Recall: The tangent plane for $f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ is defined by

$$T(x, y) = f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0) + f(x_0, y_0)$$

Idea: For $x \approx x_0$ and $y \approx y_0$, one has $T(x, y) \approx f(x, y)$

The total differential of $z = f(x, y)$ is

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

Meaning:
change in $z \approx f_x(x_0, y_0) \cdot$ change in $x + f_y(x_0, y_0) \cdot$ change in $y$
Example: For $z = y \cos(x + y)$:

- find the total differential
- approximate the change in $z$ if $x$ changes from 0 to 0.04 and $y$ changes from $\frac{\pi}{2}$ to $\frac{\pi}{2} + 0.03$

Solution: The partial derivatives are

$$\frac{\partial z}{\partial x} = -y \sin(x + y) \quad \frac{\partial z}{\partial y} = -y \sin(x + y) + \cos(x + y)$$

The total differential is

$$dz = (-y \sin(x + y)) \, dx + (-y \sin(x + y) + \cos(x + y)) \, dy$$

The change in $z$ is approximately

$$\left( -\frac{\pi}{2} \sin(0 + \frac{\pi}{2}) \right) \cdot 0.04 + \left( -\frac{\pi}{2} \sin(0 + \frac{\pi}{2}) + \cos(0 + \frac{\pi}{2}) \right) \cdot 0.03$$

$$= \frac{\pi}{2}(-0.07) \approx -0.1000$$

(for the sake of comparison:

$$f(0.04, \frac{\pi}{2} + 0.03) - f(0, \frac{\pi}{2}) \approx -0.1119$$)
A function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) has a **local maximum** at \((x_0, y_0)\) if \( f(x_0, y_0) \geq f(x, y) \) for all \((x, y)\) near \((x_0, y_0)\).

**Example:**
Point \((0, 0)\) for
\[
f(x, y) = \frac{A}{2}x^2 + \frac{B}{2}y^2 + Cxy
\]
\(A < 0, B < 0, AB > C^2\)

**Example:**
Point \((0, 0)\) for
\[
f(x, y) = \frac{A}{2}x^2 + \frac{B}{2}y^2 + Cxy
\]
\(A > 0, B > 0, AB > C^2\)
A point \((x_0, y_0)\) is a **saddle point** of \(f\) if 
\[ f_x(x_0, y_0) = 0 = f_y(x_0, y_0) \]
but \((x_0, y_0)\) is no local extremum.

Example:

Point \((0, 0)\) for

\[ f(x, y) = \frac{A}{2}x^2 + \frac{B}{2}y^2 + Cxy \]

\[ AB < C^2 \]
The critical points of \( f(x, y) \) are those points \((x_0, y_0)\) in the domain of \( f \) such that

- Either: \( f_x(x_0, y_0) = f_y(x_0, y_0) = 0 \)
- Or: At least one of \( f_x(x_0, y_0), f_y(x_0, y_0) \) is undefined

Critical points are candidates for local/global optima.

The 2nd Derivative Test

Suppose \((x_0, y_0)\) is in the domain of \( f \) and \( f_x(x_0, y_0) = f_y(x_0, y_0) = 0 \). Let

\[
D(x, y) = \begin{vmatrix}
  f_{xx}(x, y) & f_{xy}(x, y) \\
  f_{xy}(x, y) & f_{yy}(x, y)
\end{vmatrix}
\]

- If \( D(x_0, y_0) > 0 \) and \( f_{xx}(x_0, y_0) > 0 \), then \( f \) has a local minimum at \((x_0, y_0)\)
- If \( D(x_0, y_0) > 0 \) and \( f_{xx}(x_0, y_0) < 0 \), then \( f \) has a local maximum at \((x_0, y_0)\)
- If \( D(x_0, y_0) < 0 \), then \( f \) has a saddle point at \((x_0, y_0)\)

Remark: \( D(x, y) \) is the determinant of the Hessian matrix

Intuition:

Consider case \((x_0, y_0) = (0, 0)\) and \( f(0, 0) = 0 \).

Which polynomial \( q \) has the same derivatives as \( f \)?

\[
q(x, y) = \frac{1}{2} f_{xx}(0, 0) \cdot x^2 + \frac{1}{2} f_{yy}(0, 0) \cdot y^2 + f_{xy}(0, 0) \cdot xy
\]

Then \( q(x, y) \approx f(x, y) \) for \((x, y) \approx (0, 0)\).

If \( q \) has local max at \((0, 0)\) \( \Rightarrow \) \( f \) has local max at \((0, 0)\)
Let \( f(x, y) = \frac{9}{x} + 3xy - y^2 \). Find and classify all critical points using appropriate partial derivative tests.

\[
\begin{align*}
    f_x(x, y) &= -\frac{9}{x^2} + 3y = 0 \implies y = \frac{3}{x^2} \\
    f_y(x, y) &= 3x - 2y = 0 \implies y = \frac{3}{2x}
\end{align*}
\]

Combining gives

\[
\frac{3}{x^2} = \frac{3}{2x} \implies 2 = x^3 \implies x = 2^{1/3}
\]

Then \((x_0, y_0) = (2^{1/3}, \frac{3}{2}2^{1/3})\) is only critical point.

\[
\begin{align*}
    f_{xx} &= \frac{18}{x^3}, \quad f_{yy} = -2, \quad f_{xy} = 3
\end{align*}
\]

Then

\[
D(x_0, y_0) = \begin{vmatrix} 18/2 & 3 \\ 3 & -2 \end{vmatrix} = -27 < 0
\]

Hence the critical point is a saddle point.
Finding global optimum

Example:
Find the absolute maximum and minimum values of the function $h(x, y) = y^2 - 3x^2$ on the domain $R = \{(x, y) : x^2 + y^2 \leq 9\}$.

Recipe:

1) Find all critical points that lie in $R$
2) Consider the boundary of $R$ as a 1-dimensional function and determine optima
3) Compare function value on all found points
Solution of example:

• **Step 1:**

\[
\begin{align*}
    h_x(x, y) &= -6x = 0 \Rightarrow x = 0 \\
    h_y(x, y) &= 2y = 0 \Rightarrow y = 0
\end{align*}
\]

We found critical point \((0, 0)\) (which lies in \(R\)).

• **Step 2:** For a point \((x, y)\) with \(x^2 + y^2 = 9\), we can write

\[
h(x, y) = \frac{y^2}{x^2} - 3x^2 = 9 - 4x^2 =: g(x)
\]

Then \(g'(x) = -8x = 0 \Rightarrow x = 0\).

This gives critical point \((0, \pm 3)\).

But \(\max\{g(x) : -3 \leq x \leq 3\}\) might be attained for \(x = \pm 3\).

• **Step 3:**

<table>
<thead>
<tr>
<th>point</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>0</td>
</tr>
<tr>
<td>(0,-3)</td>
<td>9</td>
</tr>
<tr>
<td>(0,3)</td>
<td>9</td>
</tr>
<tr>
<td>(-3,0)</td>
<td>-27</td>
</tr>
<tr>
<td>(3,0)</td>
<td>-27</td>
</tr>
</tbody>
</table>

The **global maximum** is 9. The **global minimum** is \(-27\).