NORMAL AND BINORMAL VECTORS (§13.3)

Consider curve in \( \mathbb{R}^3 \) defined by \( \vec{r}(t) \).
Recall: \( \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \) is the unit tangent vector

Claim: For all \( t \), \( \vec{T}(t) \) and \( \vec{T}'(t) \) are orthogonal.

Proof: Write \( \vec{T}(t) = (x(t), y(t), z(t)) \).
Since \( \vec{T}(t) \) is unit vector, \( \vec{T}(t) \cdot \vec{T}(t) = 1 \).
Differenciating both sides gives
\[
0 = \frac{d}{dt}1 = \frac{d}{dt}(x(t)^2 + y(t)^2 + z(t)^2) \\
= 2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) \\
= 2 \cdot \vec{T}(t) \cdot \vec{T}'(t)
\]

Thus \( \vec{T}(t) \perp \vec{T}'(t) \).
**Def:** The **unit normal vector** of \( \vec{r}(t) \) is

\[
\mathbf{\hat{N}}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}
\]

Observe: \( \mathbf{\hat{N}}(t) \perp \mathbf{T}(t) \) for all \( t \).

The **osculating circle** that is tangent to curve at \( \vec{r}(t) \) and has same curvature, has radius \( \frac{1}{\kappa(t)} \) and center \( \vec{r}(t) + \frac{1}{\kappa(t)} \mathbf{\hat{N}}(t) \).
For a curve $\vec{r}(t)$ we have the following definitions:

- The **unit normal vector** of $\vec{r}(t)$ is

  $$\hat{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

- The **binormal vector** $\vec{B}(t) = \vec{T}(t) \times \hat{N}(t)$

- The plane spanned by vectors $\vec{T}(t)$ and $\hat{N}(t)$ and containing $\vec{r}(t)$ is called the **osculating plane**.
  Its equation is $\{ \vec{P} \in \mathbb{R}^3 : \vec{B}(t) \cdot \vec{P} = \vec{B}(t) \cdot \vec{r}(t) \}$

- The plane spanned by vectors $\hat{N}(t)$ and $\vec{B}(t)$ is called the **normal plane**.
  Its equation is

  $$\{ \vec{P} \in \mathbb{R}^3 : \vec{T}(t) \cdot \vec{P} = \vec{T}(t) \cdot \vec{r}(t) \}$$

**Remark:** $\vec{T}(t), \hat{N}(t), \vec{B}(t)$ are pairwise orthogonal unit vectors.
Example: For curve $\vec{r}(t) = (\sin(t), \cos(t), t)$ and $t = \pi$, find the unit tangent vector, the unit normal vector, the binormal vector, the osculating plane and the normal plane.

\[
\vec{r}'(t) = (\cos(t), -\sin(t), 1) \\
|\vec{r}'(t)| = \sqrt{2} \\
\vec{T}(t) = \left(\frac{1}{\sqrt{2}} \cos(t), -\frac{1}{\sqrt{2}} \sin(t), \frac{1}{\sqrt{2}}\right) \\
\vec{T}'(t) = \left(-\frac{1}{\sqrt{2}} \sin(t), -\frac{1}{\sqrt{2}} \cos(t), 0\right) \\
|\vec{T}'(t)| = \sqrt{\frac{1}{2} \sin^2(t) + \frac{1}{2} \cos^2(t)} = \frac{1}{\sqrt{2}} \\
\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = (-\sin(t), -\cos(t), 0) \\
\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \left(\frac{1}{\sqrt{2}} \cos(t), -\frac{1}{\sqrt{2}} \sin(t), -\frac{1}{\sqrt{2}}\right) \\
\vec{T}(\pi) = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \\
\vec{N}(\pi) = (0, 1, 0) \\
\vec{B}(\pi) = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)
Example (cont.)

Osculating plane: Contains $\vec{r}(\pi) = (0, -1, \pi)$ and has normal vector $\vec{B}(\pi) = (-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$. The plane is

$$-\frac{1}{\sqrt{2}}x + 0y - \frac{1}{\sqrt{2}}z = -\frac{\pi}{\sqrt{2}} \iff x + z = \pi$$

Normal plane: Contains $\vec{r}(\pi) = (0, -1, \pi)$ and has normal vector $\vec{T}(\pi) = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. The plane is

$$-\frac{1}{\sqrt{2}}x + 0y + \frac{1}{\sqrt{2}}z = \frac{\pi}{\sqrt{2}} \iff -x + z = \pi$$
Example (Midterm II, Lieblich, 2013, Ex 3)

For the curve \( \vec{r}(t) = (t \cos(t), t \sin(t), t) \),

a) Find the \textbf{unit tangent vector} \( \vec{T}(t) \)

b) Is the \textbf{binormal vector} \( \vec{B}(t) \) ever parallel to the \( z \)-axis?

\[
\vec{r}'(t) = (\cos(t) - t \sin(t), \sin(t) + t \cos(t), 1)
\]
\[
|\vec{r}'(t)| = \sqrt{(\cos(t) - t \sin(t))^2 + (\sin(t) + t \cos(t))^2 + 1}
\]
\[
= \sqrt{(t^2 + 1) \cos^2(t) + (t^2 + 1) \sin^2(t) + 1}
\]
\[
= \sqrt{t^2 + 2}
\]
\[
\vec{T}(t) = \left( \frac{\cos(t) - t \sin(t)}{\sqrt{t^2 + 2}}, \frac{\sin(t) + t \cos(t)}{\sqrt{t^2 + 2}}, \frac{1}{\sqrt{t^2 + 2}} \right)
\]

\textbf{Method 1:} take derivative \( \vec{T}'(t) \), normalize and take cross product with \( \vec{T}(t) \)

\( \rightarrow \) possible, veeery messy and tedious).

\textbf{Method 2:} Suppose \( \vec{B}(t_0) = (0, 0, z) \) with either \( z = 1 \) or \( z = -1 \) for some \( t_0 \) (recall that \( \vec{B}(t_0) \) is a unit vector).

We know that always \( \vec{B}(t_0) \perp \vec{T}(t_0) \).

But \( \vec{B}(t_0) \cdot \vec{T}(t_0) = z \cdot \frac{1}{\sqrt{t_0^2 + 2}} \neq 0 \) which is a contradiction.

\textbf{Method 3:} Use that \( \vec{B}(t) = \frac{\vec{r}'(t) \times \vec{r}''(t)}{|\vec{r}'(t) \times \vec{r}''(t)|} \)