\gamma(t)) \in \mathbb{T} = \partial\mathbb{D}$ (see [5], Chapter 3.3). It is known that $\lambda(t)$ is continuous, and conversely that (1.3) generates a sequence of increasing domains whenever λ is continuous. In fact, even (1.1) generates conformal maps onto increasing domains assuming only measurability of p in t, see [14]. The paper [11] contains a topological characterization of increasing domain families Ω_t for which (1.1) simplifies to (1.3) with continuous λ.

It is known [8] that the domains Ω_t obtained from (1.3) are not necessarily slit domains if λ is only assumed to be continuous. The main result of this paper is a rather sharp condition on λ that guarantees slit domains.

We will first consider (1.2) with $\Omega = \mathbb{C}$, so that $t_1 = \infty$. A *quasiarc* is the image of $[0, \infty)$ under a quasiconformal homeomorphism of $\mathbb{C}$. Piecewise smooth arcs without zero angle cusps are quasiarcs, as well as for instance the van Koch snowflake. A simple geometric characterization of quasiarcs is the Ahlfors three-point condition (the diameter of any subarc is bounded above by a bounded multiple of the distance between the endpoints). Denote $\text{Lip}(\frac{1}{2})$ the space of Hölder continuous functions λ with exponent $1/2$, $|\lambda(s) - \lambda(t)| \leq c|s-t|^{1/2}$, and let $||\lambda||_{\frac{1}{2}}$ denote the smallest such c.

Theorem 1. *If γ is a quasiarc, then $\lambda \in Lip(\frac{1}{2})$. Conversely, there is a constant $C > 0$ such that if $\lambda \in Lip(\frac{1}{2})$ with $||\lambda||_{\frac{1}{2}} < C$, then $\mathbb{C} \setminus f_t(\mathbb{D})$ is a quasiarc for all t.*

Theorem 1 is quantitative in the sense that the quasiconformal constant K of $\mathbb{C} \setminus f_t(\mathbb{D})$ depends on C only, and $K \to 1$ as $C \to 0$. If the assumption $||\lambda||_{\frac{1}{2}} < C$ is replaced by the slightly weaker local condition

$$\inf_{\epsilon > 0} \sup_{|t-s| < \epsilon} \frac{|\lambda(t) - \lambda(s)|}{\sqrt{|t-s|}} < C,$$

then $\mathbb{C}\setminus f_t(\mathbb{D})$ still is a quasiarc. This follows from Lemma 2.7. The assumption $||\lambda||_{\frac{1}{2}} < C$ cannot be omitted: We show that there is a function $\lambda \in \mathrm{Lip}(\frac{1}{2})$ such that $\partial\Omega_t$ is not even locally connected (and $\Omega_t \nearrow \Omega = \mathbb{C}$).

By means of a simple transformation, the Loewner equation becomes an ODE: For $t \leq s$, set

$$\phi_{s,t} = f_s^{-1} \circ f_t.$$

Then $\phi_{s,t}$ maps $\mathbb{D}$ into $\mathbb{D}$, satisfies $\phi_{s,t}(0) = 0$ and $\phi'_{s,t}(0) = e^{t-s}$. By differentiating the identity $f_s \circ \phi_{s,t} = f_t$ with respect to s, (1.3) becomes

$$\partial_s \phi_{s,t}(z) = -\phi_{s,t}(z)\frac{\lambda(s) + \phi_{s,t}(z)}{\lambda(s) - \phi_{s,t}(z)}. \tag{1.4}$$

Notice that $\phi_{t,t} = \mathrm{id}$. Fix t and set $g_s = \phi_{t+s,t}$ for $0 \leq s \leq s_0 := t_1 - t$, and let $\xi(s) = \lambda(t+s)$. Then

$$\partial_s g_s = -g_s \frac{\xi(s) + g_s}{\xi(s) - g_s} \quad \text{for} \quad s \in [0, s_0), \quad g_0 = \mathrm{id}. \tag{1.5}$$

Notice that the solutions to the initial value problem (1.5) are unique, whereas the solutions to (1.1) and (1.3) are unique only up to post-composition with a conformal map of $\cup_t f_t(\mathbb{D})$, simply because no initial conditions are imposed (if $t_1 = \infty$, a certain terminal condition is automatically satisfied and hence solutions are unique in this case). Therefore it is more natural to work with (1.5). Theorem 1 is a simple consequence of

Theorem 2. *Let $\xi : [0, s_0] \to \partial\mathbb{D}$ be continuous and g_s the solution to (1.5). If $g_{s_0}(\mathbb{D})$ is a quasislit-disc, then $\xi \in Lip(\frac{1}{2})$. Conversely, if $||\xi||_{\frac{1}{2}} < C$, then $g_s(\mathbb{D})$ is a quasislit-disc for all s.*

A *quasislit-disc* is, by definition, the image of a radially slit disc under a quasiconformal map fixing $\mathbb{D}$ and 0. See Section 2 for basic properties. Theorems 1 and 2 are quantitative in the sense that the quasiconformal constant K and the $\mathrm{Lip}(\frac{1}{2})-$ norm depend on each other, and K is close to 1 if and only if $||\xi||_{\frac{1}{2}}$ is close to 0.

The idea of the proof of Theorem 2 is as follows: The Loewner equation can be viewed as a decomposition of a conformal map f into infinitesimal conformal factors. This suggests approximating f by a composition of (finitely many) conformal maps. Given $\xi \in \mathrm{Lip}(\frac{1}{2})$, it is not hard to approximate ξ by functions $\xi^{(\varepsilon)} \in \mathrm{Lip}(\frac{1}{2})$ for which it is easy to see that the solutions $g_s^{(\varepsilon)}$ generate slits. If $||\xi^{(\varepsilon)}||_{\frac{1}{2}}$ is small, then the conformal welding in these slits can be shown to be quasisymmetric, by analyzing (1.5) on $\mathbb{T}$. This easily implies that the slits are quasiconformal slits. By compactness of the space of quasislit domains, one obtains quasislits in the limit $\varepsilon \to 0$.

In (1.1-1.5), f_t and g_s are normalized in an interior point. A different important normalization requires f or g to fix a boundary point. Then it is more convenient to consider the upper halfplane $\mathbb{H}$ instead of $\mathbb{D}$, and to choose ∞ as this boundary point. The normalization $f'(t) = e^t$ is then to be replaced by the hydrodynamic normalization $f_t(z) = z - 2t/z + O(1/z^2)$ as $z \to \infty$ $(t \geq 0)$, and the equation corresponding to (1.5) is

$$\partial_s g_s = \frac{2}{\xi(s) - g_s} \quad \text{for} \quad s \in [0, s_0), \quad g_0 = \text{id}, \tag{1.6}$$

where $\xi : [0, s_0) \to \mathbb{R}$. See [17] (a derivation of this equation can be found in [11]). Our Theorems 1 and 2 remain true in this situation, and the proofs simplify slightly. There is one significant difference between (1.5) and (1.6). This is apparent by observing that the condition $\xi \in \text{Lip}(\frac{1}{2})$ for maps into $\mathbb{T}$, resp. $\mathbb{R}$, give the same *local* regularity, but not *global*: the lift of $\xi : \mathbb{R} \to \mathbb{T}$ to $\mathbb{R}$ is not necessarily in $\text{Lip}(\frac{1}{2})$ if ξ is. See the comment after the statement of Lemma 2.7.

There are many examples of functions $\xi(s)$ which are not $\text{Lip}(\frac{1}{2})$ but which generate simple arcs. For instance, it is not hard to see that a disc slit by a smooth arc that meets $\mathbb{T}$ tangentially cannot have $\xi \in \text{Lip}(\frac{1}{2})$. A more interesting class of examples is SLE_κ : If $0 \leq \kappa \leq 4$, then (1.6) with $\xi(t) = B(\kappa t)$ generates a simple arc in $\mathbb{H}$ almost surely, where B is Brownian motion on $\mathbb{R}$. The same is true for the disc versions (1.1-1.5), with $\lambda(t) = e^{iB(\kappa t)}$. This is proved in [16]. Previously O. Schramm [17] has proved that for $\kappa > 4$, the Loewner equation almost never generates arcs in $\mathbb{H}$ (the boundary of $g_s(\mathbb{D})$ is still locally connected and therefore a curve, see [16], but these curves are not simple curves in $\mathbb{H}$).

For a slit domain with analytic slit, it is known that the driving term ξ in the Loewner equation is real analytic. Recently, C. Earle and A. Epstein [4] have given a new proof of this fact, and also showed that for slits which are C^n, ξ is at least C^{n-1}. Using the methods of our paper, one can obtain yet another proof for analytic arcs, and also show that C^n curves yield ξ in $C^{n-1/2}$. Details of this will appear elsewhere.

The Loewner equation is related to a certain algorithm for finding the conformal map onto a given simply connected domain, discovered independently by R. Kühnau [9] and the first author. This algorithm has been implemented (see http:\\www.math.washington.edu\ ~marshall) and converges extremely well in practice, even for conformal maps onto snowflake-like domains. In a forthcoming paper, we will prove convergence of this algorithm.

The paper is organized as follows: Section 2 contains a discussion of quasislit discs. In Section 3, we consider the Loewner equation on $\mathbb{T}$ and show quasisymmetry of the welding for small $\text{Lip}(\frac{1}{2})-$ norm. The main Theorems are proved in Section 4. In Section 5, a pathological example is constructed.

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2 Slit discs

In this section we introduce the concept of quasislit-discs and establish their basic properties as needed in the proof of the main theorem.

Let $\Gamma \subset \overline{\mathbb{D}}$ be a Jordan arc with endpoints a and b. We are only interested in arcs that begin in the circle $\mathbb{T}$ and are contained in $\mathbb{D} \setminus \{0\}$, $\Gamma \cap \mathbb{T} = \{a\}$, say. Call such a domain a *slit-disc* with *initial point* a and *tip* b. For every $0 < t < 1$ the domain $D_t = \mathbb{D} \setminus [t, 1]$ is a slit disc.

A slit disc D is called a *K-quasislit-disc* if there is a $K-$quasiconformal homeomorphism f of the sphere and $0 < t < 1$ such that $f(D_t) = D$ and $f(0) = 0$. See the usual references [1] and [13] for background on quasiconformal maps, and [7], [15] for geometric properties of quasicircles (=images of circles or lines under quasiconformal maps). The requirement $f(D_t) = D$ implies $f(\mathbb{D}) = \mathbb{D}$, and since quasiconformal homeomorphisms of the disc extend to the plane by reflection (with the same maximal dilatation K), it suffices to consider $K-$quasiconformal homeomorphism f of the disc.

Denote $QSD(K)$, resp. $QSD(K, r)$ the set of all K-quasislit-discs, resp. those with conformal radius $|g'(0)| = r$, where g is a conformal map from $\mathbb{D}$ with $g(0) = 0$. By the compactness of the family of K-quasiconformal maps fixing $\mathbb{D}$ and 0 we immediately obtain

Lemma 2.1. *Equipped with the Caratheodory topology of domain convergence, $QSD(K, r)$ is compact.* □

It is easy to see that the Caratheodory topology coincides with the topology induced by the Hausdorff distance $d(\partial D_1, \partial D_2)$ of the boundaries of domains of $QSD(K, r)$. This is not true in the larger space of all slit discs.

We now give an analytic characterization in terms of the conformal welding. The conformal welding on a slit disc D is defined as follows. Let $g : \mathbb{D} \to D$ be conformal with $g(0) = 0$ and $g(1) = b$. Then there are two preimages $\alpha^+, \alpha^- \in \mathbb{T}$ of the initial point a. Let I^+ be the oriented arc $< 1, \alpha^+ >$ and I^- the arc $< \alpha^-, 1 >$ on $\mathbb{T}$ and set $I = I^+ \cup I^-$. Then the welding homeomorphism $\phi : I \to I$ is the unique continuous function with $\phi(\alpha^+) = \alpha^-$ and

$$g(x) = g(\phi(x)) \quad \text{for all x} \in I.$$

Notice that $\phi(1) = 1$.

Lemma 2.2. *$\mathbb{D} \setminus \Gamma$ is a quasislit-disc if and only if there is a constant $1 \leq M < \infty$ such that*

$$M^{-1} \leq \frac{|\phi(x) - 1|}{|x - 1|} \leq M \tag{2.1}$$

for all $x \in I^+$ and

$$M^{-1} \leq \frac{|\phi(x) - \phi(y)|}{|\phi(y) - \phi(z)|} \leq M \tag{2.2}$$

whenever $x, y, z \in I^+$ (in this order) with $|x-y| = |y-z|$. Furthermore, the quasislit constant K of $\mathbb{D} \setminus \Gamma$ depends on M only, with $K \to 1$ if and only if $M \to 1$.

Proof: Suppose (2.1) and (2.2) hold. Let $T : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a quasiconformal map with $T(i) = \alpha^+, T(-i) = \alpha^-, T(1) = 1$ and $T^{-1}(0) \in [-1, 1]$. For instance, T can be constructed as a composition of a quasiconformal map that sends α^+ and α^- to symmetric points, followed by a Möbius transformation. It is easy to verify that $\varphi = T^{-1} \circ \phi \circ T$ satisfies (2.1) and (2.2) on $< 1, i >$ and $< -i, 1 >$ (with a possibly larger constant M'). We extend φ to $\mathbb{T}$ by reflection in $i\mathbb{R}$, $\varphi(x) = \iota\varphi(\iota x)$, where $\iota x = -\overline{x}$. Next, define $\psi : \mathbb{T} \to \mathbb{T}$ by $\psi(x) = x$ on the upper halfcircle $< 1, -1 >$, and $\psi(x) = \varphi(\overline{x})$ on $< -1, 1 >$. It is not hard to check that ψ is an orientation-preserving quasisymmetric homeomorphism, i.e. (2.2) holds for ψ (with a constant M'') and *all* points x, y, z on $\mathbb{T}$ with $|x - y| = |y - z|$. Furthermore M'' is close to 1 if M is close to 1. Since ψ is quasisymmetric, it has a quasiconformal extension (again denoted ψ) to $\mathbb{D}$, and we can choose this extension such that $\psi([-1, 1]) = [-1, 1]$. It follows that $\psi^{-1}(T^{-1}(0)) \in (-1, 1)$. Hence there is a unique $0 < t < 1$ such that a conformal map h from $\mathbb{D}$ to D_t exists with $h(\psi^{-1}(T^{-1}0)) = 0$, $h(i) = h(-i) = 1$ and $h(1) = t$. Now $f = g \circ T \circ \psi \circ h^{-1}$ is a $K-$quasiconformal homeomorphism from D_t onto $\mathbb{D} \setminus \Gamma$ that extends to a homeomorphism of $\mathbb{D}$. Since the interval [t,1] is removable for quasiconformal maps, $\mathbb{D} \setminus \Gamma$ is a quasislit-disc.

Conversely, if D is a quasislit-disc and $f(D_t) = D$, consider the conformal maps $h : \mathbb{D} \to D_t$ with $h(0) = 0, h(1) = t$ and $g : \mathbb{D} \to D$ with $g(0) = 0, g(1) = b$ (the tip of D). Then $\psi = g^{-1} \circ f \circ h$ is a $K-$quasiconformal selfmap of the disc fixing 0 and 1, and therefore extends quasisymmetrically to $\mathbb{T}$. Since $\phi(x) = \psi(\overline{\psi(x)})$ on A, the Lemma follows. $\square$

Notice that the slit of a K-quasislit-disc obviously is a *K-quasiarc* (i.e. the image of a line segment under some K-quasiconformal map), whereas the converse is not true. Here is a geometric characterization.

Lemma 2.3. *The slit disc $\mathbb{D} \setminus \Gamma$ is a quasislit-disc if and only if Γ is a quasiarc that approaches $\mathbb{T}$ nontangentially.*

Proof: If $\mathbb{D} \setminus \Gamma$ is a quasislit-disc, then Γ is a quasiarc by definition, and the nontangential approach follows from the quasiinvariance of the hyperbolic distance.

Conversely, a disc slit by a nontangential quasiarc is a John-domain (see [15], Chapter 5 for the definition), with constant depending on the quasiarc-constant and the nontangential approach only. It follows from the doubling property of harmonic measure in John domains that (2.1) and (2.2) hold. From Lemma 2.2 we obtain that $\mathbb{D} \setminus \Gamma$ is a quasislit-disc. $\square$

Remark: The proof shows that Lemma 2.3 is quantitative in the following sense: If Γ is contained in the sector $\{z : -\alpha < \arg(1 - \overline{a}z) < \alpha$ and $|a - z| < \frac{1}{2}\}$, then the quasi-slit constant K of $\mathbb{D} \setminus \Gamma$ depends on $\alpha < \frac{\pi}{2}$ and the quasiarc constant L of Γ only. One can also show that $K \to 1$ as $\alpha \to 0$ and $L \to 1$. In the other direction, it is easy to see that $\alpha \to 0$ as $K \to 1$.

Lemma 2.4. *Let $D = \mathbb{D} \setminus \Gamma$ be a K-quasislit-disc with initial point a and tip b. Let c and d be points on Γ such that the arc Γ_c of Γ from c to a contains Γ_d. Let ϕ_c, ϕ_d be conformal maps of $\mathbb{D}$ onto $\mathbb{D} \setminus \Gamma_c, \mathbb{D} \setminus \Gamma_d$ fixing 0. Then $\phi_d^{-1}(\phi_c(\mathbb{D}))$ is a K-quasislit-disc, too.*

Proof: There exists $0 < t < 1$ and a K-quasiconformal selfmap f of $\mathbb{D}$ fixing 0 such that $f(D_t) = D$. Set $t_c = f^{-1}(c)$ and $t_d = f^{-1}(d)$ so that $t \le t_c < t_d \le 1$. Let $h : \mathbb{D} \to D_{t_d}$ be conformal with $h(0) = 0$ and $h'(0) > 0$. Then $F = \phi_d^{-1} \circ f \circ h$ is a K-quasiconformal homeomorphism of $\mathbb{D}$ fixing 0 with $F(D_{h^{-1}(t_c)}) = \phi_d^{-1}(\phi_c(\mathbb{D}))$. $\square$

Remark: Except for the sharp bound of the quasislit constant, Lemma 2.4 can also be obtained from the subinvariance principle of [6].

Lemma 2.5. *Let $D = \mathbb{D} \setminus \Gamma$ be a K-quasislit-disc with initial point a and tip b. Let $f : \mathbb{D} \to D$ be the conformal map with $f(0) = 0$ and assume that $r := f'(0) > 1/2$. Then*

$$|f^{-1}(b) - a| \le C\sqrt{1-r}, \tag{2.3}$$

where C depends continuously on K with $C(1) = 0$.

Proof: Set $A = f^{-1}(\Gamma)$. We are first going to show that

$$\text{diam } \ A \asymp \text{diam } \ \Gamma \asymp \sqrt{1-r}. \tag{2.4}$$

Let Γ^* denote the reflection of Γ in $\mathbb{T}$ and set $B = \Gamma \cup \Gamma^*$. Extend f by reflection in $\mathbb{T}$ to a conformal map $f : \hat{\mathbb{C}} \setminus A \to \hat{\mathbb{C}} \setminus B$. With the usual notation $\Delta = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, consider the conformal maps $\phi : \Delta \to \hat{\mathbb{C}} \setminus A$ and $\psi : \Delta \to \hat{\mathbb{C}} \setminus B$ with positive derivative at ∞. The derivatives are $\mathrm{cap\,A}$ and $\mathrm{cap\,B} = (1/r)\,\mathrm{cap\,A}$, where cap denotes logarithmic capacity. Since $\mathrm{cap\,X} \asymp \mathrm{diam\ X}$ for connected sets X, we obtain

$$\text{diam } A \asymp \ \mathrm{cap\,A} = \mathrm{r\,cap\,B} \asymp \mathrm{diam\ B} \asymp \text{diam } \Gamma.$$

Note also that diam $\Gamma \asymp 1 - \text{dist } (0, \Gamma)$ by Lemma 2.3. Thus

$$\log \frac{1}{f'(0)} = \int_A \log \left| \frac{e^{i\theta}}{f(e^{i\theta})} \right| \frac{d\theta}{2\pi} \le \frac{|A|}{2\pi} \log \frac{1}{\text{dist } (0, \Gamma)}$$

and we obtain

$$1 - r \le C\frac{|A|}{2\pi} \text{ diam } \Gamma \asymp (\text{diam } A)^2.$$

It remains to prove diam $A \le C\sqrt{1-r}$. To this end, let $A' = f^{-1}(B(b, \frac{1-|b|}{2}))$, then $|f| \le (1+|b|)/2$ on A'. Since $1 - |b| \asymp \text{diam } \Gamma$, we obtain

$$\log \frac{1}{f'(0)} \ge \frac{|A'|}{2\pi} \log \frac{2}{1+|b|} \asymp |A'|(1-|b|) \asymp |A'|\text{diam } \Gamma.$$

The doubling property of harmonic measure in John domains implies $|A'| \asymp |A|$ and we conclude diam $A \leq C\sqrt{1-r}$, finishing the proof of (2.4).

Since $\psi = f \circ \phi$ and $f(0) = 0$, we have

$$z_0 := \phi^{-1}(0) = \psi^{-1}(0).$$

Now $\tilde{\phi} = (\phi - f^{-1}(b))/\operatorname{cap A}$ and $\tilde{\psi} = (\psi - a)/\operatorname{cap B}$ are normalized univalent maps in Δ that omit the value 0. Since diam $\hat{\mathbb{C}} \setminus g(\Delta) \leq 4$ for normalized univalent maps, the functions $\tilde{\phi}(z) - z$ and $\tilde{\psi}(z) - z$ are analytic in Δ with boundary values of absolute value bounded above by 5. The maximum principle yields

$$|z_0 + \frac{f^{-1}(b)}{\operatorname{cap A}}| \leq 5 \quad \text{and} \quad |z_0 + \frac{a}{\operatorname{cap B}}| \leq 5.$$

Hence

$$|\frac{f^{-1}(b)}{\operatorname{cap A}} - \frac{a}{\operatorname{cap B}}| \leq 10,$$

and (2.3) follows since $|f^{-1}(b)| = |a| = 1$.

The claim $C(1) = 0$ can be proved by a normal families argument: Let $D_n = \mathbb{D} \setminus \Gamma_n$ be a sequence of K_n-quasislit-discs and $K_n \to 1$. We may assume that $a_n = 1$, then A_n (resp. B_n) "converge" (after rescaling) to a horizontal (resp. vertical) line segment. Therefore $\tilde{\psi}_n(z) \to z + 1/z$ and $\tilde{\phi}_n(z) \to z - 1/z$. Since $z_n = \phi_n^{-1}(0) = \psi_n^{-1}(0) \to \infty$, it follows that $\tilde{\psi}_n(z_n) - \tilde{\phi}_n(z_n) \to 0$ and $|f_n^{-1}(b_n) - a_n| = o(\operatorname{cap A_n})$ follows easily. □

The proof of the Hölder continuity of the Loewner parameter for quasislit-discs is an immediate consequence of Lemmas 2.4 and 2.5 and is explained in Section 4. For the converse, we need to construct quasislit-discs with prescribed $a, f^{-1}(b)$ and $f'(0)$. That this is possible is the content of the next lemma.

Lemma 2.6. *Given $a, a' \in \mathbb{T}$ and $0 < r < 1$, there is a slit-disc $D = D_{a,a',r}$ with initial point a and tip b such that the conformal map $f : \mathbb{D} \to D$ with $f(0) = 0$ satisfies*

$$f'(0) = r \quad \textit{and} \quad f(a') = b.$$

Moreover, for $r \geq 1/2$, $D_{a,a',r}$ can be chosen as a K-quasislit-disc with

$$K \leq C(\frac{|f^{-1}(b) - a|}{\sqrt{1-r}})$$

where $C : [0, \infty) \to [1, \infty)$ is continuous with $C(0) = 1$.

Proof: Replacing $f(z)$ by $af(a'z)$ we see that it suffices to construct for each $w \in \mathbb{D} \setminus \{0\}$ a conformal map f to a slit disc with initial point 1 such that 1 maps to the tip and $f(0) = 0, f'(0) = w$ holds.

Define the *canonical slits* S_b from points $b \in \mathbb{D} \setminus [-1, 0]$ to 1 as follows: Denote by $T(z) = (1-z)/(1+z)$ the conformal map from $\mathbb{D}$ to the right half plane, then

$$S_b = T^{-1}[0, T(b)].$$

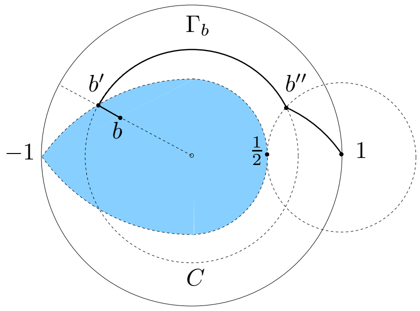

Figure 2.1: The region $d < 1 - |b|$ and a slit Γ_b.

Construct a family $D_b = \mathbb{D}\backslash\Gamma_b$ of slit discs with endpoint $b \in \mathbb{D}\backslash\{0\}$, together with the conformal maps f_b from $\mathbb{D}$ to D_b normalized by $f_b(0) = 0, f_b(1) = b$, with the following properties:

1. The function $m(b) = f_b'(0)$ has a continuous extension to $\overline{\mathbb{D}}$, with $m(0) = 0$ and $m|_{\mathbb{T}} =$ identity .
2. For $b \notin [-1, 0]$, Γ_b has initial point 1.
3. For $|b - 1| < 1/2$, $\Gamma_b = S_b$.

A particular family with these properties is constructed as follows: For $|b-1| < 1/2$ we have to set $\Gamma_b = S_b$. For $|b-1| \geq 1/2$ but $b \notin [-1, 0]$, let d be the distance from b to $[-1, 0]$. If $d < 1-|b|$, set $b' = b/(d+|b|)$ and let Γ_b be composed of the line segment $[b, b']$, the shorter arc of the circle $C = \{|z| = |b'|\}$ from b' to the intersection b'' of C with the circle $\{|z - 1| = 1/2\}$, and the canonical slit from $S_{b''}$. If $d \geq 1 - |b|$, Γ_b is composed of two arcs only, namely the circular arc of $C = \{|z| = |b|\}$ from b to the intersection b'' of C with the circle $\{|z-1| = 1/2\}$, and $S_{b''}$. Finally set $\Gamma_b = [-1, b]$ for $b \in [-1, 0]$. Then (2) and (3) are trivially satisfied, and (1) is an exercise in Caratheodory kernel convergence.

By topology, $m : \mathbb{D} \to \mathbb{D}$ is surjective and the existence claim of the lemma follows for $w \notin [-1, 0]$. For the general case, repeat the above construction with an exceptional line segment [0,i] instead of [-1,0], say.

The estimate of the lemma follows from the fact that $\frac{|f^{-1}(b)-a|}{\sqrt{1-r}}$ is small only if $|b - 1| < 1/2$, and that the ratio is close to zero if and only if b is close to $[1/2, 1]$.

□

Finally we will prove a partial converse of Lemma 2.4. This is useful for reducing the proof of Theorem 2 to a bounded time interval ($s_0 = 1$, say). If $D_i, 1 \leq i \leq n$, are slit-discs with initial points a_i and tips b_i define the *conformal concatination*

$$D_1 * D_2 * ... * D_n = g_n \circ ... \circ g_2 \circ g_1(\mathbb{D}),$$

where $g_i : \mathbb{D} \to D_i$ are conformal maps such that $g_i(a_{i-1}) = b_i$ and $g_i(0) = 0$. It is a slit disc too. The concationation of quasislit discs is a quasislit disc, with constant depending on the constants of D_i and on n. The next lemma says that if the conformal radii $|g_i'(0)|$ are bounded away from 1, the constant of the concatination can be bounded independently of n.

Lemma 2.7. *If D_i are $K-$ quasislit discs and $|g_i'(0)| \leq 1-\varepsilon$ for all $1 \leq i \leq n$, then $D_1 * ... * D_n$ is a $K'-$ quasislit disc with K' depending on ε and K only, and $K' \to 1$ as $K \to 1$ with ε fixed.*

It is worthwhile to mention that the corresponding statement is *not* true for quasislit *planes*, if concatination is defined using conformal maps with the hydrodynamic normalization instead of normalization in an interior point. This explains a difference between the "radial Loewner equation" (1.3) and the "chordal Loewner equation" (1.6) mentioned in the introduction: Whereas the function $\lambda(t) = e^{it}$ produces quasislit discs with bounded K (a logarithmic spiral), its lift $\xi(t) = t$ to $\mathbb{R}$ does not. The reason for this difference is that 0 is a hyperbolic (attractive) fixpoint for g_i, whereas ∞ is parabolic for conformal maps of $\mathbb{H}$ with hydrodynamic normalization.

Proof: We may assume that ε is sufficiently small and all D_i have conformal radius $\in [1-2\varepsilon, 1-\varepsilon]$ (write g_i as a composition of quasislit-maps, using Lemma 2.4). Let $\Gamma = \mathbb{D} \setminus (D_1 * D_2 * ... * D_n)$. Set $\gamma_j = \mathbb{D} \setminus D_j$, $\Gamma_j = g_n \circ ... \circ g_{j+1}(\gamma_j)$ and $\Gamma_n = \gamma_n$ so that $\Gamma = \cup_{j=1}^n \Gamma_j$. There are $r_0 < 1, r_1 < 1$ (depending on ε on K) such that $g_{j+1}(\gamma_j) \subset D(0, r_0)$ and $|g_j(z)| \leq r_1 |z|$ on $D(0, r_0)$. Hence there is an integer k_0 such that

$$\Gamma_j \subset B(0, \frac{1}{2}\text{dist } (\Gamma_{j+k}, 0)) \tag{2.5}$$

for all $k \geq k_0$ and all $j \leq n-k$. We may assume that ε is small enough to guarantee

$$\text{diam } \ \Gamma_j \leq \frac{1}{4}\text{dist } (\Gamma_j, 0). \tag{2.6}$$

To prove the Lemma if n is bounded is easy using compactness. Hence we may use the Lemma for k_0 and assume $n > k_0$. Then

$$\Gamma_j \cup ... \cup \Gamma_{j+k_0-1} \text{ is a quasiarc for all } j \tag{2.7}$$

with constant depending on K and ε. To see this, first apply the Lemma to $D_j * D_{j+1} * ... * D_{j+k_0-1}$ to obtain a quasislit disc, then apply g_{j+k_0-1} which changes the constant only by a bounded amount, and finally apply $g_n \circ ... \circ g_{j+k_0}$, using that every conformal map in $\mathbb{D}$ has a $(1+r_0)/(1-r_0)-$ qc extension from $D(0, r_0)$ to $\mathbb{C}$ (this is an easy consequence of the $\lambda-$Lemma). It is routine to check that every arc Γ satisfying (2.5),(2.6) and (2.7) also satisfies the Ahlfors three point property, hence is a quasiarc. It also follows that the set $\Gamma \cup \{0\}$ has the three point property, and from (2.7) it follows that Γ meets $\mathbb{T}$ nontangentially. This implies that $\mathbb{D} \setminus \Gamma$ is a $K'-$quasislit disc.

To prove that $K' \to 1$ as $K \to 1$ is easy (using compactness) for each fixed n. To prove independence of n goes along the same lines as the above proof. We omit the details. $\square$

3 The Loewner equation and conformal welding

Let $\xi : [0, s_0] \to \mathbb{T}$ be continuous and let g_s be the solution to (1.5),

$$\partial_s g_s(z) = -g_s(z)\frac{\xi(s) + g_s(z)}{\xi(s) - g_s(z)} \quad , \quad g_0(z) = z. \tag{3.1}$$

There is no loss of generality in assuming $\xi(0) = 1$. For ease of notation, extend ξ to $[0, \infty)$ by $\xi(s) \equiv \xi(s_0)$ for $s > s_0$. Notice that $g_s'(0) = e^{-s}$. First let us assume that $g_{s_0}(\mathbb{D}) = \mathbb{D} \setminus \Gamma_{s_0}$ is a slit disc. Then $g_s(\mathbb{D}) = \mathbb{D} \setminus \Gamma_s$ is a slit disc with initial point $\xi(s)$ for all $s \in [0, s_0]$. By Schwarz reflection, g_s is a conformal map of $\hat{\mathbb{C}} \setminus g_s^{-1}(\Gamma_s)$ onto $\hat{\mathbb{C}} \setminus (\Gamma_s \cup \Gamma_s^*)$, where $*$ denotes reflection in $\mathbb{T}$. Furthermore, (3.1) holds throughout $\mathbb{C} \setminus g_s^{-1}(\Gamma_s)$. Roughly speaking, the slit Γ_{s_0} is formed as follows: Points on $\mathbb{T}$ move along $\mathbb{T}$ towards the singularity $\xi(s)$ according to (3.1), until they actually meet the singularity. At this time, they leave the circle and move into $\mathbb{D}$. Points hit the singularity pairwise (from both sides of $\xi(s)$), hence points on $\mathbb{T}$ are identified under the welding homeomorphism if and only if they hit the singularity at the same time and before time s_0. More formally, for $z \in \mathbb{T}$ denote

$$T(z) = \sup\{s > 0 : g_t(z) \neq \xi(t) \text{ for all } t \in [0, s]\}. \tag{3.2}$$

If $T(z) < \infty$ we have $g_{T(z)}(z) = \xi(T(z))$, and we could have defined $T(z)$ by setting $T(z) = s$ for both preimages $g_s^{-1}(\xi(s))$.

As in Section 2, set $I = g_{s_0}^{-1}(\Gamma_{s_0}) \subset \mathbb{T}$ and consider the welding $\phi : I \to I$ that interchanges the endpoints of I and satisfies

$$g_{s_0}(x) = g_{s_0}(\phi(x)) \quad \text{ for all x} \in I,$$

in particular $\phi(1) = 1$. For $z \in I$ we then have

$$T(z) = T(z') \quad \text{ if and only if } \quad \phi(z) = z' \text{ or } z = z'. \tag{3.3}$$

In the special case $\xi(t) = \xi_0(t) \equiv 1$, calculation shows

$$g_t(z) = e^t \left(1 + z\,(2 - 2e^{-t} + z) - (1+z)\sqrt{1 + z(2 - 4e^{-t} + z)}\right)/(2z)$$

so that

$$g_t^{-1}(1) = 2e^{-t} - 1 \pm i2e^{-t}\sqrt{e^t - 1}$$

and

$$T_0(e^{i\varphi}) = -\log\frac{1 + \cos\varphi}{2}. \tag{3.4}$$

For later use, write

$$G(t,x) = \arg g_t(e^{ix}).$$

Now let us drop the assumption of $g_{s_0}(\mathbb{D})$ being a slit disc. Then we can still define $T(z)$ by (3.2), and we would like to show that under the assumption of $||\xi||_{\frac{1}{2}}$ being small, pairs of points z, z' hit the singularity in finite time.

Lemma 3.1. *There is a constant C_0 such that if $||\xi||_{\frac{1}{2}} < C_0$, then*

$$C^{-1} \leq \frac{T(z)}{T_0(z)} \leq C$$

for all $z \in \mathbb{T}$ with $\Re z > 0$, where $C = C(||\xi||_{\frac{1}{2}}) \to 1$ as $||\xi||_{\frac{1}{2}} \to 0$.

Proof: The idea is quite simple and best explained for the half-plane version (1.6), $\partial_s g_s(x) = 2/(\xi(s) - g_s(x))$: If $\xi(0) = 0, x > 0$ and $s = \varepsilon x^2$ for some small fixed ε, then $|\xi(t)| < x\sqrt{\varepsilon}||\xi||_{\frac{1}{2}}$ for $t \leq s$ so that $g_s(x) - \xi(s) \sim x - 2s/x - \xi(s) \sim (1 - C')x$, so that $g_s - \xi$ made a definite advance towards 0. Repeating this argument one can estimate the time T for which $g_s - \xi = 0$. This idea can easily be made precise, but we need some notation.

Fix $z = e^{i\varphi_0}$ with $0 < \varphi_0 \leq \pi/2$ and set $\varphi(s) = \arg g_s(z) = -i \log g_s(z)$, so $\varphi(0) = \varphi_0$. Writing $\xi(s) = e^{i\psi(s)}$ with $\psi(0) = 0$ we have

$$\varphi'(s) = -i\frac{\partial_s g_s(z)}{g_s(z)} = i\frac{\xi(s) + g_s(z)}{\xi(s) - g_s(z)} = -\cot\frac{\varphi(s) - \psi(s)}{2} \tag{3.5}$$

and in particular (the case $\psi \equiv 0$)

$$\partial_s G(s,\tau) = -\cot\frac{G(s,\tau)}{2}. \tag{3.6}$$

Define for $0 < \varphi_1 < \varphi_0$

$$T(\varphi_0, \varphi_1) = \inf\{t : \varphi(t) - \psi(t) = \varphi_1\}, \tag{3.7}$$

and for $0 < \varphi_1 < \varphi_2 \leq \pi/2$

$$T_0(\varphi_2, \varphi_1) = \inf\{t : G(t, \varphi_2) = \varphi_1\} = -\log\frac{1 + \cos\varphi_2}{2} + \log\frac{1 + \cos\varphi_1}{2}.$$

We claim that there is a universal constant $\varepsilon > 0$ and $C = C(||\xi||_{\frac{1}{2}})$ with $C(t) \to 1$ as $t \to 0$, such that

$$\frac{1}{C} \leq \frac{T_0(\varphi_0, (1-\varepsilon)\varphi_0)}{T(\varphi_0, (1-\varepsilon)\varphi_0)} \leq C. \tag{3.8}$$

Repeated application of this estimate and summation yields the Lemma.

To proof the estimate, fix a small constant ε and set

$$\sigma := T_0(\varphi_0, (1-\varepsilon)\varphi_0) \leq C\varepsilon\varphi_0^2,$$

where we have used (3.4). Then $|\psi(s)| \leq \delta := \sqrt{\sigma}||\xi||_{\frac{1}{2}} = o(\varphi_0)$ for $0 \leq s \leq \sigma$, where the $o(\varphi_0)$ refers to $||\xi||_{\frac{1}{2}} \to 0$. From (3.5) and (3.6) we get

$$-\cot\frac{\varphi(s)-\delta}{2} \leq \varphi'(s) \leq -\cot\frac{\varphi(s)+\delta}{2}, \quad \partial_s G(s,\varphi_0 \pm \delta) = -\cot\frac{G(s,\varphi_0 \pm \delta)}{2}$$

and obtain

$$G(s,\varphi_0-\delta)+\delta \leq \varphi(s) \leq G(s,\varphi_0+\delta)-\delta.$$

Using $|\partial_x G(\sigma,x)| < C_1$ for $\sigma < C_2 x^2$ we obtain

$$\varphi(\sigma)-\psi(\sigma) \leq G(\sigma,\varphi_0+\delta) \leq G(\sigma,\varphi_0)+C_1\delta = (1-\varepsilon)\varphi_0+C_1\delta = (1-\varepsilon+o(1))\varphi_0$$

as $||\xi||_{\frac{1}{2}} \to 0$. The lower bound $\varphi(\sigma)-\psi(\sigma) \geq (1-\varepsilon-o(1))\varphi_0$ is obtained in the same way, and (3.8) and the Lemma follows easily. □

Lemma 3.2. *There is a constant C_0 such that if $||\xi||_{\frac{1}{2}} < C_0$, $z_1, z_2, z_3 \in I^+ \cap \{\Re z > 0\}$ and*

$$|z_1 - z_2| = |z_2 - z_3|,$$

then

$$\frac{1}{C} \leq \frac{|\phi(z_1)-\phi(z_2)|}{|\phi(z_2)-\phi(z_3)|} \leq C$$

where $C \to 1$ as $||\xi||_{\frac{1}{2}} \to 0$.

Proof: Write $z_j = e^{i\varphi_j}$, $g_s(z_j) = e^{i\varphi_j(s)}$ and $\xi(t) = e^{i\psi(t)}$. After relabeling we may assume $0 < \varphi_1 < \varphi_2 < \varphi_3$. If $\varphi_3 - \varphi_1 \geq \varphi_1$, then the lemma follows from Lemma 3.1 and (3.4) since $T(z_j) = T(\phi(z_j))$ and since we may assume that $||\xi||_{\frac{1}{2}}$ is close to zero. Hence we may assume $\varphi_3 - \varphi_1 < \varphi_1$.

Since $(\varphi_1(t)-\psi(t))/(\varphi_3(t)-\varphi_1(t)) \to 0$ as $t \to T(z_1)$, there is T such that $\varphi_1(T)-\psi(T) = \varphi_3(T)-\varphi_1(T)$. Set

$$q(t) = \frac{\varphi_3(t)-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}.$$

We claim that $q(t)$ is bounded for $0 \leq t \leq T$. In fact, we will prove that $q(t)/q(0)$ is bounded for $0 \leq t \leq T$ whenever $0 < \varphi_1 < \varphi_2 < \varphi_3$ and $\varphi_3 - \varphi_1 < \varphi_1$, without assuming that $\varphi_3 - \varphi_2 = \varphi_2 - \varphi_1$: Set

$$d_j(t) = \varphi_j(t) - \psi(t) > 0 \quad \text{and} \quad \delta(t) = \varphi_3(t) - \varphi_1(t) > 0.$$

From (3.5) and $d_1(t) < d_3(t) \leq 2d_1(t)$ for $0 \leq t \leq T$ we have

$$\partial_t \delta = \frac{\sin(\varphi_3-\varphi_1)/2}{\sin(d_3/2)\sin(d_1/2)} \asymp \frac{\varphi_3-\varphi_1}{d_1^2}$$

and

$$\begin{aligned}\partial_t \log q &= \frac{1}{\sin(d_1/2)}\left(\frac{\sin(\varphi_3-\varphi_1)/2}{(\varphi_3-\varphi_1)\sin(d_3/2)} - \frac{\sin(\varphi_2-\varphi_1)/2}{(\varphi_2-\varphi_1)\sin(d_2/2)}\right) \\ &\asymp \frac{\varphi_3-\varphi_2}{d_1 d_2 d_3} + O(1),\end{aligned}$$

again using $d_3(t) \leq 2d_1(t)$.

Since $\partial_t \delta > 0$ we have $\varphi_3(t) - \varphi_2(t) < \delta(t) < \delta(T) = d_1(T)$, and we obtain

$$|\partial_t \log q(t)| \leq C d_1(T)/d_1(t)^3 + O(1).$$

From Lemma 3.1 and (3.4) we know $T(e^{i\varphi}) \asymp \varphi^2$, hence $T \leq C$. Write $T(z) = s + \tilde{T}_s(g_s(z))$ with $\tilde{T}_s$ being the time where the image of $g_s(z)$ under the Loewner flow associated with the function $t \mapsto \xi(t+s)$ hits the singularity $\xi(t+s)$. Lemma 3.1 yields $\tilde{T}_s(e^{i\varphi}) \asymp (\varphi - \psi(s))^2$ and we obtain

$$d_1(s)^2 \asymp \tilde{T}_s(g_s(z_1)) = T(z_1) - s = T + \tilde{T}_T(g_T(z_1)) - s \geq T - s + Cd_1(T)^2.$$

Thus

$$\begin{aligned}\log q(T)/q(0) &\leq C \int_0^T \left(\frac{d_1(T)}{d_1(t)^3} + O(1)\right) dt \\ &\leq C d_1(T) \int_{Cd_1(T)^2}^{\infty} s^{-\frac{3}{2}} ds + O(1) = O(1).\end{aligned}$$

Similarly one can proof that $(\varphi_3(t) - \varphi_1(t))/(\varphi_3(t) - \varphi_2(t))$ is bounded.

To finish the proof of the lemma, we can argue as in the case $\varphi_3 - \varphi_1 \geq \varphi_1$ to obtain that $|(g_T(\phi(z_1)) - g_T(\phi(z_2)))/(g_T(\phi(z_2)) - g_T(\phi(z_3)))|$ is bounded above and below, and then use the boundedness of the quotient (now running the equation backwards from time T to 0) to conclude that $|(\phi(z_1) - \phi(z_2))/(\phi(z_2) - \phi(z_3))|$ is bounded above and below. □

4 Proof of the Theorems

In the introduction we have described how to pass from a Loewner chain f_t associated with a slit domain to the equation (1.5). This can be reversed: If g_s solves (1.5), $0 \leq s \leq s_0$, then

$$f_t = g_{s_0} \circ g_{s_0+t}^{-1} \quad , \quad -s_0 \leq t \leq 0 \tag{4.1}$$

is a Loewner chain (more precisely, f_t has an analytic extension to $\mathbb{D}$ that constitutes a Loewner chain) satisfying (1.3), and $f_{-s_0} = g_{s_0}$, $f_0 = id$, and $\lambda(t) = \xi(s_0 + t)$. This follows from observing that all f_t are well-defined on some subdomain of $\mathbb{D}$, that (1.3) holds there because of (1.5), and that (1.3) has analytic solutions in all of $\mathbb{D}$. See ([14], Theorem 6.2). Set

$$\phi_{s,t} = f_s^{-1} \circ f_t.$$

For $-s_0 = t_0 < t_1 < ... < t_n = 0$ we have

$$g_{s_0} = f_{-s_0} = \phi^{(n)} \circ \phi^{(n-1)} \circ ... \circ \phi^{(1)} \quad , \quad \phi^{(j)} = \phi_{t_j, t_{j-1}}.$$

The $\phi^{(j)}$ are the (unique) solutions to

$$\partial_s \phi_s^{(j)}(z) = -\phi_s^{(j)}(z) \frac{\lambda_j(s) + \phi_s^{(j)}(z)}{\lambda_j(s) - \phi_s^{(j)}(z)} \quad , \quad \lambda_j(s) = \lambda(t_{j-1} + s) \quad , \quad \phi_0^{(j)} = id \tag{4.2}$$

with $0 \leq s \leq t_j - t_{j-1}$. Conversely, if solutions $\phi^{(j)}$ to (4.2) are given with $\lambda_j(t_j - t_{j-1}) = \lambda_{j-1}(0)$, then the solutions f_{-s_0} resp. g_{s_0} to (1.3) resp. (1.5) are just $\phi^{(n)} \circ \phi^{(n-1)} \circ ... \circ \phi^{(1)}$. In particular, if all $\phi^{(j)}$ map to slit discs, then f_t and g_s map to slit discs.

Proof of Theorem 2: Let us first assume that $g_{s_0}(\mathbb{D}) = \mathbb{D} \setminus \Gamma$ is a $K-$quasislit disc. Consider the Loewner chain (4.1) described above. By Lemma 2.4, $f = f_s^{-1} \circ f_t = g_{s+s_0} \circ g_{t+s_0}^{-1}$ maps onto a $K-$ quasislit disc. Since the initial point of $f(\mathbb{D})$ is $a = \xi(s)$ and the tip is $b = f(\xi(t))$, we have

$$|\xi(t) - \xi(s)| \leq C\sqrt{1 - e^{t-s}} \leq C\sqrt{s-t}$$

by Lemma 2.5, which proves $\xi \in \mathrm{Lip}(\frac{1}{2})$.

Now let $\xi : [0, s_0) \to \mathbb{T}$ with small $||\xi||_{\frac{1}{2}}$ be given. We will first assume $s_0 \leq \sigma_0$ where σ_0 is a small enough to ensure that $T(z) > \sigma_0$ for $|z| = 1, \Re z < 0$ (see Lemma 3.1). In order to approximate ξ by functions $\xi_n \in \mathrm{Lip}(\frac{1}{2})$, set

$$t_j = \frac{j}{n} s_0 \quad , \quad \xi_j = \xi(t_j) \quad , \quad r = e^{-\frac{s_0}{n}}$$

for $0 \leq j \leq n$, and let

$$D_j = D_{\xi_{j+1}, \xi_j, r}$$

be the slit disc provided by Lemma 2.6. Thus the normalized conformal map $\phi^{(j)} : \mathbb{D} \to D_j$ maps ξ_j to the tip of D_j, and ξ_{j+1} is the base. Let $\lambda_j : [0, \frac{s_0}{n}] \to \mathbb{T}$ be the Loewner parameter of D_j (i.e. the function obtained from (1.3) for $\Omega = D_j$). Then $\lambda_j(0) = \xi_j$, $\lambda_j(\frac{s_0}{n}) = \xi_{j+1}$ and $||\lambda_j||_{\frac{1}{2}} \leq C(||\xi||_{\frac{1}{2}})$. The latter follows from D_j being a quasislit disc by Lemma 2.6, combined with the already proven part of the theorem. Notice that $C(t) \to 0$ as $t \to 0$.

As explained above, the composition

$$g_{s_0}^{(n)} = \phi^{(n-1)} \circ ... \circ \phi^{(0)} \tag{4.3}$$

is the solution to (1.3) with parameter $\xi_n : [0, s_0] \to \mathbb{T}$ given by

$$\xi_n|_{[t_j, t_{j+1}]}(t) = \lambda_j(t - t_j).$$

It follows that

$$||\xi_n||_{\frac{1}{2}} \leq 2C(||\xi||_{\frac{1}{2}}) + ||\xi||_{\frac{1}{2}},$$

and the right hand side is small if $||\xi||_{\frac{1}{2}}$ is small.

From Lemmas 2.2, 3.1 and 3.2 it follows that $D(n) = g_{s_0}^{(n)}(\mathbb{D})$ is a quasislit disc. Indeed, by the assumption $s_0 \leq \sigma_0$ we have $I^+ \subset \{Rez > 0\}$. Hence (2.1) follows from Lemma 2.2, (3.4), and Lemma 3.1, whereas (2.2) follows from Lemma 3.2. Finally, the case $s_0 > \sigma_0$ follows from Lemma 2.7.

By Lemma 2.1, $g_s^{(n)}(\mathbb{D})$ converges to a conformal map g_s onto a quasislit disc. Since ξ_n converges to ξ, passing to the limit in (1.5) on $\mathbb{D}$, we conclude that g_s must also satisfy (1.5) and Theorem 2 follows. $\square$

Proof of Theorem 1: If γ is a quasiarc, then the proof of Lemma 2.4 shows that $\phi_{s,t} = f_s^{-1} \circ f_t$ maps $\mathbb{D}$ onto quasislit discs, and $\lambda \in \mathrm{Lip}(\frac{1}{2})$ follows from Lemma 2.5.

Conversely, if $||\lambda||_{\frac{1}{2}}$ is small, then the solutions g_s of (1.3) with $\xi(s) = \lambda(s)$ map to quasislits by Theorem 2. It is well-known that $e^s g_s \to f_0$ locally uniformly in $\mathbb{D}$ as $s \to \infty$ (see [14], Theorem 6.3). By compactness of the space of quasislit planes with conformal radius 1, the theorem follows. $\square$

5 Examples

We want to construct a connected unbounded set that is not locally connected such that the driving term $\lambda(t)$ in the Loewner equation is in $\mathrm{Lip}(\frac{1}{2})$. To this end, let γ_1 be a logarithmic spiral, for instance $\gamma_1(t) = te^{i \log|t|}, -1 \leq t \leq 1$. Let γ_2 be a smooth curve joining one endpoint of γ_1 to ∞ within $\mathbb{C} \setminus \gamma_1$. Let $F : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \mathbb{D}$ be the homeomorphism $F(z) = (|z|+1)z/|z|$. Then $\Gamma = F(\gamma_1 \cup \gamma_2)$ has the desired properties: Clearly Γ is not locally connected. Fix a point $z_0 \in \mathbb{C} \setminus (\Gamma \cup \mathbb{D})$. Then $\mathbb{C} \setminus \Gamma$ can be embedded in a Loewner chain (with respect to z_0) as follows: Fix a parametrization γ of $\gamma_1 \cup \gamma_2$ and set $t_0 = \gamma^{-1}(0)$. The domains $\hat{D}_t = \mathbb{C} \setminus \gamma[t, \infty)$ are continuously increasing. Set $D_t = F(\hat{D}_t)$ for $t \leq t_0$ and $D_t = F(\hat{D}_t) \cup \mathbb{D}$ for $t > t_0$. It is easy to check that for $s < t$, there is a crosscut $c_{s,t}$ of D_t separating z_0 from $D_t \setminus D_s$ with diam $c_{s,t} \to 0$ as $s \to t$. Thus the condition of ([11], Theorem 2.6) is satisfied and D_t together with the conformal maps $f_t : \mathbb{D} \to D_t$, $f_t(0) = z_0$ form a Loewner chain satisfying (1.3) with continuous λ (assuming that (D_t) is parametrized such that $f_t'(0) = e^t$). In order to show $\lambda \in \mathrm{Lip}(\frac{1}{2})$, it suffices to show

$$|\lambda(t) - \lambda(t_0)| \leq C|t - t_0|^{\frac{1}{2}}. \tag{5.1}$$

Assume $t < t_0$ (the proof for $t > t_0$ is similar). Define a sequence $T_n \nearrow t_0$ inductively by choosing $T_0 < t_0$ and setting

$$T_{n+1} = \inf\{t > T_n : |\Gamma(t) - \Gamma(T_n)| = |\Gamma(T_n)| - 1\}.$$

Since consecutive turns of the logarithmic spiral γ_1 have geometrically decreasing distances from 0, we have that $|\Gamma(T_{n+1}) - \Gamma(T_n)|$ is comparable to the distance of $\Gamma(T_n)$ to the "previous turn" of Γ. Then it is easy to see that $\Gamma_n = \Gamma[T_n, T_{n+1}]$ has harmonic measure

$$\omega_n = \omega(z_0, \Gamma_n, D_{T_n})$$

satisfying

$$\omega_{n+k} \leq C_1 e^{-C_2 k} \omega_n \quad \text{for all} \quad k \geq 1. \tag{5.2}$$

Furthermore, $f_{T_n}^{-1}(\Gamma_n)$ is a smooth quasislit (with uniformly bounded K). From (2.4) we obtain

$$|\lambda(T_{n+1}) - \lambda(T_n)| \leq C \mathrm{diam}\, f_{T_n}^{-1}(\Gamma_n) \leq C\omega_n.$$

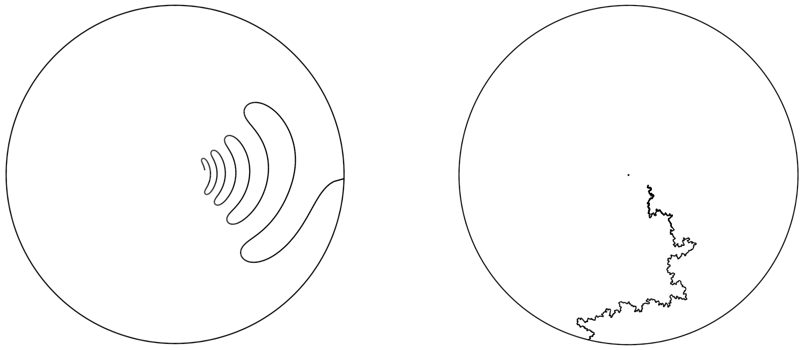

Figure 5.1: $\lambda(t) = \sin(8\pi t)$ $\lambda(t) =$ random walk

By Lemma 2.5

$$T_{n+1} - T_n \geq C\omega_n^2,$$

and the claim (5.1) follows from (5.2) by summation. $\square$

Finally we would like to mention that our proof of Theorem 2 leads to a simple algorithm for approximating the solutions of (1.3): The compositions (4.3) are converging to g_s as proved above, so the problem reduces to finding the maps $\phi^{(j)}$. Using the conformal maps $z \mapsto (z+a)^a(z+1-a)^{1-a}$ of the upper half plane to the half plane slit by a line segment of angle $a\pi$, this problem essentially reduces to finding the exponents a. To do this numerically is not hard. This algorithm has been implemented by our undergraduate research assistants Tarn Adams, Gary Look and Julie Rowlett, whom we would like to thank for their effort. The two pictures were produced with this algorithm. In the second picture, $\lambda(t)$ is the random walk

$$\lambda(\frac{j}{n}) = \exp(i\sqrt{\frac{2}{n}}\sum_{k=0}^{j} \pm 1), \quad 0 \leq j \leq n.$$

Since this approximates $B(2t)$, this should (and does) look like a "typical" path of the Loop Erased Random Walk by [17].

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