QUASICIRCLES MODULO BILIPSCHITZ MAPS

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Abstract. We give an explicit construction of all quasicircles, modulo bilipschitz maps. More precisely, we construct a class $\mathcal{S}$ of planar Jordan curves, using a process similar to the construction of the van Koch snowflake curve. These snowflake-like curves are easily seen to be quasicircles. We prove that for every quasicircle Γ there is a bilipschitz homeomorphism f of the plane and a snowflake-like curve $S \in \mathcal{S}$ with $\Gamma = f(S)$. In the same fashion we obtain a construction of all bilipschitz-homogeneous Jordan curves, modulo bilipschitz maps, and determine all dimension functions occuring for such curves. As a tool, we construct a variant of the Konyagin-Volberg uniformly doubling measure on Γ.

1. Introduction

Quasicircles are images of circles under quasiconformal maps of the plane, see Section 2 for definitions and basic properties. They appear in many different settings in analysis, for instance as Julia sets of some rational maps, as limit sets of some Kleinian groups, or as boundaries of those domains for which every BMO-function extends. There are a large number of characterizations of quasicircles, see [G]. In this paper we present a simple construction of Jordan curves that yields *all* quasicircles, up to applying a bilipschitz map of the plane.

To give a rough description of our snowflake-like curves S, proceed as in the inductive construction of the standard van-Koch snowflake, with the main difference that there are two replacement options instead of just one: Each of the 4^n line segments of the n-th generation can be replaced by a rescaled and rotated copy of

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one of the two polygonal arcs of Figure 1.1. The sidelength p of the first alternative is a parameter that is fixed throughout the construction of each individual S. See Section 3 for a more precise description. Denote $\mathcal{S}$ the collection of all curves S obtained in this way.

Figure 1.1 The two polygonal arcs allowed in forming a snowflake-like curve

Theorem 1.1. *A Jordan curve $\Gamma \subset \mathbb{R}^2$ is a quasicircle if and only if there are $S \in \mathcal{S}$ and a bilipschitz map f of $\mathbb{R}^2$ such that*

$$\Gamma = f(S).$$

If Γ is a $K-$quasicircle, then there is $p = p(K)$ and a bilipschitz f with $\Gamma = f(S)$. If in addition $\operatorname{diam}\Gamma = 1$, then the bilipschitz norm of f depends on K only.

As a possible application, consider a domain property that is invariant under bilipschitz maps. To decide if such a property holds for all quasidiscs (domains bounded by quasicircles), it is sufficient to test all snowflake-like curves. To illustrate what we have in mind, notice that the domains bounded by our snowflake-like curves are easily seen to be John domains (every point x in the boundary can be joined to an interior point x_0 by a curve γ such that for every point $y \in \gamma$, the distance of y to the boundary is comparable to the diameter of the arc of γ between x and y). Since this John property is obviously preserved under bilipschitz maps, we conclude from Theorem 1.1 the (well-known) fact that quasidiscs are John-domains.

The proof of Theorem 1.1 is based on the construction of a uniformly doubling measure μ on Γ which, in a scaling invariant way, is bounded above resp. below by 1-dimensional resp. α-dimensional Hausdorff content, where $\alpha < 2$. More specifically, we prove

Theorem 1.2. *Let Γ be a K-quasicircle. Then there are a probability measure μ on Γ and constants $C > 0, \alpha < 2$ depending only on K such that*

$$C^{-1}\frac{r}{s} \leq \frac{\mu(B(x,r))}{\mu(B(x,s))} \leq C(\frac{r}{s})^{\alpha}$$

for all $s < r \leq \operatorname{diam}\Gamma = 1$ and all $x \in \Gamma$.

Measures satisfying the upper bound have been constructed in arbitrary metric spaces by Konyagin and Volberg [KV], with any exponent larger than the Assouad dimension of the space. A simpler construction for arbitrary compact sets in $\mathbb{R}^n$ was given by Wu [W]. It is clear that measures having the lower bound do not exist in such generality, a minimal (though not sufficient) requirement being that the Hausdorff dimension of Γ is 1.

A Problem: Our construction of the measure of Theorem 1.2 is not canonical. Natural measures such as harmonic measure or Hausdorff measures don't work in general. Is there a natural (for instance Möbius invariant) construction?

The idea of the proof of Theorem 1.1 is as follows: Given Γ and μ as above, we obtain a quasisymmetric homeomorphism $f : \mathbb{T} \to \Gamma$ such that $|I| \asymp \mu(f(I))$ for all arcs $I \subset \mathbb{T}$, where $\mathbb{T}$ is the unit circle and $|I|$ denotes normalized length. Here and in what follows we write $a \asymp b$ if the ratio a/b is bounded above and below away from zero. We construct a snowflake-like curve S together with a natural parametrization $g : \mathbb{T} \to S$ satisfying $|I| \asymp \mu(g(I))$. Then we use the trivial but useful observation that quasiconformal maps are determined by their Jacobian determinant, up to composition by bilipschitz maps, Lemma 2.1 below. Applied to extensions of f and g this shows that $f \circ g^{-1}$ is a bilipschitz homeomorphism mapping S to Γ.

The same idea can be applied to bilipschitz-homogeneous curves. A Jordan curve Γ is called *bilipschitz-homogeneous* if there is a constant L such that for every pair of points $a, b \in \Gamma$ there is a L-bilipschitz homeomorphism $f : \Gamma \to \Gamma$ satisfying $f(a) = b$. These curves have been extensively studied by Mayer [M], Ghamsari and Herron [GH],[HM]. Recently Bishop [B] succeeded in proving that they are always quasicircles. Now consider the class $\mathcal{HS}$ of homogeneous snowflake-like curves S defined by requiring that during the construction of S *all* of the 4^n line segments of the n-th generation are replaced by *the same* (rescaled and rotated) polygonal arc of Figure 1.1. Our next theorem says that these curves are precisely the bilipschitz-homogeneous curves, modulo bilipschitz maps.

Theorem 1.3. *Let $\Gamma \subset \mathbb{R}^2$ be a Jordan curve. Then the following statements are equivalent:*

(i) Γ is bilipschitz-homogeneous.

(ii) There is $S \in \mathcal{HS}$ and a bilipschitz map f of $\mathbb{R}^2$ such that $\Gamma = f(S)$.

(iii) There is a quasiconformal map F of $\mathbb{R}^2$ with $\Gamma = F(\mathbb{T})$ such that the Jacobian determinant JF satisfies

$$C^{-1} \leq \frac{JF(w)}{JF(z)} \leq C \left(\frac{1-|z|}{1-|w|}\right)^{\alpha}$$

for some constants $C > 0$, $0 \leq \alpha < 1$ and all $z, w \in \mathbb{D}$ with $|z| \leq |w|$.

(iv) There is a quasiconformal map F of $\mathbb{R}^2$ with $\Gamma = F(\mathbb{T})$ such that JF is almost radial (i.e. $JF(x) \asymp JF(|x|)$ for all $x \in \mathbb{R}^2$).

It is an open problem to characterize Jacobian determinants of quasiconformal maps (up to a bounded factor, say). David and Semmes conjectured that a weight $\omega : \mathbb{R}^2 \to \mathbb{R}^+$ is comparable to a Jacobian determinant if and only if ω is a strong $A^\infty-$ weight. In this context, part (iv) of Theorem 1.3 can be viewed as a characterization of sufficiently regular almost radial Jacobian determinants of quasiconformal maps:

Corollary 1.4. *Let $\omega : [0,1) \to \mathbb{R}^+$ be non-decreasing. There is a quasiconformal map F of $\mathbb{R}^2$ with $JF(z) \asymp \omega(|z|)$ in $\mathbb{D}$ if and only if*

$$C^{-1} \leq \frac{\omega(s)}{\omega(r)} \leq C(\frac{1-r}{1-s})^{\alpha}$$

for some $C > 0, \alpha < 1$ and all $0 \leq r \leq s < 1$.

For a compact set $A \subset \mathbb{R}^2$, denote $N_A(r)$ the minimal number of discs of radius r needed to cover A. Then $\delta(r) = N_A(r)^{-1}$ is a canonical choice of a dimension function in order to obtain a Hausdorff measure supported on A. Part (iii) of Theorem 1.3 solves the problem posed in [HM] about characterizing the dimension functions $\delta : [0,1] \to [0,1]$ that can occur for bilipschitz homogeneous curves:

Corollary 1.5. *Let $\delta : [0,1] \to [0,1]$ be non-decreasing. Then δ is comparable to $N_\Gamma(r)^{-1}$ for a bilipschitz homogeneous curve Γ if and only if*

$$\frac{\delta(s)}{\delta(r)} \leq C(\frac{s}{r})^\beta$$

for some $C > 0, \beta < 2$ and all $0 < r \leq s \leq 1$.

Organization of the paper : Section 2 provides the (well-known) background concerning quasiconformal maps. The snowflake-like curves and their parametrizations are described in Section 3. Section 4 contains the construction of the doubling measure and is independent from the rest of the paper. Theorem 1.1 is proved in Section 5. Section 6 is devoted to bilipschitz homogeneous curves. There we prove Theorem 1.4 and the corollaries.

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2. Quasiconformal maps and their Jacobians

In this section we collect the facts about quasiconformal maps needed throughout the rest of the paper. The expert may safely skip it. Let $K \geq 1$ and consider an orientation preserving homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$. Then f is K-quasiconformal if $f \in W^{1,2}_{loc}$ (first order distributional derivatives being locally square-integrable) and if the inequality $|Df(x)|^2 \leq KJf(x)$ between the operator-norm of the derivative Df and the Jacobian determinant Jf holds a.e. We have $K > 1$ unless f is conformal. The standard references to the basic theory are [A] and [LV].

Recall that homeomorphisms f of $\mathbb{R}^2$ are called L - *bilipschitz* if

$$\frac{|x-y|}{L} \leq |f(x) - f(y)| \leq L|x-y|$$

for all $x, y \in \mathbb{R}^2$. The smallest such L is refered to as the bilipschitz norm of f. It is clear that bilipschitz maps are quasiconformal, whereas the converse is false in general.

Quasiconformal maps are quasisymmetric (if $|x-y| = |x-z|$, then $|f(x)-f(y)| \leq C|f(x) - f(z)|$) and vice versa. If $f : \mathbb{T} \to f(\mathbb{T})$ is quasisymmetric, then there is a quasiconformal extension $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\operatorname{diam} f(I) \asymp |I||Df(x)| \tag{2.1}$$

for every arc $I \subset \mathbb{T}$ and every point $x \in \mathbb{R}^2$ for which $\operatorname{dist}(x, \mathbb{T}) \asymp \operatorname{dist}(x, I) \asymp |I|$. We may further assume that $|Df(x)| \asymp 1$ for $|x| > 2$.

Lemma 2.1. *If f, g are quasiconformal homeomorphisms of $\mathbb{R}^2$ and if*

$$Jf(x) \asymp Jg(x) \quad a.e.,$$

then

$$F = f \circ g^{-1}$$

is bilipschitz.

Proof. By the chainrule $JF(x) \asymp 1$ a.e. Since F is quasiconformal, we obtain $|DF| \asymp 1$ a.e. The lemma follows from $F \in W^{1,2}_{loc}$. □

The images of circles under quasiconformal maps of the plane are called quasicircles. A simple closed curve (Jordancurve) Γ is a quasicircle if and only if

$$\sup_{x,y\in\Gamma} \frac{\operatorname{diam} \Gamma(x, y)}{|x - y|} < \infty, \tag{2.2}$$

where $\Gamma(x, y)$ denotes the subarc between x and y of smaller diameter. This is the Ahlfors three-point condition.

3. Snowflake-like curves

To describe the construction, fix a parameter $1/4 \leq p < 1/2$ defining the first arc γ of Fig. 1.1. Denote by γ' the second arc (the line segment) of Figure 1.1. Inductively define polygons S_n consisting of 4^n line segments as follows: Denote the unit square by S_1. To pass from S_n to S_{n+1}, for each of the 4^n edges $[x, y]$ of S_n replace $[x, y]$ by a scaled copy of γ or γ'. Here we assume that x follows y in the positive orientation of S_n, that the scaling map is orientation preserving, and that it maps the left endpoint of γ resp. γ' onto x. See Fig. 3.1 for a possible S_3.

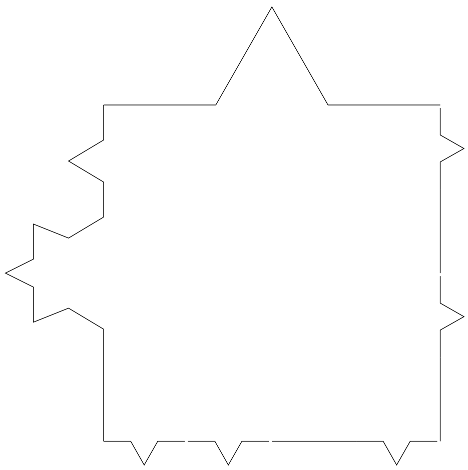

Figure 3.1 A possible S_3.

For a given S_n there are 2^{4^n} possibilities for choosing S_{n+1}. It is clear that each sequence S_n thus obtained converges (geometrically) to a closed limit curve S. Below we will show that these limit curves are quasicircles, in particular they are Jordan curves. Denote $\mathcal{S}(p)$ the collection of all limit curves S, and set

$$\mathcal{S} = \bigcup_{1/4 \le p < 1/2} \mathcal{S}(p).$$

Next, consider the class

$$\mathcal{HS} = \bigcup_{1/4 \le p < 1/2} \mathcal{HS}(p)$$

of homogeneous snowflake-like curves defined as follows: A curve $S \in \mathcal{S}(p)$ belongs to $\mathcal{HS}(p)$ if and only if the approximating curve S_{n+1} is formed from S_n by replacing *all* edges of S_n by a scaled copy of the *same* arc γ or γ'. Hence there are only two choices of S_{n+1} for a given S_n.

Lemma 3.1. *Every curve $S \in \mathcal{S}(p)$ is a K-quasicircle, with K depending on p only.*

Proof. For an edge I of some S_n, denote $T(I)$ the isosceles triangle with base I and height $\sqrt{p - 1/4}|I|$. So $T(I)$ is the convex hull of the rescaled arc γ. Then the (smaller) arc $S(I)$ of S with the same endpoints as I is contained in $T(I)$. If J is another edge (of some S_m), one easily proves by induction that either $I \cap J \neq \emptyset$ or

$$\operatorname{dist}(T(I), T(J)) \geq c_p \min(\operatorname{diam} T(I), \operatorname{diam} T(J)).$$

Using the Ahlfors three-point condition (2.2), the lemma easily follows. □

Next we will describe a one to one correspondence between $\mathcal{S}(p)$ and certain labelled graphs. Let $G = (V, E)$ be the infinite planar graph depicted in Fig. 3.2. It is obtained from a rooted homogeneous tree of degree 7 by cyclically joining the 4^n vertices $v \in V$ of graph-distance $d(v) = d(v, v_0) = n$ from the root v_0.

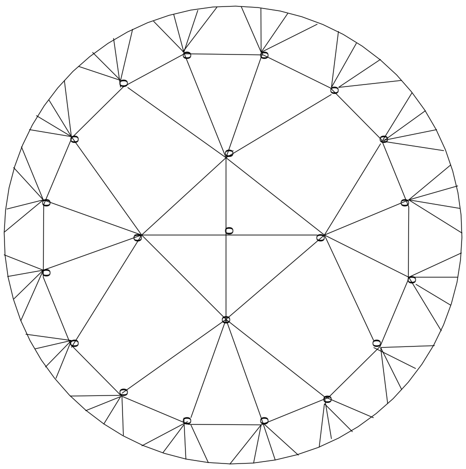

Figure 3.2 The graph G

The correspondence between a vertex v and an arc $S(v)$ of S is characterized by the following four properties:
(i) $S(v_0) = S$.
(ii) If $d(v) = n$, then $S(v)$ is an arc obtained from an edge of S_n.
(iii) If $d(v) = d(v') = n$ and if v, v' are adjacent in G, then $S(v)$ and $S(v')$ have a common endpoint.
(iv) If v' is a descendent of v (i.e. $d(v, v') = d(v') - d(v)$) then $S(v') \subset S(v)$.

Define the labelling $\ell_S : V \to \mathbb{R}_+$ by

$$\ell_S(v) = \operatorname{diam} S(v'), \tag{3.1}$$

where v' is any child of v (that is $d(v,v') = d(v') - d(v) = 1$).

This process of passing from S to ℓ clearly is reversible: If $\ell : V \to \mathbb{R}_+$ is given and has the property that

$$\frac{\ell(v')}{\ell(v)} \in \{p, \frac{1}{4}\} \tag{3.2}$$

whenever v' is a child of v, then there is a curve $S = S_\ell \in \mathcal{S}(p)$, unique up to rotation, such that $\ell = \ell_S$.

Given $S \in \mathcal{S}(p)$, there is a canonical homeomorphism $\phi_S : S_1 \to S$, where S_1 is the unit square. It is the map that sends a four-adic interval $S_1(v)$ on S_1 onto the corresponding arc $S(v)$. More formally, the labelling $\ell_1(v) \equiv 4^{-d(v)}$ satisfies the above assumption (3.2) and obviously yields $S_1 = S_{\ell_1}$. With this interpretation, ϕ_S is given by

$$\phi_S(S_1(v)) = S(v) \tag{3.3}$$

for every $v \in V$.

The next lemma can be proved in the same way as Lemma 3.1.

Lemma 3.2. *Given $S \in \mathcal{S}(p)$, the homeomorphism*

$$\phi_S : S_1 \to S$$

is quasisymmetric if and only if there is C such that

$$C^{-1} \leq \frac{\ell(v')}{\ell(v)} \leq C$$

for all adjacent vertices $v, v' \in V$.

Notice that for *every* $S \in \mathcal{S}(p)$ there exists a quasisymmetric parametrization $\phi : S_1 \to S$. But the natural parametrization described above need not be quasisymmetric.

4. The doubling measure

This section is devoted to the proof of Theorem 1.2. Throughout this section Γ is a K- quasicircle. We are first going to show that the uniform metric dimension (Assouad dimension) of Γ is bounded away from 2, depending only on K. More precisely, we have

Lemma 4.1. *There are constants $C(K) > 0$ and $a(K) < 2$ such that, for every $q > 0$, every arc $I \subset \Gamma$ contains at most*

$$n \leq \frac{C}{q^a}$$

disjoint subarcs $I_1, ..., I_n$ of diameters $d_m \geq q \operatorname{diam} I$.

Proof. It is well-known (and easily follows from quasisymmetry) that quasicircles are porous: There is a constant $c(K)$ such that for every disc $D(x, r)$ there is a disc $D(y, cr) \subset D(x, r) \setminus \Gamma$. Let S be a square of sidelength l, subdivided into k^2 subsquares S_j with sidelength l/k. Then porosity and induction shows that Γ meets at most Ck^a of the S_j, where $a < 2$ depends only on c. Setting $l = \operatorname{diam} I$ and $k = [1/q]$, the lemma follows from the fact that only a bounded number of the arcs I_m can meet a fixed S_j, by the three-point property.

Proof of Theorem 1.2. We may assume $\operatorname{diam} \Gamma = 1$. Let a be the constant from Lemma 4.1, pick any $a < b < 2$ and choose a sufficiently small number $q < 1$, specified during the course of the proof.

First choose a sequence $\mathcal{I}_n = \{I_{n,j}\}$ of subdivisions of Γ into disjoint half-open arcs $I_{n,j}$ with the following two properties:
(a) $q^n \leq \operatorname{diam} I_{n,j} < 2q^n$ for all n, j.
(b) For $I \in \mathcal{I}_n$ and $J \in \mathcal{I}_{n+1}$, either $J \subset I$ (in this case we write $J < I$), or $J \cap I = \emptyset$.

Such a sequence is easy to find by successive "bisection" of arcs. Next, define a sequence μ_n of probability measures on Γ by specifying $\mu_n(I_{n,j})$ for each n, j. Our measure μ will be the weak limit of μ_n. The μ_n will have the following properties:

(1) For all n and all pairs of adjacent arcs $I, I' \in \mathcal{I}_n$,

$$\frac{1}{10} \leq \frac{\mu_n(I)}{\mu_n(I')} \frac{\operatorname{diam} I'}{\operatorname{diam} I} \leq 10.$$

(2) For all n and all arcs $I \in \mathcal{I}_n$, the mass $\mu_n(I)$ is distributed over its 'children' $J < I$, i.e. no mass from I is transported away from I :

$$\sum_{J<I} \mu_{n+1}(J) = \mu_n(I).$$

(3) For all n, all arcs $I \in \mathcal{I}_n$ and all arcs $J < I$ we have

$$\frac{\operatorname{diam} I}{\operatorname{diam} J} \leq \frac{\mu_n(I)}{\mu_{n+1}(J)} \leq q^{-b}.$$

It is immediate from (2) and (a) above that μ_n weakly converges to a measure μ on Γ. Before we proceed with the construction of μ_n, let us show that μ will have the required properties. To this end, consider $x \in \Gamma$ and $0 < r < R \leq 1$. By (a) and (b) there are arcs $I \in \mathcal{I}_n$ and $J \in \mathcal{I}_m$ with $x \in J \subset I$ and $\operatorname{diam} I \asymp R \asymp q^n$, $\operatorname{diam} J \asymp r \asymp q^m$. It easily follows from the three-point property, together with (1) and (2), that $\mu(B(x,R)) \asymp \mu(I)$ and $\mu(B(x,r)) \asymp \mu(J)$. Now (3) implies

$$\frac{\operatorname{diam} I}{\operatorname{diam} J} \leq \frac{\mu(I)}{\mu(J)} \leq q^{-b(m-n)} \asymp (\frac{R}{r})^b,$$

proving the theorem.

Now we describe the inductive construction of μ_n. Set

$$\mu_1(I_{1,j}) = \frac{1}{\#\mathcal{I}_1}$$

for all j, where $\#$ denotes cardinality. Then (1) is clear from (a), and (2), (3) are void.

To obtain μ_{n+1} from μ_n, let $I \in \mathcal{I}_n$ and let $J_1, ..., J_r$ denote the children of I (i.e. $J_l < I$), where $r = r(I)$ is the number of children. We assume the labeling is such that J_l and J_{l+1} are adjacent for all l. A first attempt is to set

$$m_l = m(J_l) = \mu_n(I) \frac{\operatorname{diam} J_l}{\sum_{k=1}^r \operatorname{diam} J_k}$$

and to try $\mu_{n+1}(J_l) = m_l$. Notice that $m_l \asymp \mu_n(I)/r$ so that we would roughly equidistribute the mass of I over its children. But then there is no reason for the ratio in (1) to remain bounded after some generations. To fix this, we proceed similarly to [W] and define

$$\mu_{n+1}(J_l) = w_l m_l$$

with weights $w_l = w(J_l)$ described below.

Let us denote I^- and I^+ the two arcs of $\mathcal{I}_n$ adjacent to I, and by $J_0 < I^-$, $J_{r+1} < I^+$ the arcs of $\mathcal{I}_{n+1}$ adjacent to J_1 resp. J_r. Set $m_0 = m(J_0)$ and $m_{r+1} = m(J_{r+1})$. Notice that $m_0 \asymp \mu_n(I^-)/r(I^-) \asymp \mu_n(I)/r(I^-)$.

We first define w_1 and w_r : Set

$$Q(J, J') = \frac{m(J)}{m(J')} \frac{\operatorname{diam} J'}{\operatorname{diam} J}$$

and let $w_1 = 1$ if $Q(J_1, J_0) \geq 1/10$, and $w_1 = 1/(10Q(J_1, J_0))$ if $Q(J_1, J_0) < 1/10$. In the same way define $w_r = 1$ if $Q(J_r, J_{r+1}) \geq 1/10$, else $w_r = 1/(10Q(J_r, J_{r+1}))$. This definition applies to all those $J \in \mathcal{I}_{n+1}$ that have an endpoint in common with their parent $J < I \in \mathcal{I}_n$. In particular we have defined w_0 and w_{r+1}.

Notice that $w_1 \geq 1$, and that $w_0 = 1$ if $w_1 > 1$ since $Q(J_0, J_1) = Q(J_1, J_0)^{-1}$. Next, set $w_2 = ... = w_{r-1} = 1$ if $w_1 = w_r = 1$. Otherwise we may assume $w_1 \geq w_r$ and choose a sequence $w_2, ..., w_{r-1}$ in such a way that

$$\sum_{j=1}^{r} w_j m_j = \mu_n(I), \tag{4.1}$$

$$\frac{1}{2} \leq \frac{w_j}{w_{j+1}} \leq 2 \tag{4.2}$$

and that

$$\varepsilon \leq w_j \leq w_1 \tag{4.3}$$

for $j = 1, ..., r-1$ and some universal constant ε. The existence of such a sequence is easy to establish if q is sufficiently small: Indeed, from Lemma 4.1 we have $w_1 \asymp r(I)/r(I^-) \leq Cq^{1-a} \leq C' r(I)^{a-1}$. Hence $w_1 m_1 \asymp w_1 r(I)^{-1} \mu_n(I) = o(\mu_n(I))$ as $q \to 0$. Now define $w_j = 2^{-j+1} w_1$ for $j = 1, 2, ..., j_0$, let the w_j have a constant value w for $j_0 + 1 \leq j \leq j_1$ and finally set $w_j = 2^{j-r(I)} w_r$ for $j_1 + 1 \leq j \leq r(I)$. It is clear that j_0, j_1 and w can be chosen so that (4.1) and (4.2) are fulfilled. Since the contribution to $\sum_{j=1}^r w_j m_j$ from $1 \leq j \leq j_0$ and from $j_1 \leq j \leq r$ is $o(\mu_n(I))$ as q decreases, w is bounded away from 0 and we have (4.3).

It remains to verify that $\mu_{n+1}(J_l) = w_l m_l$ satisfies (1)-(3) above. To see (1) for the pair (J_0, J_1), just notice that

$$\frac{\mu_{n+1}(J_1)}{\mu_{n+1}(J_0)} \frac{\operatorname{diam} J_0}{\operatorname{diam} J_1} = \frac{w_1}{w_0} Q(J_1, J_0) = \frac{1}{10}, Q(J_1, J_0) \text{ or } 10$$

if $Q(J_1, J_0) < 1/10, \in [1/10, 10]$ or > 10 respectively. Similarly, (1) holds for (J_r, J_{r+1}). For the pairs (J_k, J_{k+1}) with $1 \leq k \leq r-1$, the ratio in (1) equals w_k / w_{k+1} which is bounded above and below by $1/2$ and 2.

Property (2) is immediate from (4.1).

To check the lower bound of (3), let us begin with $J = J_1$: If $w_1 = 1$ (the case $Q(J_1, J_0) \geq 1/10$) this follows at once from the triangle inequality $\operatorname{diam} I \leq \sum_1^r \operatorname{diam} J_l$. Otherwise we have $w_1 = 1/(10 Q(J_1, J_0)) > 1$ and $w_0 = 1$. Hence we have the lower estimate of (3) for J_0,

$$\frac{\operatorname{diam} I^-}{\operatorname{diam} J_0} \frac{\mu_{n+1}(J_0)}{\mu_n(I^-)} \leq 1.$$

Using property (1) for $\mathcal{I}_n$, we obtain

$$\frac{\mu_n(I)}{\mu_{n+1}(J_1)} = \frac{1}{\operatorname{diam} J_1}\frac{10 \operatorname{diam} J_0}{m_0}\mu_n(I) \geq \frac{1}{\operatorname{diam} J_1}\frac{10 \operatorname{diam} J_0}{\mu_{n+1}(J_0)}\frac{\mu_n(I^-) \operatorname{diam} I}{10 \operatorname{diam} I^-} \geq \frac{\operatorname{diam} I}{\operatorname{diam} J_1}.$$

To prove the lower bound of (3) for J_l with $2 \leq l \leq r$, use $w_l \leq w_1$ to obtain

$$\frac{\mu_n(I)}{\mu_{n+1}(J_l)} \geq \frac{\mu_n(I)}{w_1 m_l} = \frac{\mu_n(I)}{\mu_{n+1}(J_1)}\frac{\operatorname{diam} J_1}{\operatorname{diam} J_l} \geq \frac{\operatorname{diam} I}{\operatorname{diam} J_l}.$$

The upper bound of (3) easily follows from Lemma 4.1 if q is small enough, since the w_j are bounded below (independently of q) by (4.3).

□

5. The proof of Theorem 1.1

Proof of Theorem 1.1. Given a quasicircle Γ, apply Theorem 1.2 to obtain the probability measure μ on Γ. Use μ to define a homeomorphism

$$\phi : S_1 \to \Gamma$$

between the unit square S_1 and Γ in such a way that the push-forward under ϕ of length on S_1 is μ : Fix points $a \in S_1$ and $b \in \Gamma$, and for $x \in S_1$ define $\phi(x)$ to be the unique point on Γ such that the (oriented) arc $\Gamma(\phi(x))$ from b to $\phi(x)$ has

$$\mu(\Gamma(\phi(x))) = \frac{\operatorname{length} S_1(x)}{\operatorname{length} S_1},$$

where $S_1(x)$ is the arc from a to x.

From ϕ we obtain a function (labelling) $\ell : V \to \mathbb{R}_+$ in the canonical way, compare (3.1) and (3.3): For vertices $v \in V$ set

$$\ell(v) = \operatorname{diam} \phi(S_1(v)).$$

We first observe that

$$\ell(v) \asymp \ell(v') \tag{5.1}$$

if v and v' are adjacent. To see this, just notice that the arcs $\Gamma(v) = \phi(S_1(v))$ and $\Gamma(v')$ have measure $\asymp 4^{-d(v)}$, that Γ is a quasicircle, and use the doubling property of μ (no uniformity is needed yet).

Next, let $\alpha < 2$ be the exponent from Theorem 1.2, set

$$A = 4^{\frac{1}{\alpha}} > 2$$

and observe that for all $v \in V$ and all descendents v' of v we have

$$C^{-1}4^{-d(v,v')} \leq \frac{\ell(v')}{\ell(v)} \leq CA^{-d(v,v')}. \tag{5.2}$$

To see this, observe that the four-adic interval $S_1(v')$ is contained in $S_1(v)$ and has $\operatorname{length} S_1(v') = 4^{-d} \operatorname{length} S_1(v)$, where $d = d(v, v')$. Then (5.2) is obtained from Theorem 1.2, applied to any $x \in \phi(S(v'))$, by choosing r, s comparable to the diameters of $\phi(S(v))$ and $\phi(S(v'))$.

For *every* labelling ℓ satisfying (5.1) and (5.2) there is a labelling

$$\ell' \asymp \ell \tag{5.3}$$

(that is $\ell'(v) \asymp \ell(v)$ for all $v \in V$) satisfying (3.2) with $p = A^{-1}$: Just set $\ell'(v_0) = 1$ and inductively define

$$\ell'(v') = \begin{cases} \dfrac{1}{4}\ell'(v) & \text{if} \quad \ell'(v) \geq \ell(v) \\ \dfrac{1}{A}\ell'(v) & \text{if} \quad \ell'(v) < \ell(v) \end{cases} \tag{5.4}$$

if v' is a child of v. Then (3.2) is obvious and $\ell' \asymp \ell$ is easy.

From (3.2) we obtain a snowflake-like curve S with $\ell_S = \ell'$. Now $\ell_S \asymp \ell$ together with (5.1) and Lemma 3.2 imply that both $\phi : S_1 \to \Gamma$ and $\phi_S : S_1 \to S$ are quasisymmetric. Let Φ resp. Φ_S be quasiconformal extensions to the plane satisfying (2.1) with $\mathbb{T}$ replaced by S_1 (the disc in (2.1) can be replaced by any chord-arc domain, as can be seen by applying a bilipschitz homeomorphism of the plane). Then (2.1) together with (5.3) implies $|D\Phi(x)| \asymp |D\Phi_S(x)|$ in $\mathbb{R}^2$, and the theorem follows from Lemma 2.1. $\square$

Bilipschitz Homogeneous Curves

Proof of Theorem 1.3. By [HM] and [B], our definition of bilipschitz-homogeneity coincides with the one used in [M] (existence of a bilipschitz group acting transitively on Γ.) To prove (i) implies (ii) we use Theorem 1.1 of [M]. Hence there is a parametrization $h : \mathbb{T} \to \Gamma$ satisfying

$$|h(x) - h(y)| \le C|h(u) - h(v)|$$

whenever $|x - y| \le |u - v|$. Set

$$a_n = \min\{|h(x) - h(y)| : |x - y| = 4^{-n}\}$$

and consider the labelled graph (G, V) of Fig. 3.2 with $l(v) = a_n$ if $d(v) = n$. We proceed as in the proof of Theorem 1.1. First we claim that there is a labelling $\ell' \asymp \ell$ satisfying (3.2). Now $|h(x) - h(y)| \asymp N(r)^{-1}$, where $N(r)$ is the minimal number of discs of radius r needed to cover Γ. So (5.2) follows from Lemma 4.1, and ℓ' can be constructed by (5.4) as in the proof of Theorem 1.1. Since $\ell'(v) = \ell'(v')$ whenever $d(v) = d(v')$, the curve $S \in \mathcal{S}$ with $\ell_S = \ell'$ belongs to $\mathcal{HS}$. As in the proof of Theorem 1.1 we observe that the Jacobian determinant of the extension of h is comparable to $J\Phi_S$ and we obtain (ii).

Now we show (ii) $\Rightarrow$ (iii). Let $S \in \mathcal{HS}(p)$. By Lemma 3.2, the canonical homeomorphism $\Phi_S : S_1 \to S$ constructed in Section 3 is quasisymmetric. Denote its quasiconformal extension satisfying (2.1) for $x \in \mathbb{D}$ by Φ_S, too. Given $z, w \in \mathbb{D}$ with $|z| \le |w|$, consider the four-adic intervals I, J with $1 - |z| \asymp \operatorname{dist}(z, I) \asymp |I| \asymp 4^{-n}$ and $1 - |w| \asymp \operatorname{dist}(w, J) \asymp |J| \asymp 4^{-m}$. It follows from (2.1) that

$$C^{-1}(\frac{1}{4})^{m-n}(1 - |z|)|D\Phi_S(z)| \le (1 - |w|)|D\Phi_S(w)| \le Cp^{m-n}(1 - |z|)|D\Phi_S(z)|.$$

We obtain (iii) with $\alpha = \log(4p)/\log 2 < 1$.

Since (iii) trivially implies (iv), it remains to show (iv) $\Rightarrow$ (i). Pick two points x, y on Γ and let R denote the rotation $R(z) = F^{-1}(x)/F^{-1}(y)z$. By assumption we have $JF \asymp J(F \circ R)$, and by Lemma 2.1 $F \circ R^{-1} \circ F^{-1}$ is bilipschitz fixing Γ and sending x to y. $\square$

Proof of Corollary 1.4. First, let ω as in the Corollary be given. Define $\eta(s) = s\omega(1-s)^{1/2}$ for $0 \leq s < 1$. Similar to the proof of (i) $\Rightarrow$ (ii) above, set $a_n = \eta(4^{-n})$ and consider the labelled graph (G, V) of Fig. 3.2 with $l(v) = a_n$ if $d(v) = n$. Then

$$\ell(v')/\ell(v) \leq C4^{-(1-\alpha/2)d(v,v')}.$$

Proceeding as above (cf. (5.2)), we obtain a quasiconformal map Φ_S onto a bilipschitz-homogeneous curve $S \in \mathcal{HS}(p)$ with $p = 4^{\alpha/2-1}$ such that $J\Phi_S \asymp \omega$ in $\mathbb{D}$.

The converse easily follows from Lemma 4.1 $\square$

Proof of Corollary 1.5. Given a bilipschitz-homogeneous Γ, let F be the quasiconformal parametrization from Theorem 1.3 (iii) and set $\omega(s) = JF(s)$ for $0 < s < 1$. From $(1-s)\omega(s)^{1/2} \asymp \operatorname{dist}(F(s), \Gamma)$ (quasisymmetry and (2.1)) we conclude

$$\delta(\rho) \asymp \frac{1}{1-s} \quad \Longleftrightarrow \quad (1-s)\omega(s)^{1/2} \asymp \rho.$$

Thus the Corollary follows from Corollary 1.4. $\square$

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