ON HAUSDORFF MEASURES AND SBR MEASURES FOR PARABOLIC RATIONAL MAPS

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Abstract: If J is the Julia set of a parabolic rational map having Hausdorff dimension $h < 1$, we show that Sullivan's h-conformal measure on J is either absolutely continuous or orthogonal with respect to the Hausdorff measures defined by the function

$$f(t) = t^h \left(\log \frac{1}{t}\right)^\tau ,$$

according whether $\tau > \tau_0$ or $\tau \leq \tau_0$ for some explicitly computable $\tau_0 > 0$.

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1. A dichotomy for SBR measures and Hausdorff measures

An analytic endomorphism T of the Riemannian sphere $\overline{\mathbb{C}}$ and of degree ≥ 2 is called parabolic if its Julia set does not contain any critical point but a rationally indifferent periodic point (i.e. its multiplier is a root of unity). It is known that the Hausdorff dimension h of the Julia set of such a transformation satisfies $\frac{1}{2} < h < 2$ (see [1]) and that there exists a unique non-atomic probability measure m determined by the Jacobian

$$\frac{dm \circ T}{dm} = |T'|^h \qquad \text{a.s.}$$

Such measures are called h-conformal in the sense of Sullivan ([11]).

In the hyperbolic case this measure is a multiple of the h-dimensional Hausdorff measure. In the parabolic case this is also true as long as $h \geq 1$ ([5]). However, if $h < 1$, the h-dimensional Hausdorff measure is zero. In this note we examine the relation between Hausdorff measures and conformal measures in more detail. In particular, we compare the conformal measure with Hausdorff measures having a gauge function different from $t \mapsto t^h$ and show a dichotomy in the behaviour of the measures (see Theorem 1.1 below).

The dynamics of parabolic rational maps is essentially determined by the behavior around rationally indifferent periodic points (also called parabolic points). Let Ω denote the set of these points (which are always contained in J, see e.g. [3]), and for $\omega \in \Omega$ let T_ω^{-p} denote the analytic inverse branch of T^p which fixes ω, where p is also so large that $(T^p)'(\omega) = 1$. Then define $p(\omega) + 1$ to be the smallest integer for which the associated coefficient in the Taylor expansion of T_ω^{-p} does not vanish. This means

$$T_\omega^{-p}(z) = z - a(z-\omega)^{p(\omega)+1} + \ldots$$

Next define

$$\alpha(\omega) = \frac{p(\omega)+1}{p(\omega)} h, \tag{1.1}$$

and

$$\alpha = \min_{\omega \in \Omega} \alpha(\omega).$$

The dynamics of parabolic rational functions under the potential $-h \log |T'|$ can be characterised using the coefficient α, which always lies in the open interval $(1, 4)$. There exists a unique T–invariant measure μ equivalent to m. This measure is called the SBR-measure of T (cf. [2]). It is finite or infinite and σ-finite according to whether $\alpha > 2$ or $\alpha \leq 2$. In particular, if the Hausdorff dimension $h \leq 1$ then the SBR measure is infinite.

We shall prove the following theorem.

Theorem 1.1. *Let T be a parabolic rational map of the Riemann sphere $\overline{\mathbb{C}}$ with Hausdorff dimension of its Julia set $h < 1$. Then Sullivan's h-conformal measure m (equivalently the*

SBR measure) is either absolutely continuous or orthogonal with respect to the Hausdorff measure associated to

$$f(t) = t^h \left(\log \frac{1}{t}\right)^{\gamma(1-h)}$$

according whether $\gamma > (\alpha - 1)^{-1}$ *or* $\gamma \le (\alpha - 1)^{-1}$.

Remark 1.2. More generally, let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a homeomorphism with $\varphi(0) = 0$ such that $\varphi(2x)/\varphi(x)$ is bounded above and away from 1 for large x, and consider the gauge function

$$f(t) = t^h \left(\varphi(\log(\frac{1}{t}))\right)^{1-h}.$$

Then our proof shows that m is absolutely continuous or orthogonal with respect to the associated Hausdorff measure depending on whether

$$\int_1^\infty \frac{\varphi^{-1}(x)}{x^\alpha} dx \tag{1.2}$$

converges or diverges. □

Comparisons of this type for Gibbs measures (with potential ϕ) of an expansive rational function and Hausdorff measures are well known. Let ϕ be Hölder continuous (with the additional property that $P(T, \phi) > \sup_{x \in \overline{\mathbb{C}}} \phi(x)$ if T is not expanding) with Gibbs measure μ_ϕ. For a function $f : [1, \infty) \longrightarrow \mathbb{R}_+$ and for sufficiently small $t > 0$ define

$$\widetilde{f}(t) = t^\kappa \exp\left(\frac{\sigma}{\sqrt{\chi}} f(-\log t)\sqrt{-\log t}\right),$$

where κ denotes the Hausdorff dimension of μ_ϕ, χ the Lyapunov exponent and σ a certain asymptotic variance (cf. [6] or [9] for a precise definition) with respect to the jump transformation (see the definition below).

Then $\mu_\phi \ll H_{\widetilde{f}}$, if f belongs to the lower class, and $\mu_\phi \perp H_{\widetilde{f}}$ if f belongs to the upper class. In particular, taking $f \equiv 0$, it follows that μ_ϕ is orthogonal to the κ–dimensional Hausdorff measure H_κ on $J(T)$.

These results originated from investigations of harmonic measures on the boundary of an open topological disc in the complex plane (see [8] and [10]). The paper [9] assumes the presence of holomorphic dynamics, treating the harmonic measure on the boundary of an RB-domain as a particular case of a projection of a Gibbs measure, and in [7], the case of a parabolic domain was considered using the theory of Markov fibred systems in the presence of a finite invariant measure equivalent to the harmonic measure.

The result we derive in the theorem is not of the type as those in [8], [9] and [7]. Instead of comparing with Hausdorff measures derived from the functions $t^\kappa \exp[c\sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}}]$ (where the dichotomy appears for some c, in particular $c = 0$ in case of the maximal $\kappa = h$), we compare with the measure obtained from $t^h \exp[c \log\log \frac{1}{t}]$, where the dichotomy appears for $c = \frac{1-h}{\alpha-1}$. This is due to the fact that fine estimates for the derivatives of T^n around parabolic points are not of the same probabilistic nature as in the case of a finite measure: that is they do not enter in the proofs in [7] at all. While for Gibbs measures it is possible to use upper and lower class results obtained from almost sure invariance principles for sums of stationary and weakly mixing sequences of random variables, this is no longer true for infinite SBR measures, and we develop a new approach to examine the relation between Hausdorff measures and the conformal measure, which is through the Schweiger formalism. This formalism is explained briefly in Section 2, where we also prove some auxiliary probabilistic results needed for the proof of Theorem 1.1. The theorem will be shown in Section 3.

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2. Convergence rates of return times

We begin by recalling some notation and definitions needed from the theory of Markov fibred systems ([1]).

Let $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a parabolic rational map. Denote by Ω the set of parabolic (periodic) points. There exists a neighbourhood $U = B(\Omega, \Theta)$ of Ω with the following properties: Define

$$S : J(T) \to \mathbb{N}$$

by

$$S(z) := \begin{cases} \min\{n \geq 1 : T^n(z) \notin U\} & \text{if } z \in U \\ 1 & \text{if } z \notin U, \end{cases}$$

and Schweiger's jump transformation $T_* : J(T) \to J(T)$ by

$$T_*(z) = T^{S(z)}(z).$$

There exists a T_*-invariant probability measure m^* on $J(T)$ which is absolutely continuous with respect to the h-conformal measure m with density satisfying $\log \frac{dm^*}{dm} \in L_\infty$. m^* also defines the invariant SBR-measure μ by Kac' formula

$$\mu(B) = \frac{\sum_{k=0}^{\infty} m^*(T^{-k}(B) \cap \{S > k\})}{m^*(S)},$$

where $m^*(S) = \int S \, dm^*$. Below, the expectation operator is always understood with respect to m^*. m^* is a probability with good mixing properties: There exists a (countable) Markov partition $\mathcal{R}$ of $J(T)$, which defines a ψ-mixing filtration with respect to T_*, such that U is a union of elements of $\mathcal{R}$ and such that S is constant on elements of $\mathcal{R}$. ψ-mixing means the following (see [1]): There exists a sequence $\psi(n) \to 0$ such that

$$\sup\left\{ \left| \frac{m^*(A \cap B)}{m^*(A) m^*(B)} - 1 \right| : A \in \mathcal{R}_0^k, B \in \mathcal{R}_{k+n}^{\infty} \right\} \leq \psi(n),$$

where $\mathcal{R}_s^t$ denotes the σ-algebra generated by the partitions $T^{-j}\mathcal{R}$ for $s \le j \le t$. Note that this property is stronger than ϕ-mixing and weaker than continued fraction mixing. Besides this notation we shall need several facts proved in [1] and [5] and which will be stated in the course of proof.

Let

$$X_0(x) = \inf\{n \ge 0 : T^n(x) \notin B(\Omega, \Theta)\}.$$

Note that $S - 1 = X_0$ on $B(\Omega, \Theta)$. Define the sequence X_k $(k \ge 1)$ of random variables by

$$X_k = X_0 \circ T_*^k.$$

Since S is constant on elements from $\mathcal{R}$, X_k is $T_*^{-k}\mathcal{R}$ measurable. It follows that this sequence $(X_k)_{k\ge 0}$ is ψ-mixing.

Let ϕ be as in Remark 1.2. In the remainder of this section we will show that $X_k \ge \phi(k)$ finitely or infinitely often (almost surely) depending on the convergence of the integral (1.2)

Lemma 2.1. *Let $\phi(n) \uparrow \infty$ as $n \to \infty$. If*

$$E(\phi^{-1}(X_0)) = \infty,$$

then

$$X_k \ge \phi(k) \quad \textit{infinitely often} \quad \textit{a.s.}$$

and, if

$$E(\phi^{-1}(X_0)) < \infty,$$

then

$$X_k \ge \phi(k) \quad \textit{finitely often} \quad \textit{a.s.}$$

Proof. Since X_k is integer valued

$$E(\sum_{k=0}^{\infty} 1_{\{X_k \ge \phi(k)\}}) = \sum_{k=0}^{\infty} m^*(\{X_k \ge \phi(k)\}) = \sum_{j=0}^{\infty} \sum_{k=0}^{\phi^{-1}(j)} m^*(X_0 = j) \sim E(\phi^{-1}(X_0)).$$

The second statement follows immediately from this.

In order to prove the first statement it suffices to show that for each $M > 0$ the set $D_M = \{\sum_{n\ge 1} 1_{\{X_n \ge \phi(n)\}} \le M\}$ has measure 0.

Define

$$B_n = \{X_n \ge \phi(n)\} \cap T_*^{-n-m} D_M$$

where m is chosen so that $\psi(m) < 1$, and

$$\nu_0(x) = \inf\{\nu \ge 1 : \ x \in B_\nu\}.$$

If $\nu_0(x) = \infty$ then $\sum_{n\geq 1} 1_{B_n}(x) = 0$ and if $\nu_0(x) < \infty$, there exists $N(x) \subset \mathbb{N}$ so that $\mathrm{card} N(x) \leq M$ and for $n \notin N(x)$

$$X_{n+m+\nu_0(x)}(x) = X_n(T_*^{\nu_0(x)+m}(x)) < \phi(n) \leq \phi(n+m+\nu_0(x)).$$

Therefore

$$\sum_{n\geq 1} 1_{B_n}(x) = \sum_{n=\nu_0(x)}^{\nu_0(x)+m} 1_{B_n}(x) + \sum_{n>\nu_0(x)+m} 1_{B_n}(x) \leq m+1+M.$$

It follows that

$$\int \sum_{n\geq 1} 1_{B_n}\ dm^* \leq M+m+1.$$

Since the set $\{X_n \geq \phi(n)\}$ is measurable with respect to $\mathcal{R}_0^n$ and $T_*^{-n-m}D_M$ is measurable with respect to $\mathcal{R}_{n+m+1}^{\infty}$ we obtain from the ψ-mixing property of m^* with respect to this filtration that

$$\begin{aligned}\int \sum_{n\geq 1} 1_{B_n}\ dm^* &= \sum_{n\geq 1} m^*(\{X_n \geq \phi(n)\} \cap T_*^{-n-m}D_M) \\ &\geq (1-\psi(m)) \sum_{n\geq 1} m^*(\{X_n \geq \phi(n)\}) m^*(T_*^{-n-m}D_M) \\ &\sim (1-\psi(m)) m^*(D_M) E(\phi^{-1}(X_0)).\end{aligned}$$

Since $E(\phi^{-1}(X_0)) = \infty$, we must have that $m^*(D_M) = 0$. ∎

Lemma 2.2. *Denote*

$$\alpha = \min_{\omega\in\Omega} \frac{p(\omega)+1}{p(\omega)} h$$

as before. Then there is a constant C *such that*

$$C^{-1} \int_1^{\infty} \frac{\phi^{-1}(x)}{x^{\alpha}} dx \leq E(\phi^{-1}(X_0)) \leq C \int_1^{\infty} \frac{\phi^{-1}(x)}{x^{\alpha}} dx.$$

In particular,

$$E(X_0^{1/\gamma}) \begin{cases} < \infty & \text{if } \gamma > \frac{1}{\alpha-1} \\ = \infty & \text{if } \gamma \leq \frac{1}{\alpha-1}. \end{cases}$$

Proof. By Lemma 8.9 in [1] there exists a constant C such that for $k \geq 1$

$$C^{-1}k^{-\alpha} \leq m(\{z : X_0(z) = k\}) \leq Ck^{-\alpha}.$$

Using that m^* is equivalent to m with Radon Nikodym derivative bounded away from 0 and ∞ we obtain

$$E(\phi^{-1}(X_0)) \sim \sum_{k=1}^{\infty} \phi^{-1}(k) k^{-\alpha} \sim \int_1^{\infty} \frac{\phi^{-1}(x)}{x^{\alpha}} dx.$$

Specializing to $\phi(x) = x^{\gamma}$ finishes the proof of the lemma. ∎

Proposition 2.3. *Let $\gamma > 1$. If*

$$E(X_0^{1/\gamma}) < \infty,$$

then

$$\lim_{n\to\infty} \frac{1}{n^\gamma} \sum_{k=0}^{n-1} X_k = 0 \qquad a.s.,$$

and if

$$E(X_0^{1/\gamma}) = \infty,$$

then

$$\limsup_{n\to\infty} \frac{1}{n^\gamma} \sum_{k=1}^{n-1} X_k = \infty \qquad a.s.$$

Proof. Assume first that $E(X_0^{1/\gamma}) < \infty$. Then

$$\begin{aligned} E\sum_{k=1}^{\infty} \frac{\min(X_k, k^\gamma)}{k^\gamma} &= \sum_{k=1}^{\infty} \frac{1}{k^\gamma} \sum_{j=1}^{k^\gamma} j m^*(X_k = j) + \sum_{k=1}^{\infty} m^*(X_k > k^\gamma) \\ &= \sum_{j=1}^{\infty} j \sum_{k \geq j^{1/\gamma}} \frac{1}{k^\gamma} m^*(X_0 = j) + \sum_{k=1}^{\infty} m^*(X_0 > k^\gamma) \\ &\ll \sum_{j=1}^{\infty} j (j^{1/\gamma})^{1-\gamma} m^*(X_0 = j) + E(X_0^{1/\gamma}) = 2E(X_0^{1/\gamma}) < \infty. \end{aligned}$$

Since $X_k \geq 0$,

$$\sum_{k=1}^{\infty} \frac{\min(X_k, k^\gamma)}{k^\gamma} < \infty \qquad \text{a.s.}$$

and by Kronecker's lemma,

$$\lim_{n\to\infty} \frac{1}{n^\gamma} \sum_{k=1}^{n} \min(X_k, k^\gamma) = 0$$

a.s. The a.s. convergence follows from this and Lemma 2.1.

Assume now that $E(X_0^{1/\gamma}) = \infty$. Then by Lemma 2.1, $X_k \geq Ck^\gamma$ for infinitely many $k \in \mathbb{N}$ where C denotes an arbitrarily large constant. Since $X_k \geq 0$, the second claim follows immediately. ∎

3. Proof of the theorem

The proof of the theorem is an easy consequence of the following lemma and the Besicovitch covering theorem.

Lemma 3.1.

(a) There exists a constant $C > 0$ such that for almost all $z \in J(T)$ and $\gamma > \frac{1}{\alpha-1}$

$$\liminf_{r\to 0} \frac{r^h (\log \frac{1}{r})^{\gamma(1-h)}}{m(B(z,r))} \geq C^{-1}.$$

(b) For almost all $z \in J(T)$ and $\gamma \leq \frac{1}{\alpha-1}$

$$\liminf_{r\to 0} \frac{r^h (\log \frac{1}{r})^{\gamma(1-h)}}{m(B(z,r))} = 0.$$

Proof.
(a) Let K denote a constant related to Koebe's distortion theorem and let $\sigma > 0$ be such that for $z \in J(T)$ all inverse analytic branches of $T^n, n = 1, 2, \ldots$, are well defined on $B(z, 2\sigma|z-\omega|)$ for each $\omega \in \Omega$.

The following facts have been proved in [5]. There exist constants $\kappa > 0$ and C_1 such that:

(i)Let $\omega \in \Omega$, $z \in J$, $r > 0$ and $N \in \mathbb{N}$ satisfying

$$T_*^N(z) \in V_\omega$$
$$\frac{\sigma}{2K}|T_*^N(z) - \omega| \leq |(T_*^N)'(z)|\; r \;\leq \frac{\kappa}{2K},$$

where V_ω denotes some fixed neighborhood of ω such that $B(\Omega,\Theta) = \bigcup_{\omega\in\Omega} V_\omega$. Then

$$\frac{r^h}{m(B(z,r))} \geq C_1^{-1}\; |T_*^N(z) - \omega|^{p(\omega)(1-h)}.$$

(ii) Let $z \in J$, $r > 0$ and $N \in \mathbb{N}$ satisfying

$$T_*^N(z) \notin B(\Omega,\Theta)$$
$$\frac{\kappa}{2K\|T'\|_\infty} \leq |(T_*^n)'(z)|\; r \;\leq \frac{\kappa}{2K}.$$

Then

$$C_1^{-1} \;\leq \frac{r^h}{m(B(z,r))} \;\leq C_1. \tag{3.1}$$

By the choice of γ (see Lemmas 2.1 and 2.2) we have that

$$T_*^N(z) \notin \{X_0 > N^\gamma\} \tag{3.2}$$

for all sufficiently large N m^*-a.e., hence also m a.e. Since $\log|(T^*)'|$ is integrable, by the ergodic theorem

$$\lim_{N\to\infty} \frac{1}{N} \log|(T_*^N)'(z)| = \chi(T_*) := \int \log|T_*'|\, dm^* \tag{3.3}$$

m-a.e. Notice that

$$\chi(T_*) > 0$$

by Theorem 4.12 in [4].

Let J_0 denote a set of full measure where both (3.2) and (3.3) hold. Let $z \in J_0$ and $r > 0$ be fixed and sufficiently small. Pick N so that

$$\frac{\kappa}{2K} \frac{1}{|(T_*^{N+1})'(z)|} \leq r \leq \frac{\kappa}{2K} \frac{1}{|(T_*^N)'(z)|}.$$

Case 1: Let $T_*^N(z) \notin B(\Omega, \Theta)$.
Since $|(T_*^{N+1})'(z)| \leq \|T'\|_\infty |(T_*^N)'(z)|$, we can apply (ii) and (3.1) follows.

Case 2: Let $T_*^N(z) \in V_\omega$ for some $\omega \in \Omega$.

Case 2a: $\frac{\sigma}{2K}|T_*^N(z) - \omega| \leq r|(T_*^N)'(z)| \leq \kappa/(2K)$.
In this case we can apply (i) to obtain ($p = p(\omega)$)

$$\begin{aligned} &\frac{r^h}{m(B(z,r))} \geq C_1^{-1}|T_*^N(z) - \omega|^{p(1-h)} \\ &\geq C_1^{-1} \mathrm{dist}(\{X_0 = N^\gamma\}, \omega)^{p(1-h)}. \end{aligned}$$

Note that by Lemma 8.3 in [1] there exists a constant C_2 so that

$$\mathrm{dist}(\{X_0 = k\}, \omega) \geq C_2 k^{-1/p}.$$

By the ergodic theorem

$$r \leq \frac{\kappa}{2K} \frac{1}{|(T_*^N)'(z)|} \leq \frac{\kappa}{2K} \exp[-N(\chi(T_*) - \epsilon)]$$

for some fixed and sufficiently small $\epsilon > 0$. Therefore

$$\begin{aligned} &\frac{r^h}{m(B(z,r))} \geq C_1^{-1} C_2 N^{-\gamma(1-h)} \\ &\geq C_1^{-1} C_2 \left(\frac{\chi(T_*) - \epsilon}{\log \frac{\kappa}{2Kr}}\right)^{\gamma(1-h)} \\ &\geq C^* \left(\log \frac{1}{r}\right)^{-\gamma(1-h)}. \end{aligned}$$

Hence we obtain

$$\frac{r^h \left(\log \frac{1}{r}\right)^{\gamma(1-h)}}{m(B(z,r))} \geq C.$$

Case 2b:

$$\frac{\kappa}{2K}\frac{1}{|(T_*^{N+1})'(z)|} \leq r \leq \frac{\sigma}{2K}|T_*^N(z)-\omega|\frac{1}{|(T_*^N)'(z)|}.$$

Let

$$r_j = \frac{1}{2K}\sigma|T^j(T_*^N(z))-\omega|\frac{1}{|(T^j\circ T_*^N)'(z)|}$$

for $j=0,1,2,...,s$, where $T^{s+1}(T_*^N(z)) = T_*^{N+1}(z)$. First note that there exists a constant C_3 such that for any r there exists a j such that $r_j \geq r \geq C_3 r_j$. By distortion it follows that

$$T^j(T_*^N(B(z,r_j))) \subset B(T^j(T_*^N(z)), \frac{1}{2}\sigma|T^j(T_*^N(z))-\omega|).$$

Hence by conformality

$$m(B(z,r)) \leq m(B(z,r_j)) \leq K^h|(T^j\circ T_*^N)'(z)|^{-h} m(B(T^j(T_*^N(z)),\sigma|T^j(T_*^N(z))-\omega|)).$$

By Lemma 2 in [5] there exists a constant C_4 such that

$$\begin{aligned}
&m(B(T^j(T_*^N(z)),\sigma|T^j(T_*^N(z))-\omega|)) \\
&\leq m(B(\omega,(1+\sigma)|T^j(T_*^N(z))-\omega|)) \\
&\leq C_4\left((1+\sigma)|T^j(T_*^N(z))-\omega|\right)^\beta,
\end{aligned}$$

where $\beta = h + p(\omega)(h-1)$ depends on $\omega\in\Omega$. Using similar arguments as in case 2a, it follows that

$$\begin{aligned}
m(B(z,r)) &\leq K^h C_4 |(T^j\circ T_*^N)'(z)|^{-h}\left((1+\sigma)|T^j(T_*^N(z))-\omega|\right)^\beta \\
&\leq C_4\left(\frac{2K^2}{\sigma}\right)^h (1+\sigma)^\beta r_j^h |T^j(T_*^N(z))-\omega|^{\beta-h} \\
&\leq C_4\left(\frac{2K^2}{C_3\sigma}\right)^h (1+\sigma)^\beta r^h |T^j(T_*^N(z))-\omega|^{p(h-1)} \\
&\leq C_4\left(\frac{2K^2}{C_3\sigma}\right)^h (1+\sigma)^\beta r^h (N^\gamma - j)^{1-h} \\
&\leq C_4'\left(\frac{2K^2}{C_3\sigma}\right)^h (1+\sigma)^\beta r^h (\log\frac{1}{r_0})^{\gamma(1-h)}.
\end{aligned}$$

Next observe that for $R_i = \frac{\kappa}{2K}\frac{1}{|(T_*^{N+i})'(z)|}$, $(i=0,1)$

$$\frac{\log\frac{1}{R_0}}{\log\frac{1}{R_1}} = \frac{\log|(T_*^N)'(z)|}{\log|(T_*^{N+1})'(z)|} \leq \frac{N(\chi(T_*)+\epsilon)}{(N+1)(\chi(T_*)-\epsilon)}.$$

Therefore there exists a constant C so that

$$m(B(z,r)) \leq Cr^h(\log\frac{1}{r})^{\gamma(1-h)}.$$

This finishes part (a) of the lemma.

(b) Fix a small number $\delta > 0$. As in part (a), let J_0 denote a subset of $J(T)$ of maximal measure so that for $z \in J_0$

$$\lim_{N\to\infty}\frac{1}{N}\log|(T_*^N)'(z)| = \chi(T_*)$$

and

$$C_2 N^{-\gamma'/p} \leq \mathrm{dist}(T_*^N(z),\Omega) \leq \delta N^{-\gamma/p}$$

for infinitely many $N \in \mathbb{N}$, where $\gamma' > \frac{1}{\alpha-1}$ is fixed, and where as above C_2 is the constant of Lemma 8.3 in [1]. By Lemmas 2.1 and 2.2 we have $m(J_0) = 1$, since $\mathrm{dist}(\{X_0 \geq N^\gamma\},\Omega) \leq C_5 N^{-\gamma/p}$ and $X_k \geq (C_5/\delta)^p k^\gamma$ ∞-often a.s.

Let $z \in J_0$ be fixed and denote $M(z)$ the subset of integers $N \in \mathbb{N}$ so that

$$|\frac{1}{N}\log|(T_*^N)'(z)| - \chi(T_*)| \leq \epsilon \quad\text{and}\quad C_2 N^{-\gamma'/p} \leq \mathrm{dist}(T_*^N(z),\Omega) \leq \delta N^{-\gamma/p}.$$

Choose a sequence of radii

$$r_N = 2K|T_*^N(z)-\omega|\frac{1}{|(T_*^N)'(z)|},$$

where $N \in M(z)$. Then by distortion, conformality and Lemma 2 in [5], for some constants C_6 and C_7 we obtain (with $p = p(\omega)$)

$$\begin{aligned}
m(B(z,r_N)) &\geq C_6 K^{-h}|(T_*^N)'(z)|^{-h} m(B(T_*^N(z),|T_*^N(z)-\omega|)) \\
&\geq C_6 C_4^{-1} K^{-h}|(T_*^N)'(z)|^{-h}|T_*^N(z)-\omega|^{h+p(h-1)} \\
&\geq C_7 r_N^h |T_*^N(z)-\omega|^{p(h-1)}.
\end{aligned}$$

Now for $N \in M(z)$

$$(\delta^{-1}|T_*^N(z)-\omega|)^{p(h-1)} \geq N^{\gamma(1-h)}$$

and

$$r_N \geq 2KC_2 N^{-\gamma'/p}\frac{1}{|(T_*^N)'(z)|} \geq 2KC_2 N^{-\gamma'/p}\exp[-N(\chi(T_*)+\epsilon)].$$

Therefore, if N is large enough,

$$N(\chi(T_*)+2\epsilon) \geq N(\chi(T_*)+\epsilon) + \frac{\gamma'}{p}\log N \geq \log\frac{2KC_2}{r_N}.$$

It follows that there is a constant C so that

$$m(B(z,r_N)) \geq C_7\delta^{p(h-1)} r_N^h N^{\gamma(1-h)}$$
$$\geq C_7\delta^{p(h-1)} r_N^h \left(\frac{\log 2KC_2 r_N^{-1}}{\chi(T_*)+2\epsilon}\right)^{\gamma(1-h)} \geq C^{-1}\delta^{p(h-1)} r_N^h (\log\frac{1}{r_N})^{\gamma(1-h)}.$$

Thus
$$\liminf_{r\to 0} \frac{r^h(\log\frac{1}{r})^{\gamma(1-h)}}{m(B(z,r))} \leq C\delta^{p(1-h)}$$

almost surely, and (b) follows by letting δ tend to zero. ■

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